Conformal dynamics of precursors to fracture

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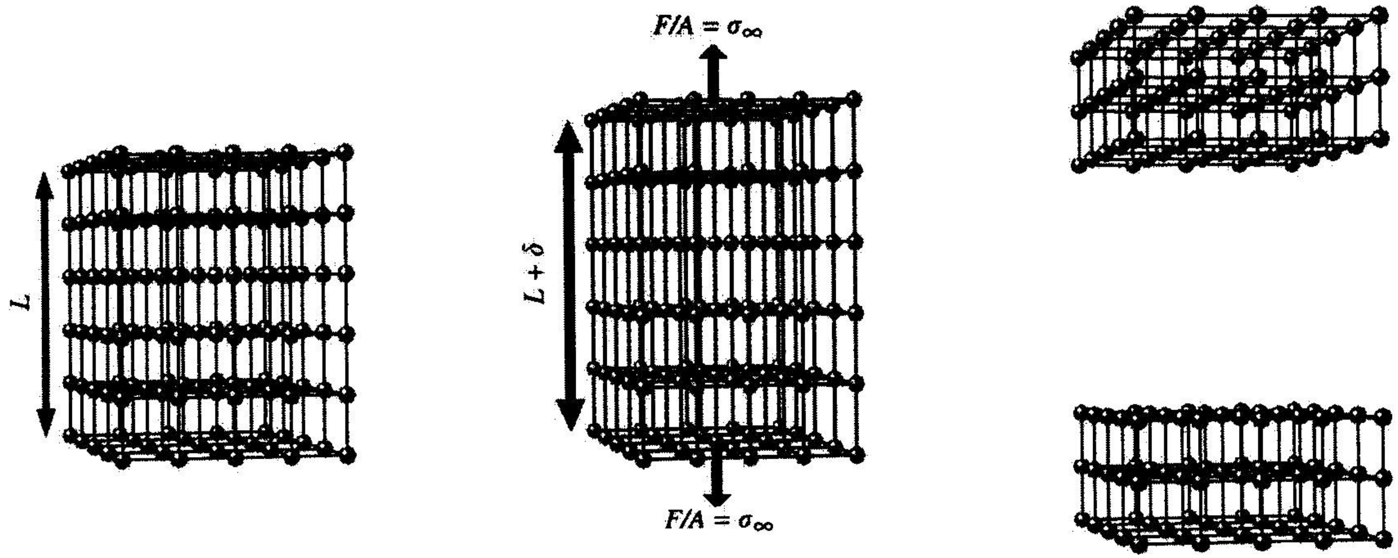
- Introduction
- Precursors to fracture and their dynamics ($2D$)
- Surface tension effects
- Conclusions

A fact: Solids break!

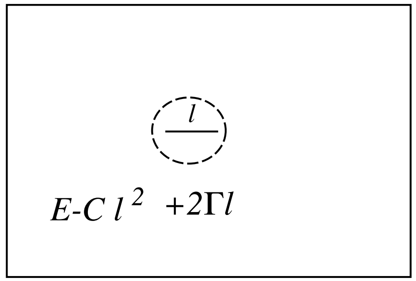
The Energy needed to break a perfect solid (bond breaking) is huge compared to experiments.



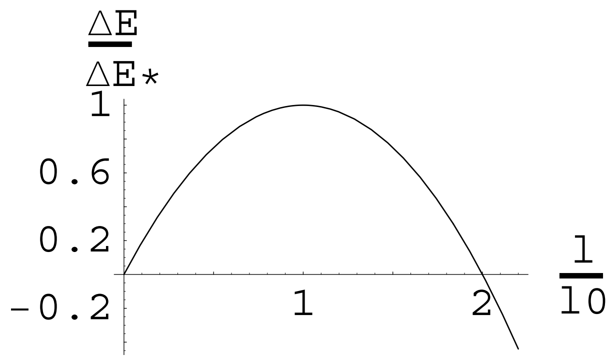
Fractures concentrate stress at their tip lowering the needed energy.

In general, fractures appears in free boundaries (borders or pores).

Lets compare the energy of a solid under tension with an without a fracture of length l.



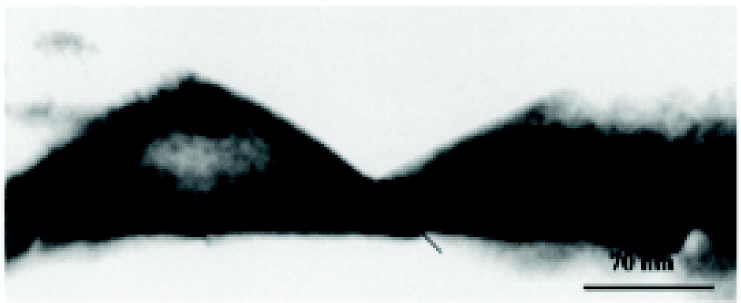
$$\Delta E = 2\Gamma l - C l^2$$



- If $l < l_0 = \Gamma / C$ a perfect solid is energetically preferred
- To nucleate a crack of length l_0 a barrier $\Delta E* = \Gamma^2 / C$ must be overcame.

Stress driven surface instability: an alternartive view

- Superface difusion as a mass transport mechanism
- Under stress a flat surface is unstable for perturbations of long wave-length: Asaro-Tiller-Grinfeld Instability
- Instability leads to the formation of grooves that seems to end in a cusp



H. Gao & W. D. Nix (Ann.Rev.Mater.Sci. **29** 173)

- It has been argued that the energy is always decresing in this process (H. Gao).

Moreover the groove tip concentrate the stress in the same way than a fracture and once the cusp is formed, fracture may proceed according to Griffith criterion (H. Gao).

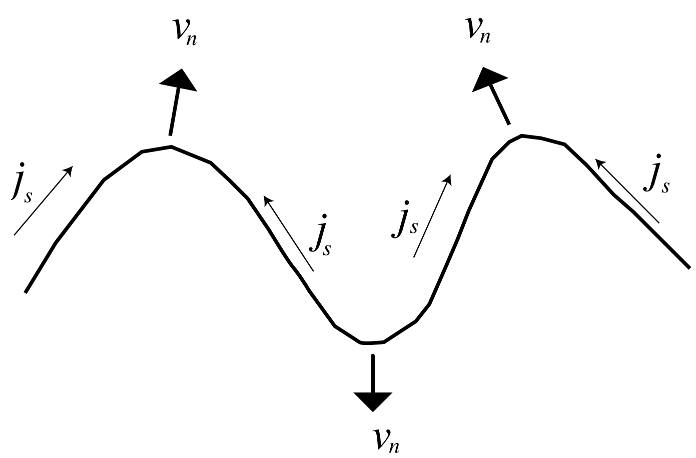
Precursors to fracture

In a large time scale, transport by diffusion can occur in the surface (and body) of a solid. This produces a movement of the interface according to Mullins

$$J_s = \frac{\nu D}{kT}\frac{\partial \mu}{\partial s}$$

Mass conservation (continuity equation) implies

$$v_n(s) = -\tilde{D}\frac{\partial^2 \mu}{\partial s^2}$$



Chemical potential

$$\mu = \mu_0 + C[S - \gamma\kappa] \ , \qquad (1)$$

- surface energy $\gamma ds \rightarrow \gamma\kappa$
  γ: dens. surface energy κ: mean curvature

- $S =$ deformation energy; $S = \frac{1}{2}\sum u_{ij}\sigma_{ij}$.
  The strain u_{ij} can be expressed in terms of the stress σ_{ij} and at the surface (plane stress and no surface tension):

$$S = \frac{1}{2E}[\mathrm{tr}\boldsymbol{\sigma}]^2$$

Finally we write

$$v_n = -\frac{\partial^2[S - \gamma\kappa]}{\partial s^2} \,. \tag{2}$$

The motion due to v_n is a slow motion. The stress field is equilibrated at a time scale corresponding to sound.

Therefore the elastic problem is considered static. This is the quasi static limit.

Newton's law:

$$\partial_i \sigma_{ij} = \rho \ddot{u}_j = 0$$

The problem is: given applied stresses on the boundary, determine the stress distribution on the body, but in 2D problems

$$\sigma_{xx} = \partial_y^2 U \ , \quad \sigma_{yy} = \partial_x^2 U \ , \quad \sigma_{xy} = -\partial_{xy} U \ ,$$

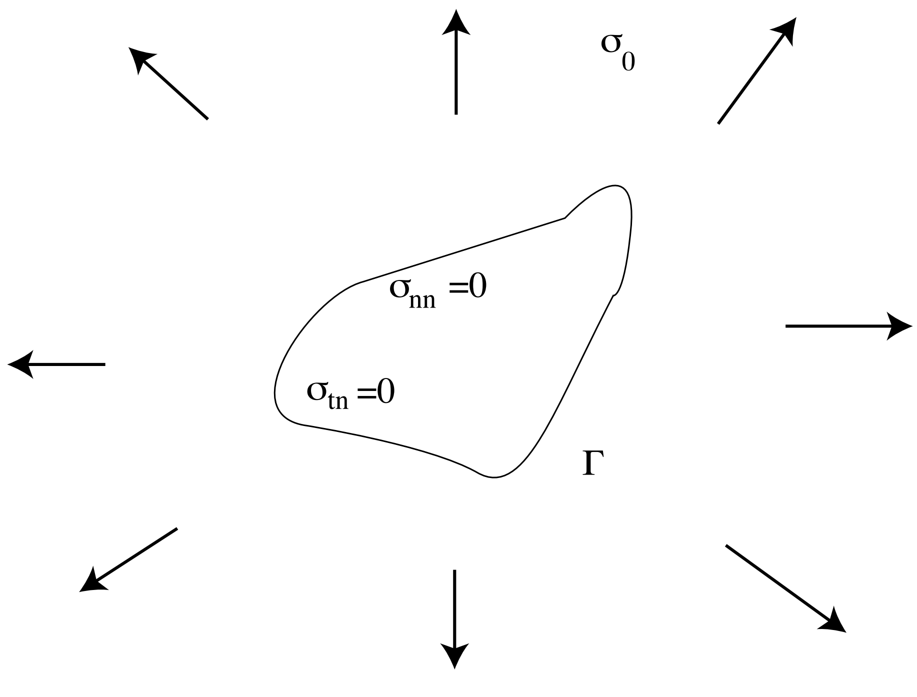
Hook$\rightarrow$ U is biharmonic: $\Delta^2 U = 0$

2D problems: Complex variables

The Gral. Sol. of the biharmonic Eq. is

$$U(z,\bar{z}) = \Re\left[\bar{z}\varphi(z) + \int \psi(z)\right] , \tag{3}$$

$\varphi(z)$ and $\int \psi(z)$: analytic functions of $z = x + iy$ determined later by boundary conditions.



$$\text{B.C. at} \quad \infty \to \begin{cases} \varphi(z) \to \frac{\sigma_0}{2} z \\ \psi(z) \to 0 \end{cases} \quad \text{as} \quad z \to \infty$$

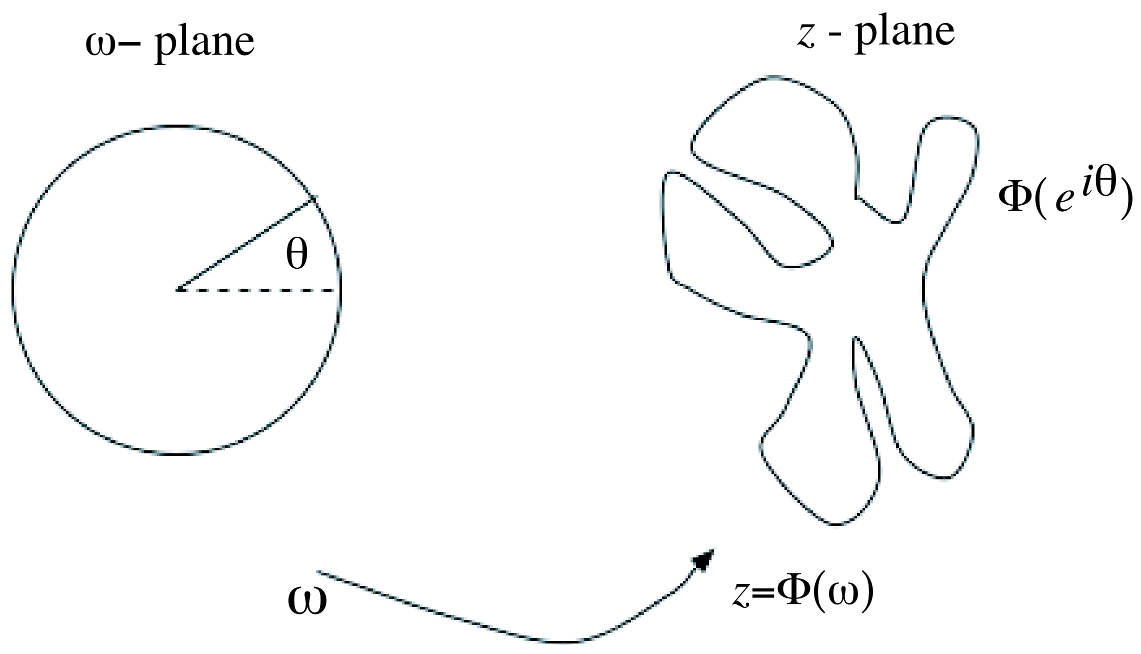
On the boundary Γ: $\sigma_{nn} = 0$ and $\sigma_{nt} = 0$ implies

$$\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} = K = 0 \quad \text{on} \quad \Gamma$$

This equation determine φ and ψ but .... The boundary is complex and is evolving!

Conformal maps: Treating the boundaries

Consider an analytic function outside the unit circle that map the exterior of the unit circle to the exterior of the object. The function satisfies $\Phi'(\omega) \neq 0$ everywhere outside the unit circle



$\tilde{\varphi}(\omega)$ $\qquad$ $\varphi(z) = \tilde{\varphi}(\Phi^{-1}(z))$

$$\Phi(\omega, t) = F_1(t)\ \omega + F_0(t) + \sum_{n=1}^{\infty} F_{-n}(t)\ \omega^{-n}\ .$$

The normal velocity $v_n(\epsilon)$

- Elastic energy on the boundary.

$$\varphi(\epsilon) + \frac{\Phi(\epsilon)}{\overline{\Phi'(\epsilon)}}\overline{\varphi'(\epsilon)} + \overline{\psi(\epsilon)} = 0 \ . \tag{4}$$

From the b.c. at ∞

$$\varphi(\epsilon) = \frac{\sigma_0}{2}F_1\epsilon + \varphi_0(\epsilon); \quad \psi(\epsilon) = \psi_0(\epsilon)$$

the unknowns are the expansion coefficients of ϕ_0 and ψ_0,

$$\varphi_0(\epsilon) = \sum_{n=0}^{\infty} u_{-n}\epsilon^{-n} \ ; \quad \psi_0(\epsilon) = \sum_{n=0}^{\infty} v_{-n}\epsilon^{-n} \ .$$

From Eq. (4) we obtain a linear system of equations for u_{-n} (and another for v_{-n}).

After solving:

$$u_n \rightarrow \varphi \rightarrow \varphi' \rightarrow \mathrm{Tr}\boldsymbol{\sigma} = 4\Re(\frac{\varphi'}{\Phi'}) \rightarrow S = \frac{1}{2E}[\mathrm{tr}\boldsymbol{\sigma}]^2$$

• The curvature κ is given in term of the conformal map as

$$\kappa = \Re\left(\frac{1}{|\Phi'|}(1 + \frac{\Phi''}{\Phi'}e^{i\theta})\right) \ . \tag{5}$$

$$v_n = -\partial_s^2(S - \gamma\kappa) = -\frac{1}{|\Phi'|}\partial_\theta\left(\frac{1}{|\Phi'|}\partial_\theta(S - \gamma\kappa)\right)$$

Motion of the interface

$$\frac{d\boldsymbol{R}(s,t)}{dt}\cdot \boldsymbol{n}(s) = v_n(s)$$

- $\boldsymbol{n}(s) \rightarrow n(s) = n_x + i\, n_y$ normal at $\Gamma(s)$

- $\boldsymbol{R}(s) \rightarrow z(s) = \Phi(e^{i\theta(s)})$ on $\Gamma(s)$

Then (we use $\epsilon = e^{i\theta}$)

$$v_n(s) = \Re\left[\frac{d\Phi(\epsilon,t)}{dt}\bar{n}\right] \; .$$

The tangent τ vector is

$$\tau = \frac{\partial\Phi(\epsilon)}{\partial s} = \frac{1}{|\Phi'(\epsilon)|}\frac{\partial\Phi(\epsilon)}{\partial\theta}$$

and the normal n is a rotation in 90^o of τ

$$n = -i\tau = \frac{\epsilon}{|\Phi'(\epsilon)|}\Phi'(\epsilon)$$

$$\frac{d\Phi(e^{i\theta},t)}{dt} = \partial_t\Phi + \Phi' e^{i\theta} i\theta_t$$

We derive:

$$v_n(s) = \Re\left(\partial_t \Phi(\epsilon)\overline{\epsilon\frac{\Phi'(\epsilon)}{|\Phi'(\epsilon)|}}\right) \quad . \tag{6}$$

We rewrite this equation,

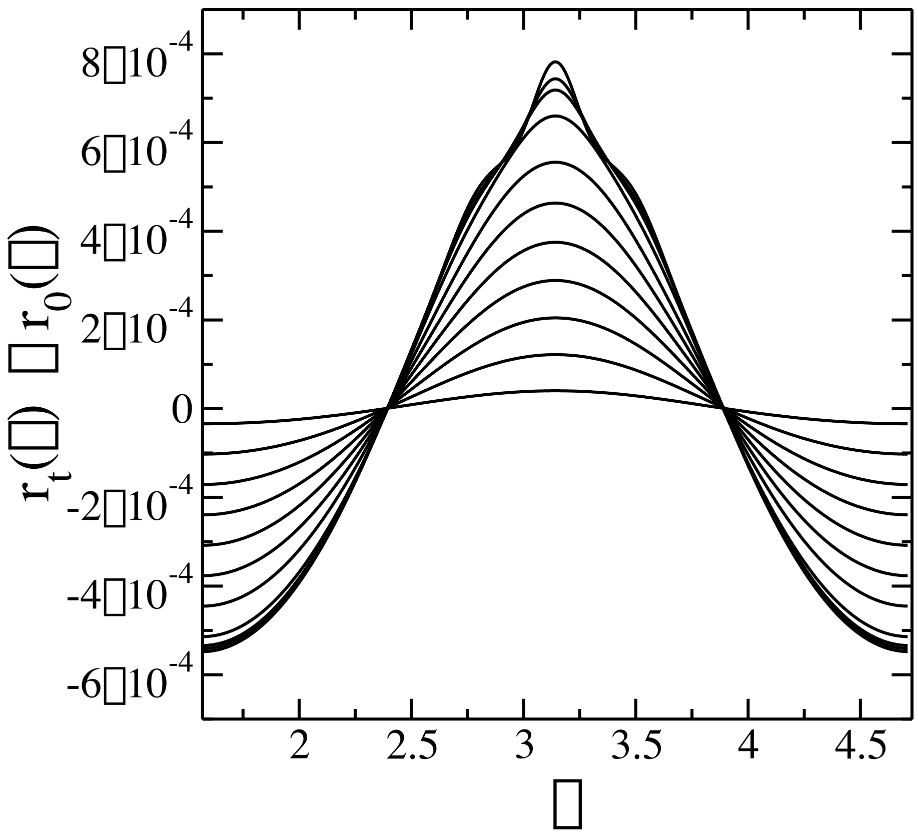
$$\partial_t \Phi = \epsilon \Phi'(\epsilon)\left(\frac{v_n}{|\Phi'|} + iC\right) \tag{7}$$

with an unknown imaginary part C.

Analytic continuation $\rightarrow$ Poisson integral formula:

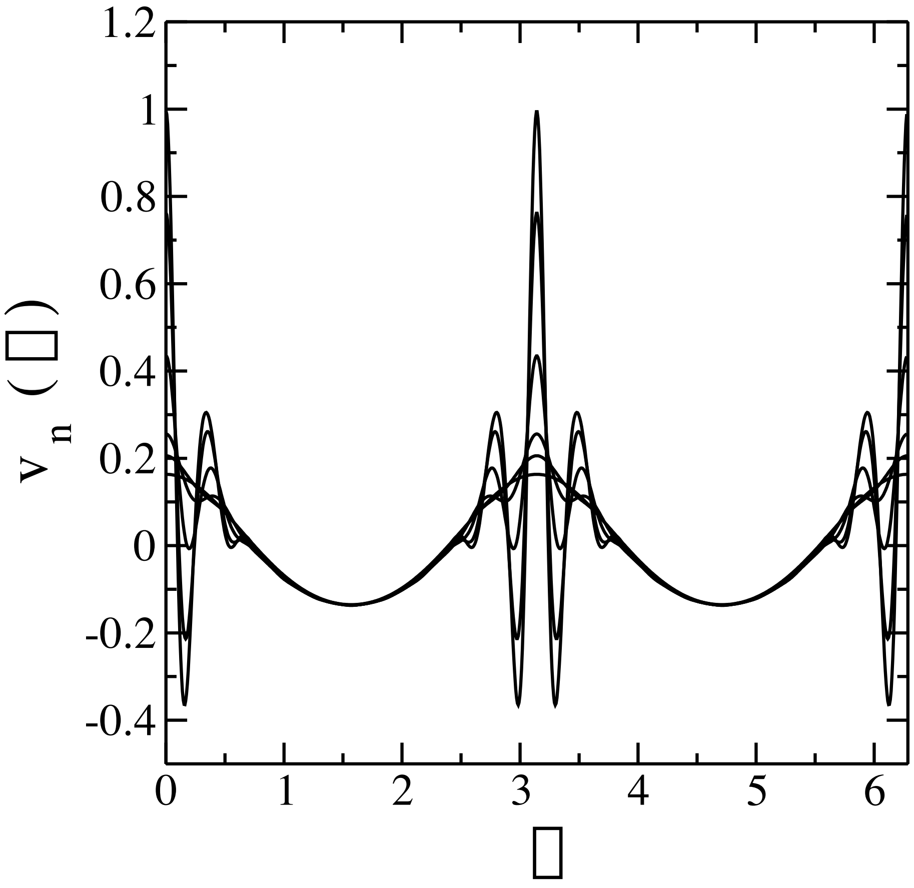
$$\partial_t \Phi = \omega \Phi'(\omega)\int_0^{2\pi} \frac{d\theta}{2\pi}\frac{\omega + e^{i\theta}}{\omega - e^{i\theta}}\frac{v_n(e^{i\theta})}{|\Phi'(e^{i\theta})|} \tag{8}$$

The equation, being analytic, must have analytic solutions which provide the dynamics of the conformal map we where seeking.



Evolution of an initial elliptical hole. Initial condition is

$$\Phi(\omega, t) = F_1(0)\omega + \frac{F_{-1}(0)}{\omega} \ , \qquad (9)$$



In the velocity we see the development of the Grinfeld instability

linear stability of a circular cavity

For a circular cavity under biaxial stress $v_n = 0$

Consider perturbations of the form

$$\Phi(\omega) = R\omega + \sum_{n=1} f_n \omega^{-n}; \quad |f_n| << 1 \quad \forall n$$

which maps the unit circle ($\epsilon = e^{i\theta}$) to a wavy shaped circle of radius R.

First: Compute linear contribution to κ

Second: Solve the elastic problem (boundary) the linear contribution to tr$\boldsymbol{\sigma}$ is

$$\mathrm{tr}\boldsymbol{\sigma} = 2\sigma_0 + 2\sum_{n=1}\left(2\frac{\sigma_0}{R}n\right)\Re[f_n\epsilon^{-n-1}]$$

To first order is correct to take $s = R\theta$, i.e.

$$v_n = -\frac{1}{R^2}\partial_\theta^2 \mu$$

v_n is linear in the perturbation amplitude.

$$g = \frac{v_n}{|\Phi'|} = \frac{v_n}{R} = \frac{1}{R^3}\sum_n (n+1)^2 A_n \Re\left[f_n \epsilon^{-n-1}\right]$$

with

$$A_n = \left(\frac{8\sigma_0^2}{ER}\right) n - \frac{\gamma}{R^2} n(n+2)$$

The analytic function G whose real part on the unit circle is g is simply

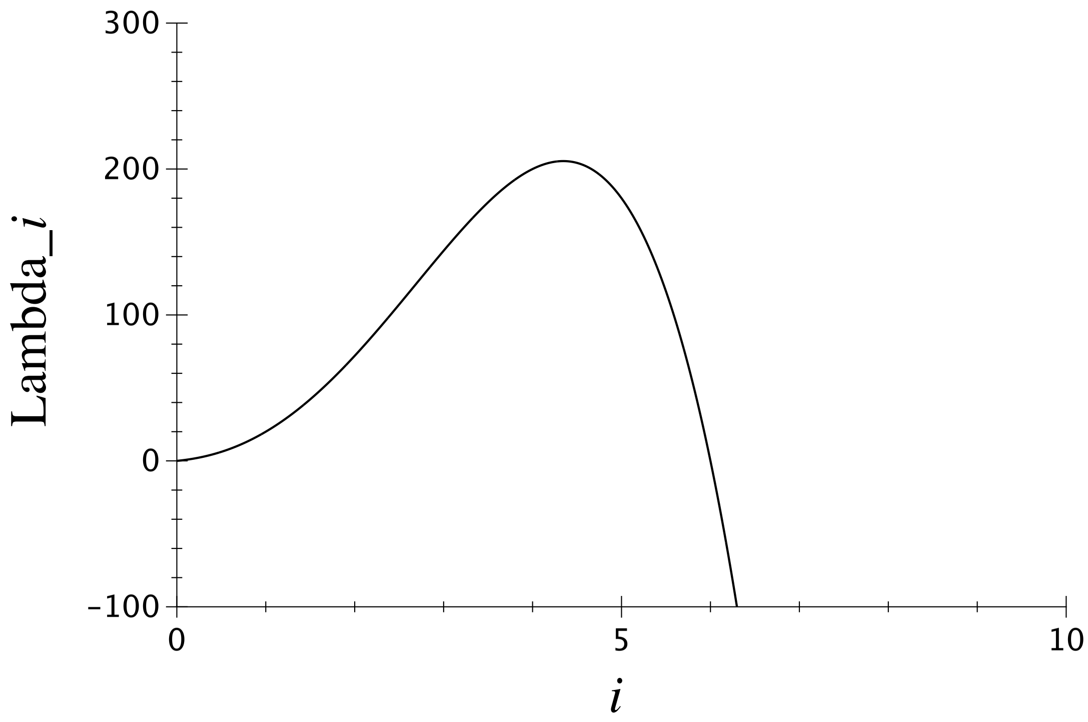
$$G = \frac{1}{R^3}\sum_n (n+1)^2 A_n f_n \omega^{-n-1}$$

and no integral must be computed.

Solutions of

$$\partial_t \Phi = \omega \Phi' G$$

of the form $f_n(t) = e^{\lambda_n t} f_n(0)$ exist with



$$\lambda_n = \frac{(n+1)^2}{R^2} A_n = \frac{(n+1)^2}{R^2} \left(\left[\frac{8\sigma_0^2}{ER} - \frac{2\gamma}{R^2} \right] n - \frac{\gamma}{R^2} n^2 \right)$$

instability occurs at a critical value $R_{crit} = \frac{E\gamma}{4\sigma_0^2}$

Surface Tension β Effects

The chemical potential $\mu = \mu_0 + C[S - \gamma\kappa + \beta\left(\frac{\partial \epsilon_{tt}}{\partial n} - \kappa\epsilon_{tt}\right)]$

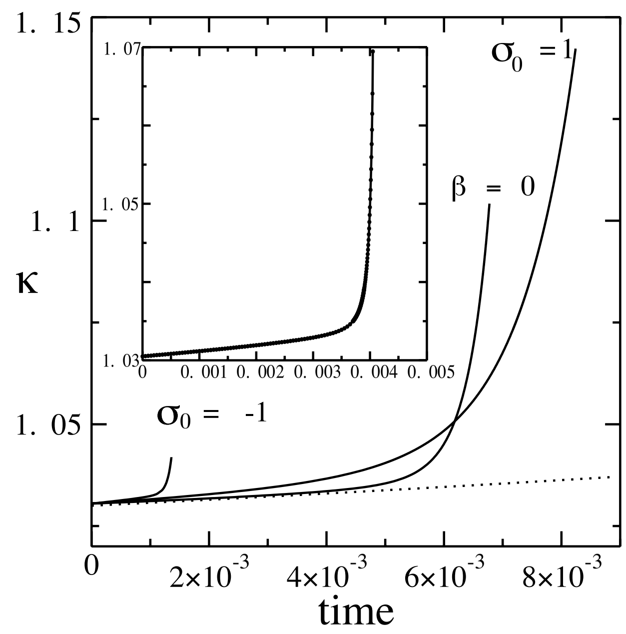
and $\sigma_{nn} = \beta\kappa$ on the boundary

Compression-Extension degeneracy is removed:

$$\frac{R^2}{(n+1)^2}\lambda_n = \frac{1}{E}\left[\left(\frac{8\sigma_0^2}{R} - \frac{12\beta\sigma_0}{R^2}\right)n - ...\right] - \frac{2\gamma}{R^2}n + ...$$

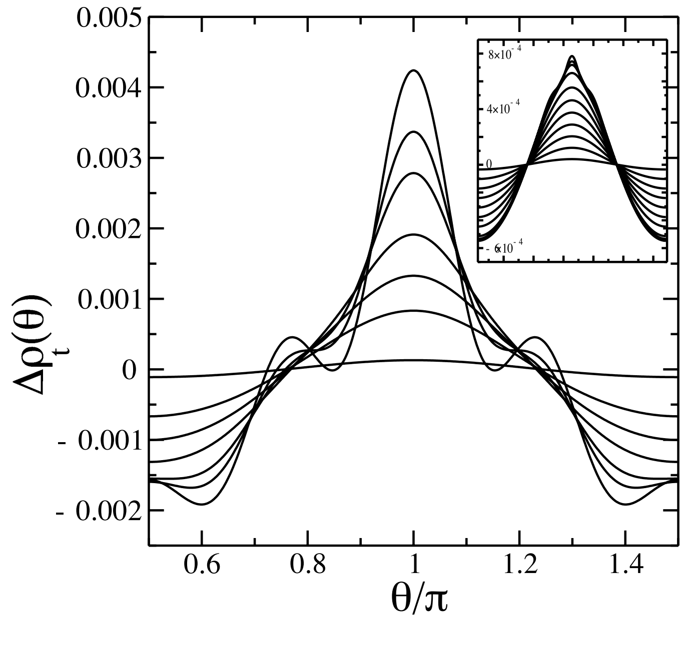
The critical value $R_{crit} = \frac{E\gamma}{4\sigma_0^2} + \frac{3\beta}{2\sigma_0}$

Nonlinear regime, we observed that grooves start to form faster for compression.

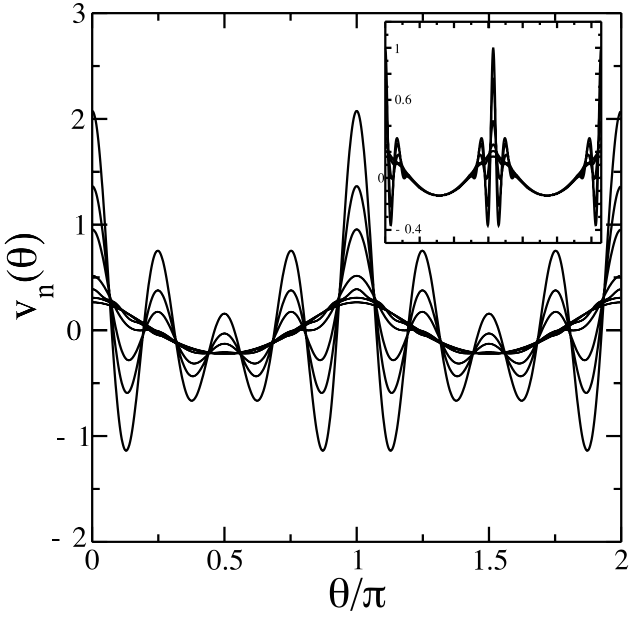


Grow faster than exponential.... finite time singularity?

Minimizing energy among a family of cycloids, minimum is attained for the cusped cycloid.



0.005
0.004
0.003
0.002
0.001
0
- 0.001
- 0.002
$\Delta\rho_t(\theta)$
0.6
0.8
1
1.2
1.4
θ/π



3
2
1
0
- 1
- 2
$v_n(\theta)$
0
0.5
1
1.5
2
θ/π
1
0.6
- 0.4

conclusions

- We have studied the dynamics of evolving cavities in stressed materials considering the effect of surface tension

- Conformal dynamics have been useful in the past to studied problems with finite time singularities (Shraiman - Bensimon dynamics for invisid fluids without surface tension)

- Analytically simple linear analysis of the instability.