Persistent patterns in nonlocal models.

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Part I. Scalar Models

Goal: Analyze pattern formation in the equation

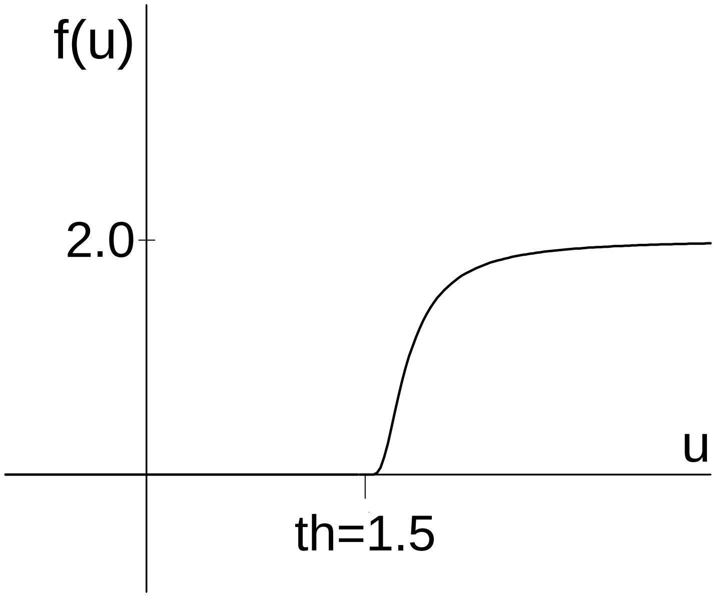
$$\frac{\partial u(x,y,t)}{\partial t} = -u + \int\int_{\mathbb{R}^2} w(x-s, y-q) f(u(s,q,t) - th) ds dq$$

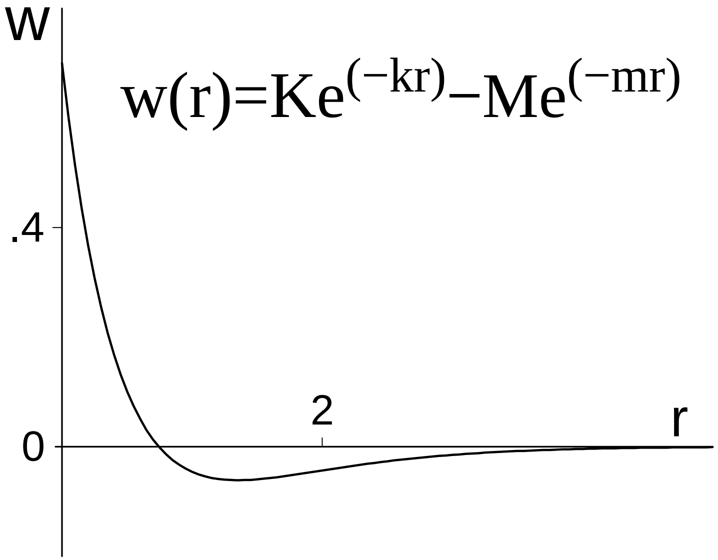
- $u(x,y,t)$ is the activity level (voltage) at position (x,y) at time t.
- $w(z)$ is the coupling weight.
- f is the firing rate function.
- $th > 0$ is the threshold.

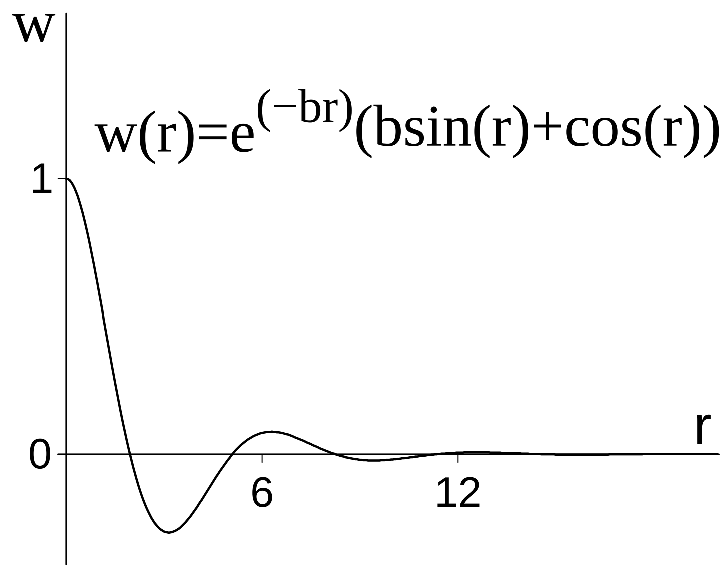
The Firing Rate

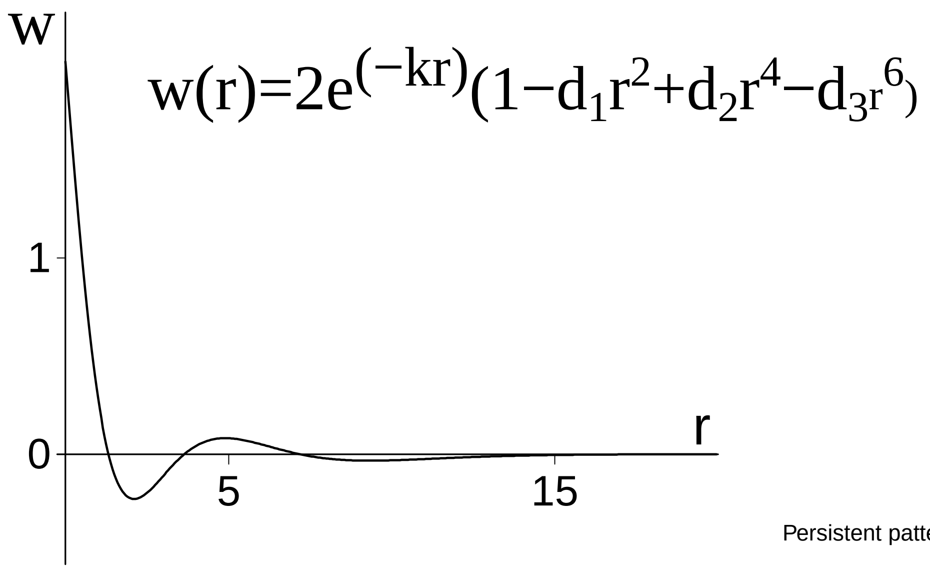
$$f(u-th) = Qexp(\frac{-\rho}{(u-th)^2})H(u-th)$$

H is the Heaviside function. Below: $Q = 2,\ \rho = .1,\ th = 1.5$




w
$w(r)=Ke^{(-kr)}-Me^{(-mr)}$
.4
0
2
r


w
$w(r)=e^{(-br)}(b\sin(r)+\cos(r))$
1
0
6
12
r


w
$w(r)=2e^{(-kr)}(1-d_1r^2+d_2r^4-d_{3r}^6)$
1
0
5
15
r

PDE Derivation

$$u_t + u = \int\int_{\mathrm{R}^2} w(\sqrt{(x-s)^2 + (y-q)^2}) f(u(s,q,t) - th) ds dq$$

Apply the two–dimensional Fourier transform defined by

$$\widehat{F}(g) \equiv (2\pi)^{-1} \int\int_{R^2} exp(-i(\alpha x + \beta y)) g(x,y) dx\, dy$$

$$\widehat{F}(u + u_t) = \widehat{F}(w)\widehat{F}(f(u - th))$$

If $w = w(r)$ then $\widehat{F}(w) = \widehat{F}(\sqrt{\alpha^2 + \beta^2})$. To obtain the PDE we approximate $\widehat{F}(w)$ by a rational function of $\sqrt{\alpha^2 + \beta^2}$.

A Lateral Inhibition Coupling

$$w(r) = 3.5e^{-2.8r} - 2.9e^{-1.9r}$$

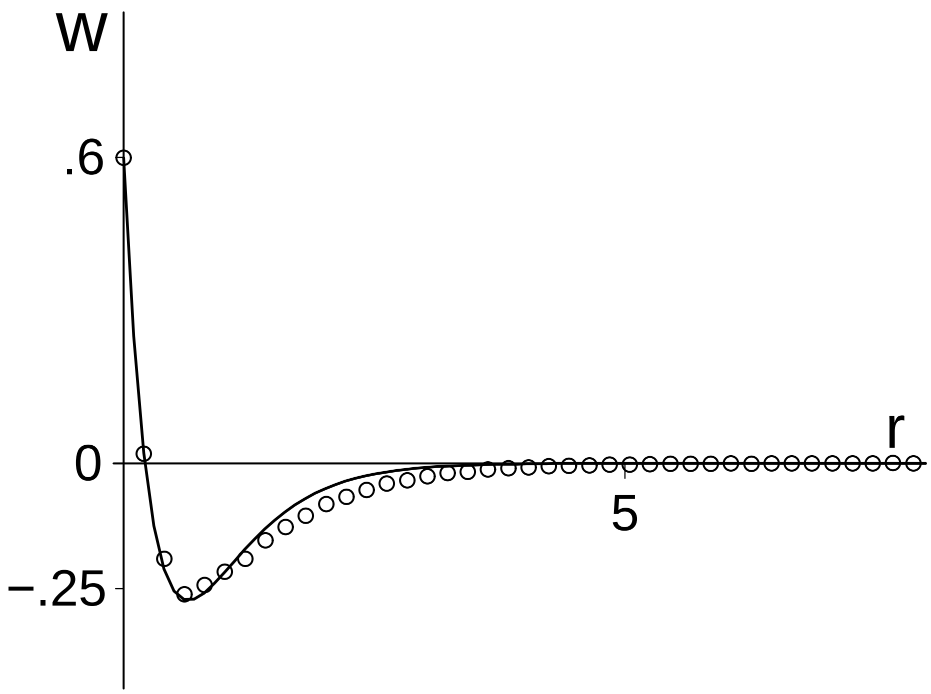
$$\hat{F}(w)(\eta) = \frac{9.8}{(7.84 + \eta^2)^{3/2}} - \frac{5.51}{(3.61 + \eta^2)^{3/2}}$$

where $\eta = \sqrt{\alpha^2 + \beta^2}$. Approximate $\hat{F}(w)$ by

$$G(\eta) = \frac{-.0808\eta^2 - .1755}{7.7592 + 4.1991\eta^2 + 3.3163\eta^4}$$

The inverse of G is

$$\tilde{w}(r) = \int_0^\infty sG(s)J_0(rs)ds$$



Original w(r) (solid) and approximation (circles).

Example

$$\widehat{F}(u + u_t) = \widehat{F}(w)\widehat{F}(f(u - th))$$

$$\widehat{F}(w) = \frac{A}{B + (\alpha^2 + \beta^2 - M)^2}$$

$$((\alpha^2+\beta^2)^2 - 2M(\alpha^2+\beta^2) + B + M^2)\widehat{F}(u+u_t) = A\widehat{F}(f(u-th))$$

Identities:

$$(\alpha^2 + \beta^2)^2\widehat{F}(g) = \widehat{F}(\nabla^4 g) \quad \text{and} \quad (\alpha^2 + \beta^2)\widehat{F}(g) = -\widehat{F}(\nabla^2 g)$$

Resultant PDE:

$$(\nabla^4 + 2M\nabla^2 + B + M^2)(u_t + u) = Af(u - th)$$

N-bump solutions.

(I) Change to polar coordinates and find symmetric solns.

$$L \equiv \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + 2M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + B + M^2$$

$$L(u_t + u) = Af(u - th),$$

(II) Find stationary solutions of the ODE problem

$$\begin{cases} Lu = Af(u - th), \\ u'(0) = u'''(0) = 0, \text{ and } \lim_{r\to\infty}(u, u', u'', u''') = (0, 0, 0, 0). \end{cases}$$

(III) Linearize the PDE around the ODE solution

Linearization

$$u(r,\theta,t) = \widetilde{u}(r) + \mu\nu(r,t)\cos(m\theta), \qquad 0 < \mu << 1$$

To first order ν satisfies

$$[\frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} + \left(\frac{2Mr^2 - 2m^2 - 1}{r^2}\right)\frac{\partial^2}{\partial r^2} + \left(\frac{2m^2 + 1 + 2Mr^2}{r^3}\right)\frac{\partial}{\partial r}$$

$$+\frac{m^4 - 4m^2 + (B + M^2)r^4 - 2Mm^2r^2}{r^4}](\nu + \frac{\partial\nu}{\partial t}) = Af'(\widetilde{u} - th)\nu$$

Let $\nu(r,0) = e^{-r^2}$. We expect that $\nu(r,t) \sim \overline{\nu}(r)e^{\lambda t}$ as $t \to \infty$,

$$\nu(r,t) \sim \overline{\nu}(r)e^{\lambda t} \text{ as } t \to \infty,$$

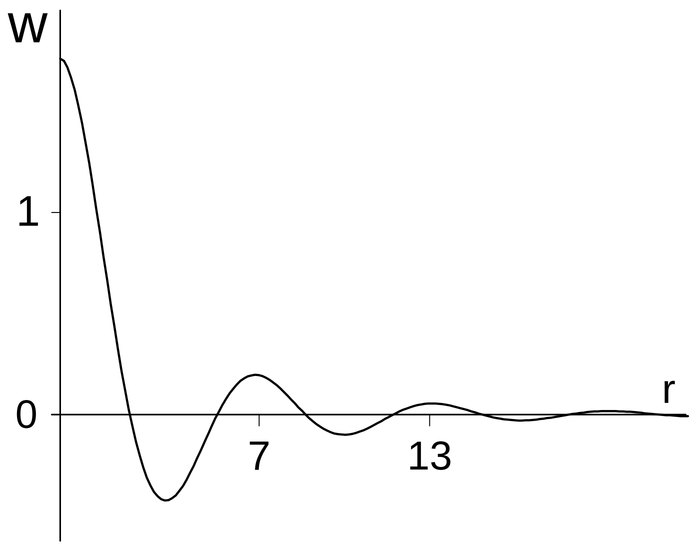
where λ is the largest eigenvalue, $\overline{\nu}(r)$ is the eigenfunction.

Example: M=1, A=.4, B=.1

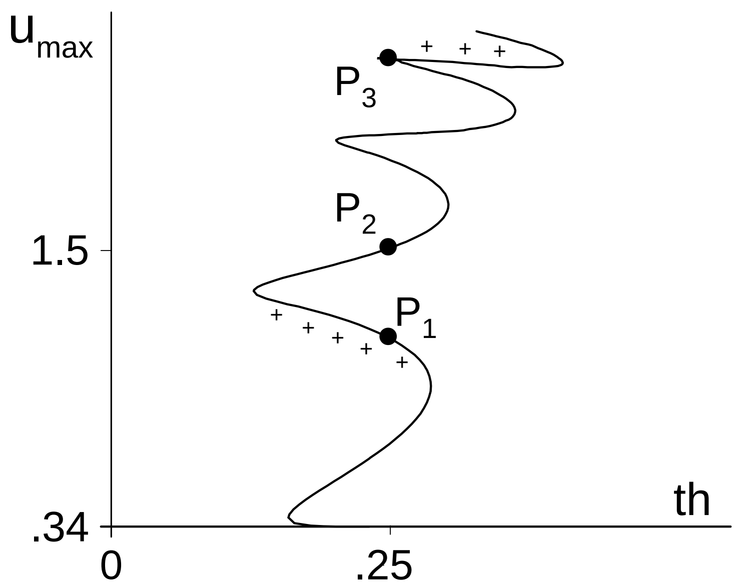
$$\widehat{F}(\sqrt{\alpha^2+\beta^2}) = \frac{A}{B+(\alpha^2+\beta^2-M)^2}$$

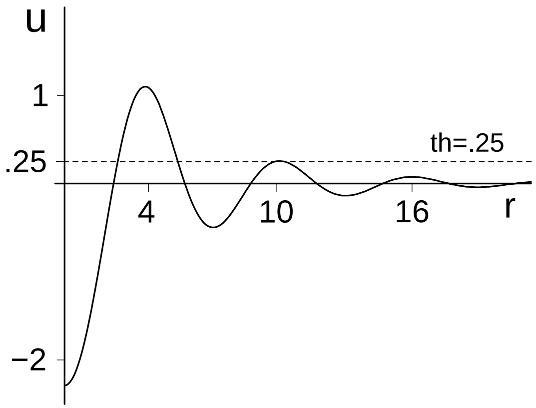
The inverse is given by

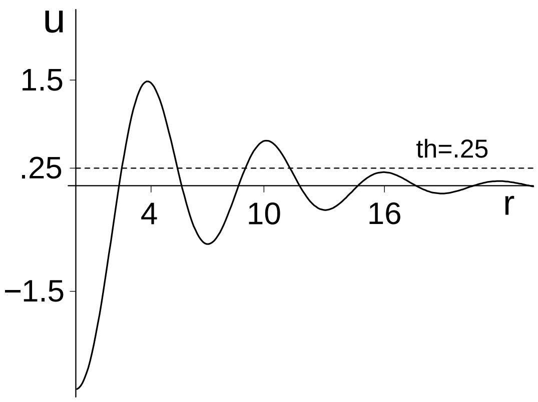
$$w(r) = \int_0^\infty s\widehat{F}(s)J_0(rs)ds$$

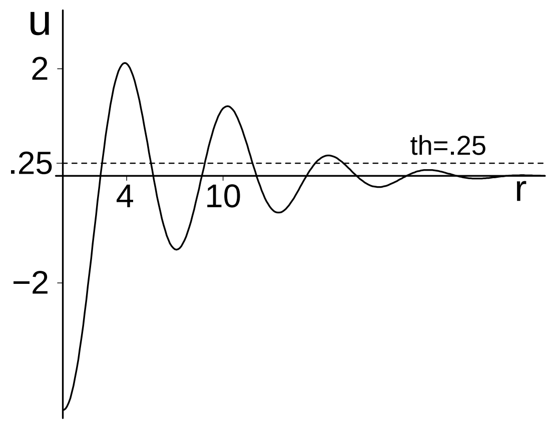


ODE Bifurcation diagram.

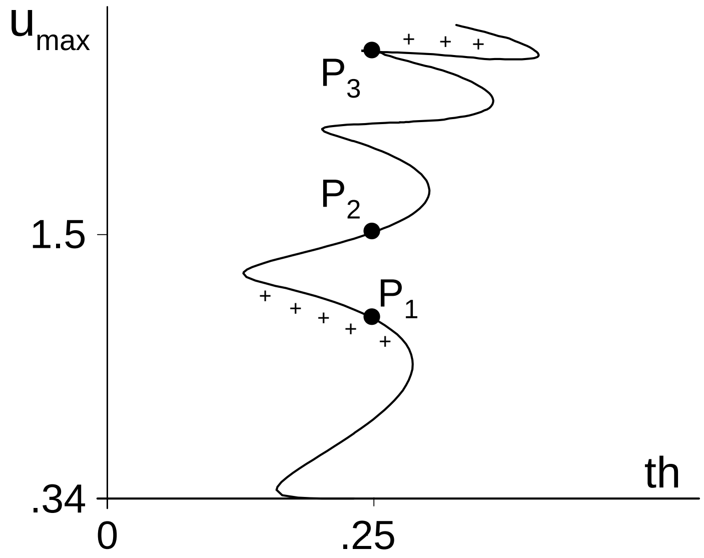


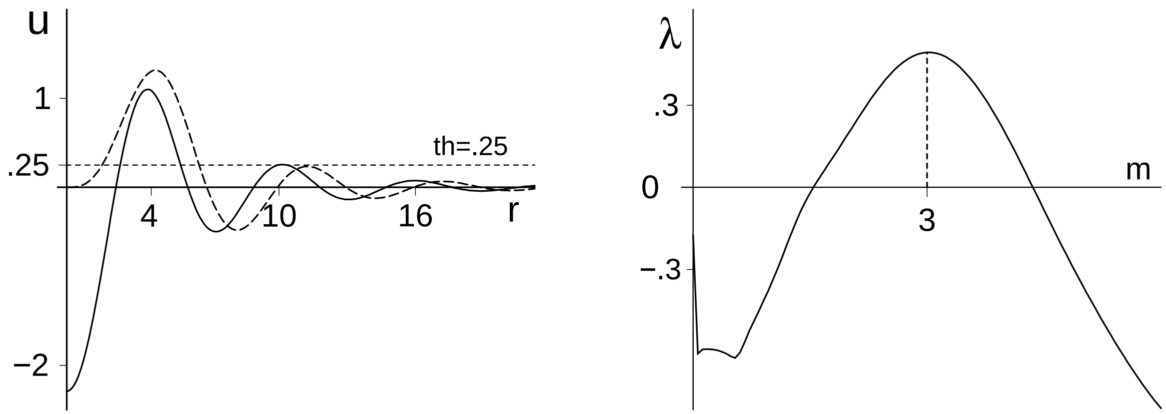




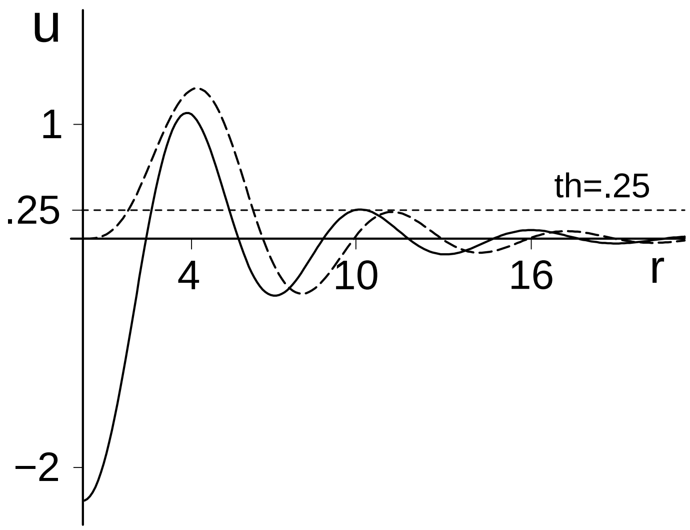


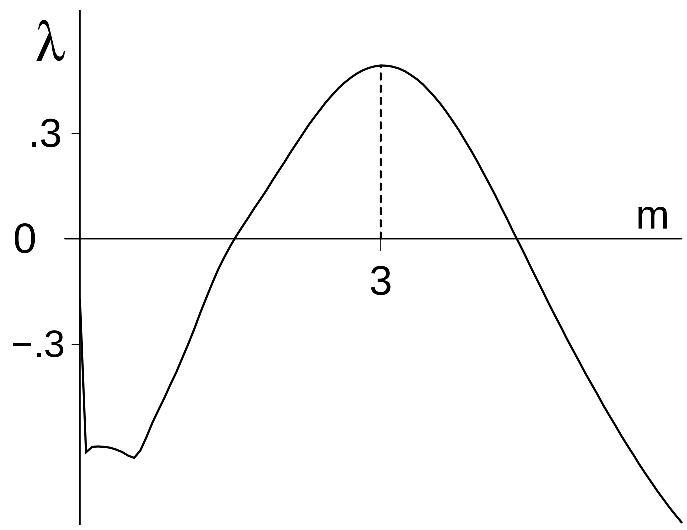
3-bump solution.

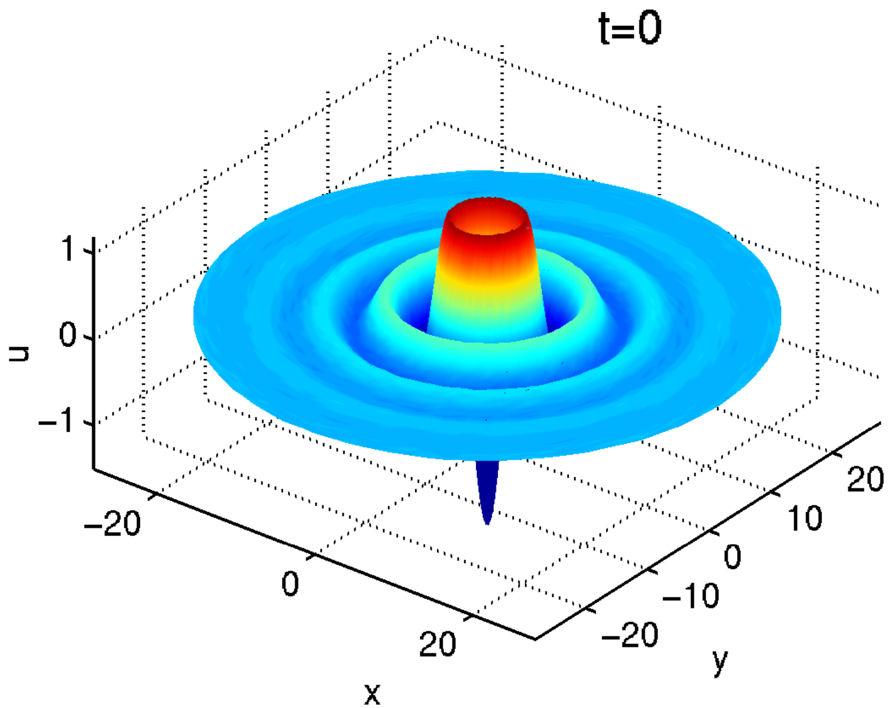


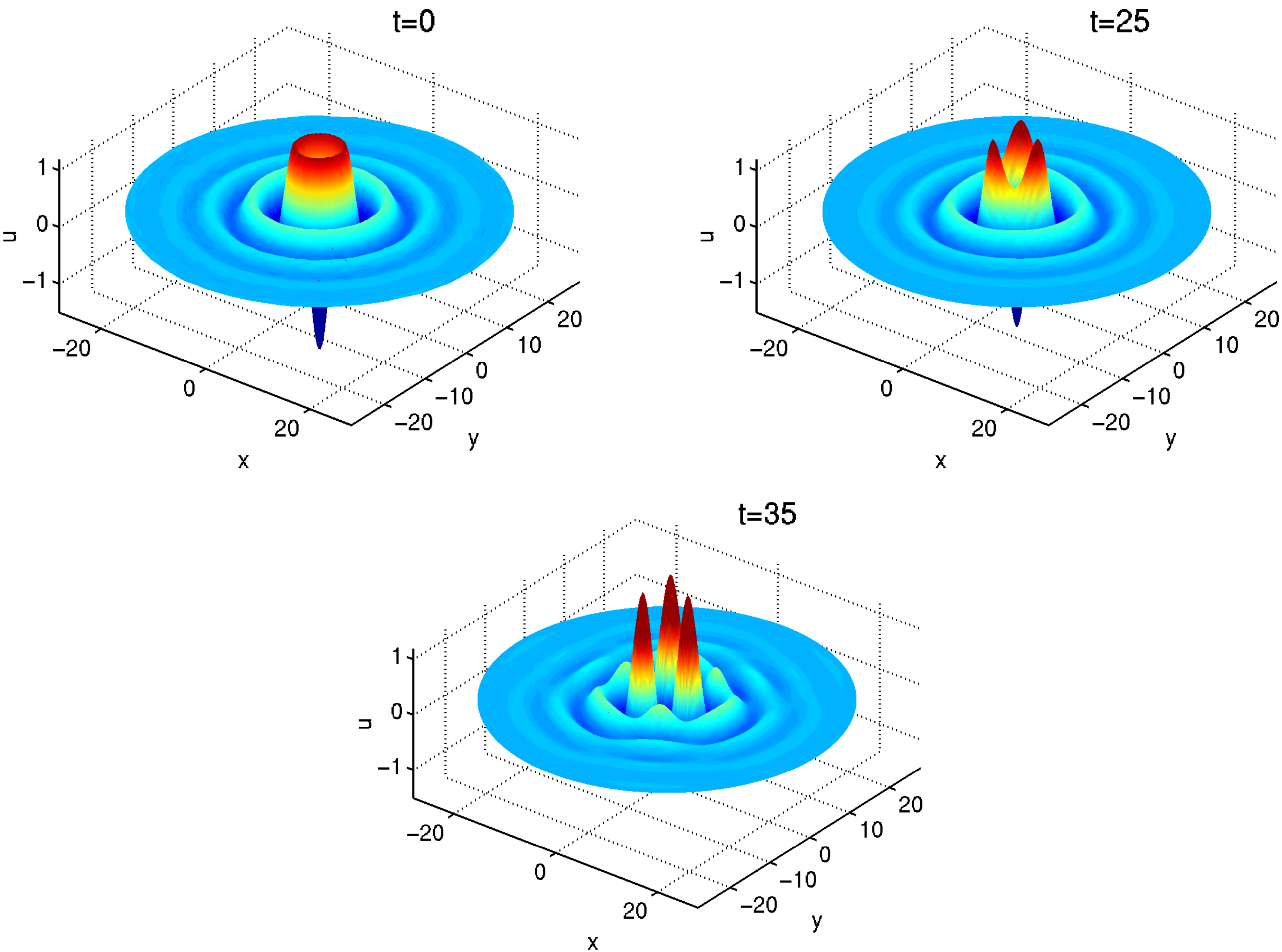


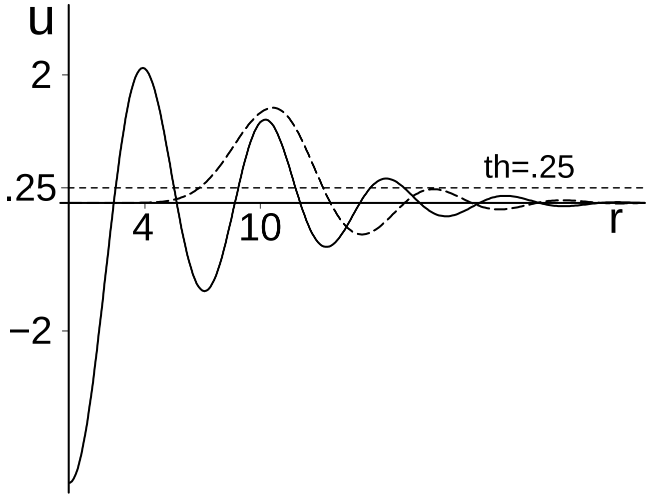
Left: ODE sol. and eigenfunction at m=3. Right: λ vs. m.

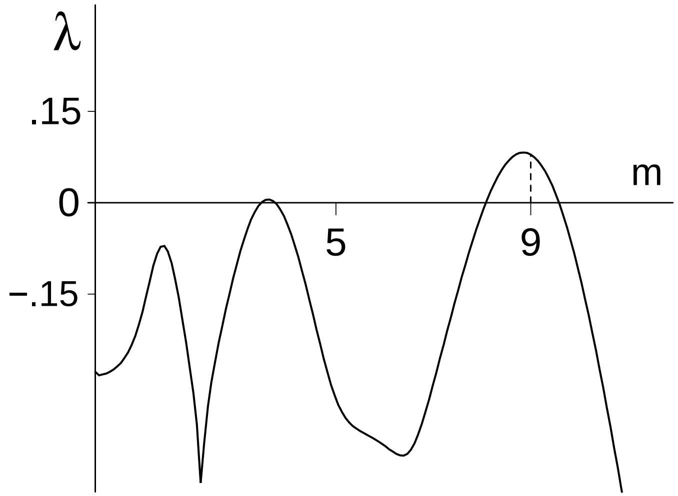

u
1
.25
−2
th=.25
4
10
16
r

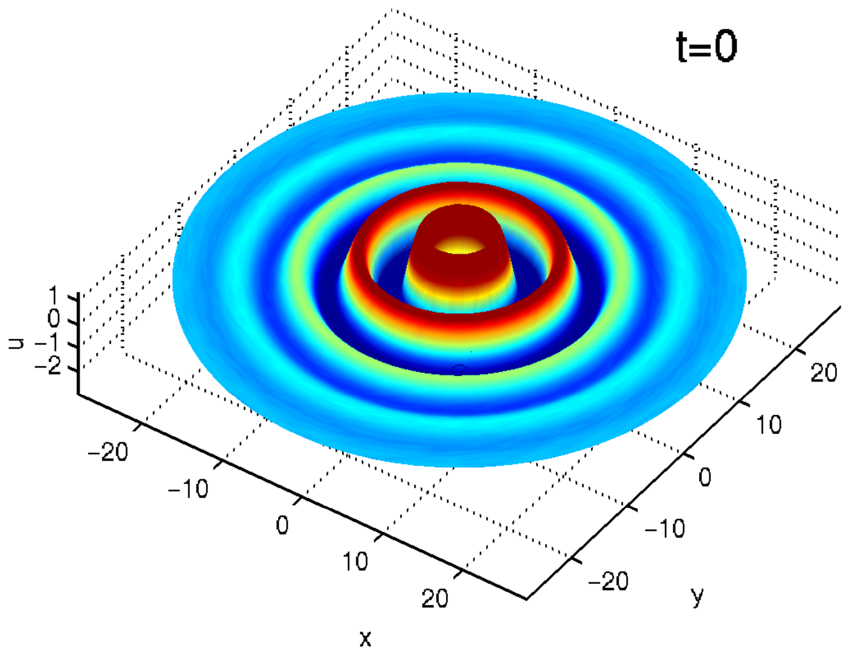

λ
.3
0
−.3
m
3

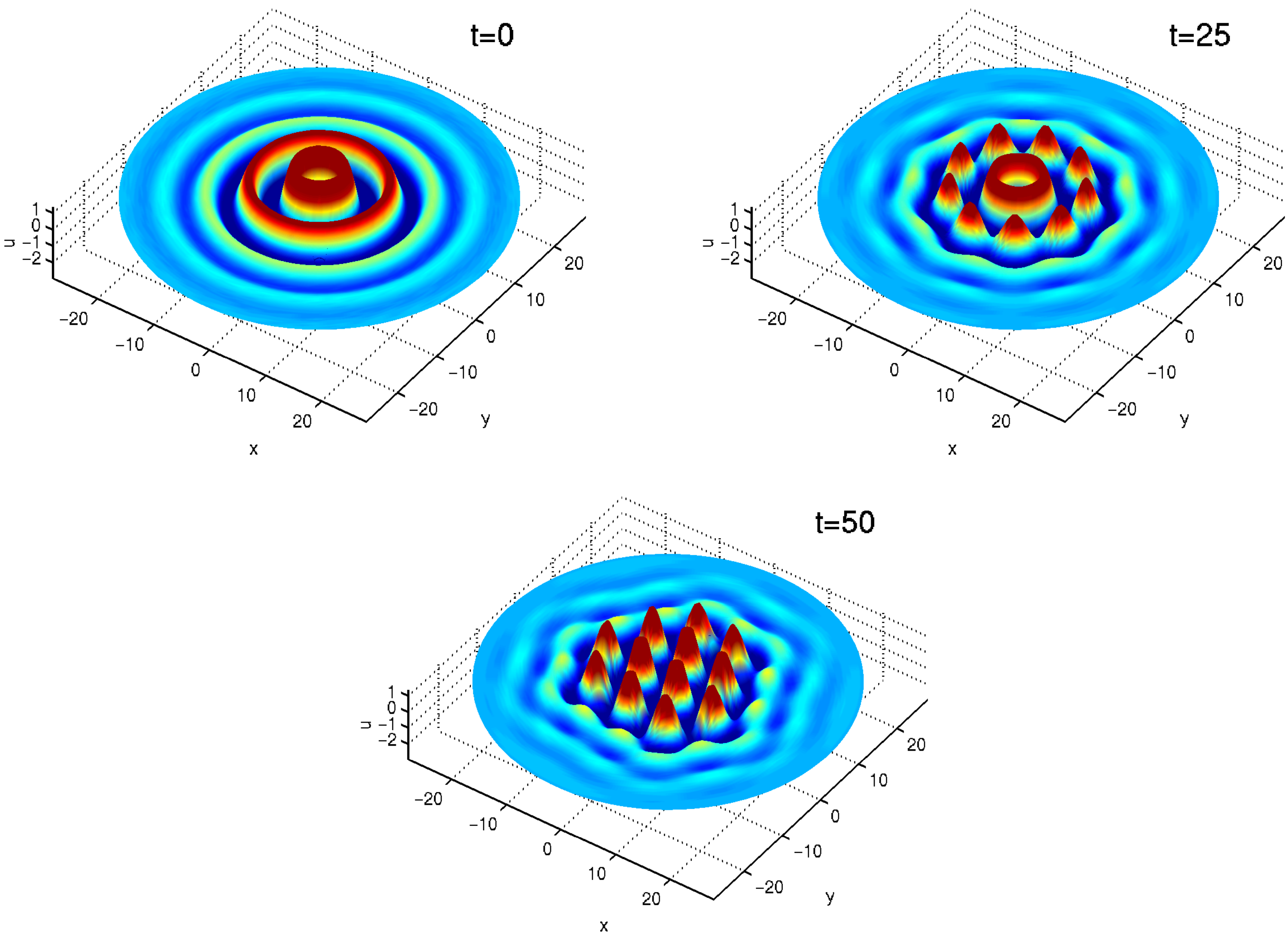

t=0
u
1
0
−1
−20
0
20
x
20
10
0
−10
−20
y

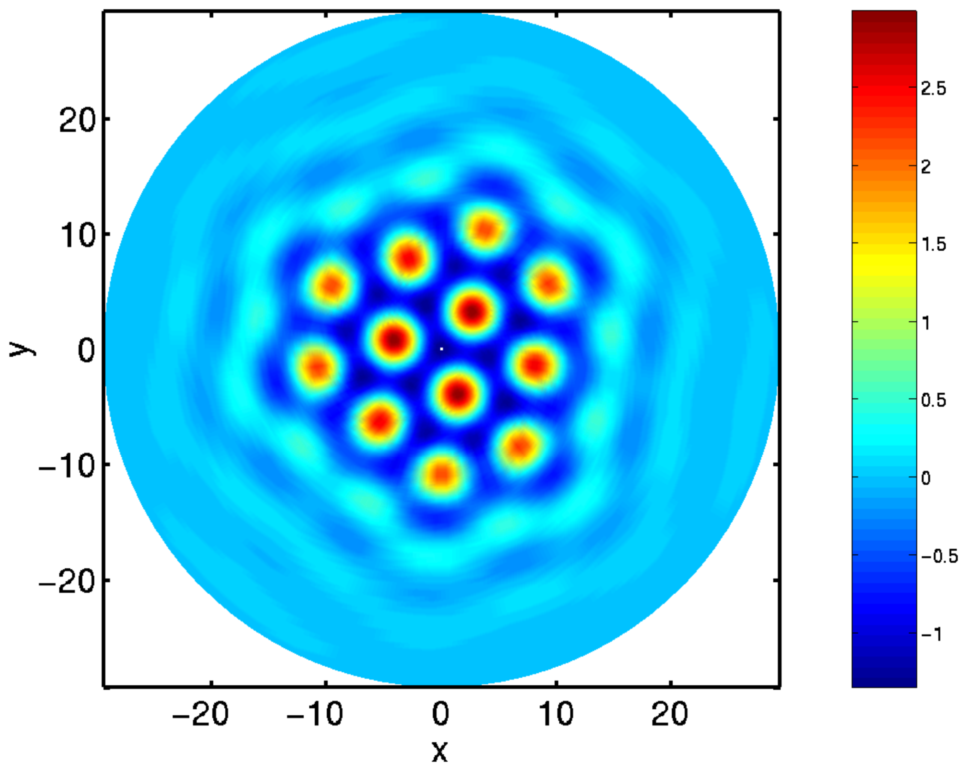

t=0
t=25
t=35
u
x
y
1
0
−1
−20
0
20
10
−10


u
2
.25
−2
4
10
r
th=.25


λ
.15
0
−.15
5
9
m

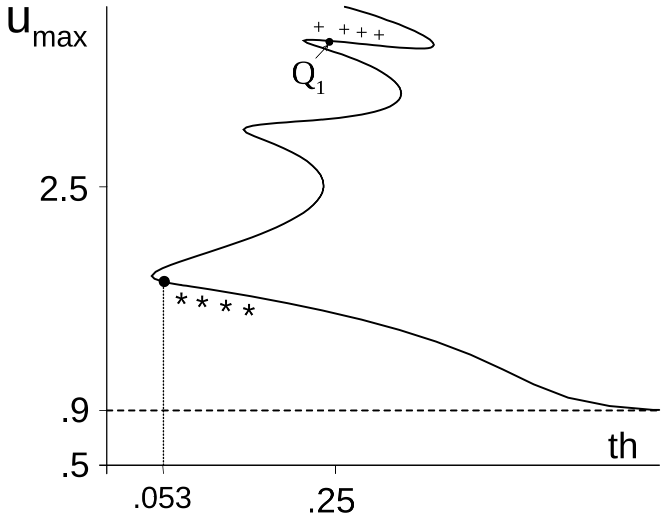

t=0
u
1
0
−1
−2
−20
−10
0
10
20
x
20
10
0
−10
−20
y

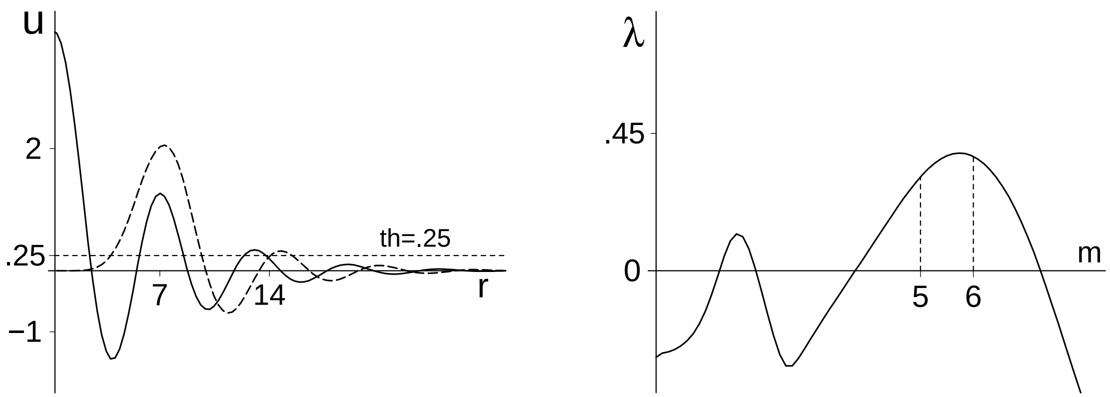

t=0
t=25
t=50
u
1
0
−1
−2
−20
−10
0
10
20
x
y



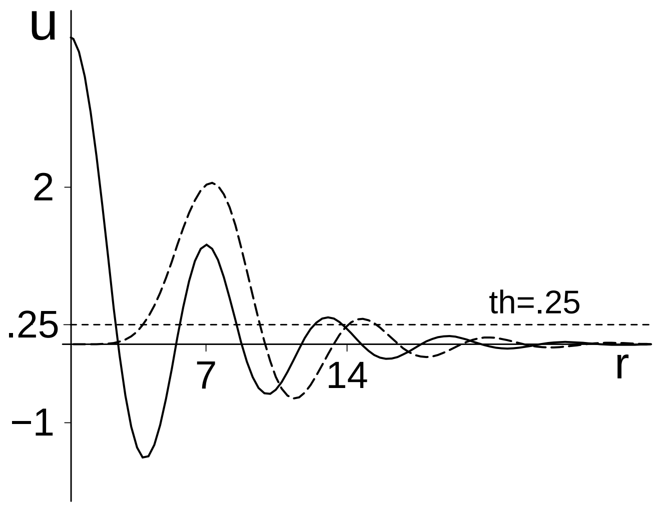
Level Curves

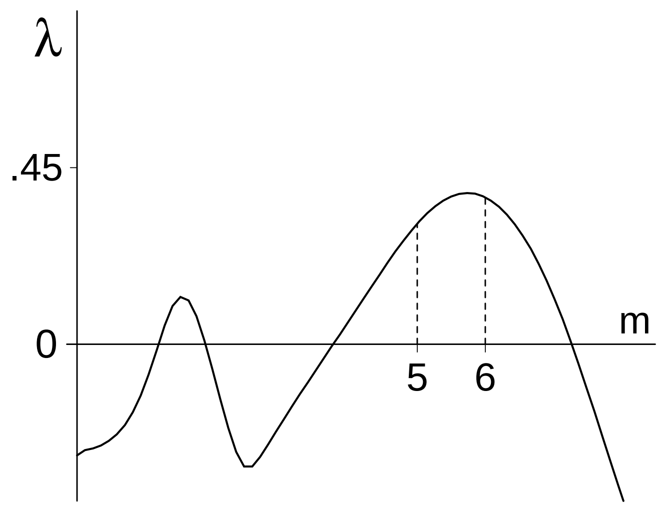
7-bump solution.

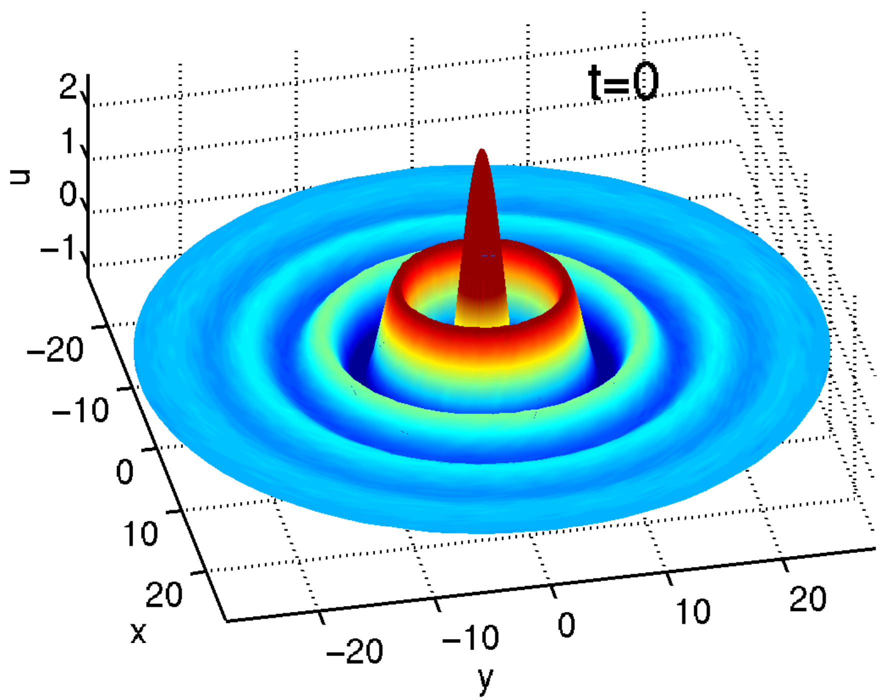


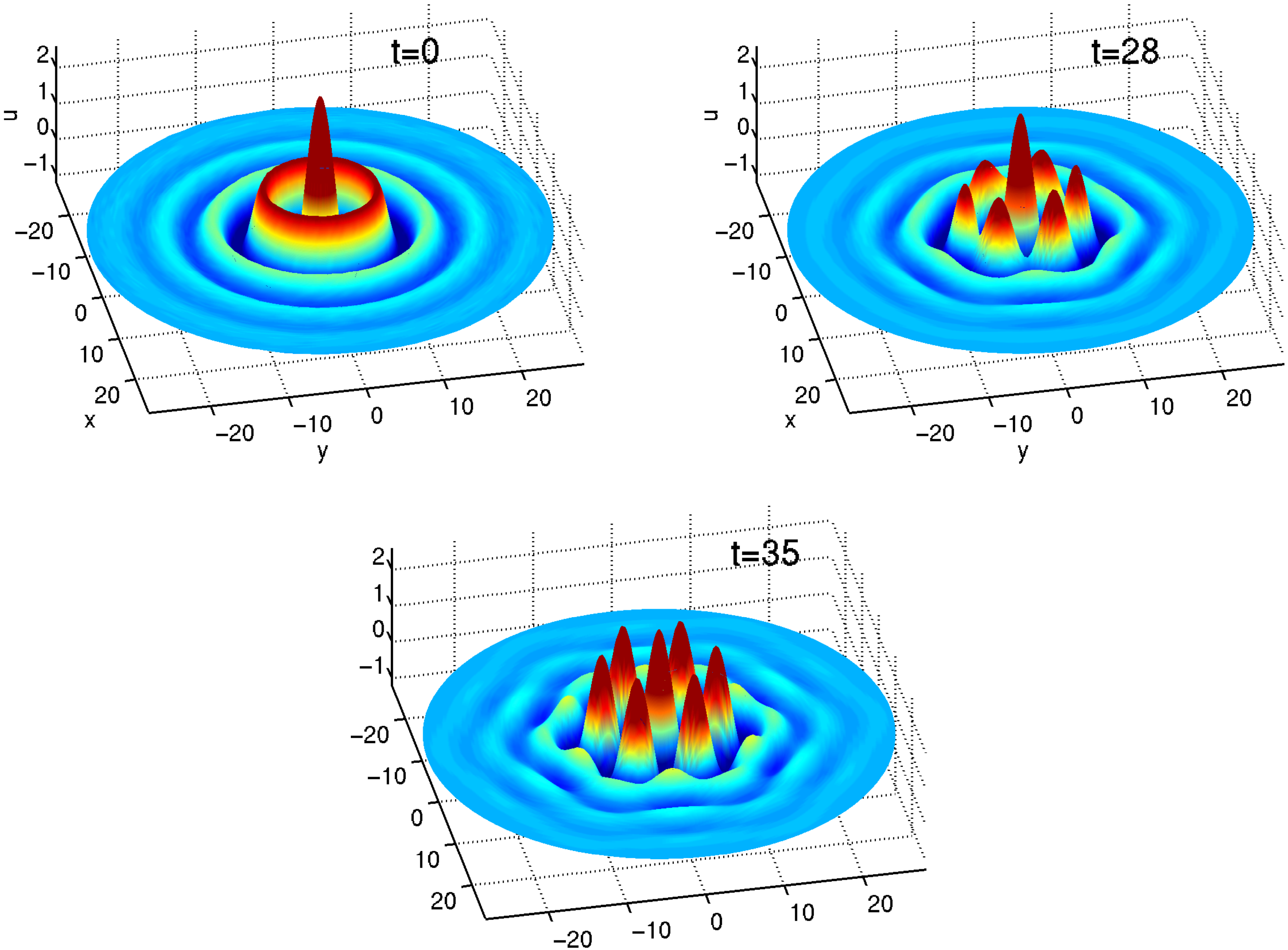


Left: ODE sol. and eigenfunction at m=6. Right: λ vs. m.


u
2
.25
−1
7
14
r
th=.25


λ
.45
0
5
6
m


t=0
u
2
1
0
−1
−20
−10
0
10
20
x
−20
−10
0
10
20
y


t=0
t=28
t=35
u
x
y
2
1
0
−1
−20
−10
0
10
20

Part II. Extension To Systems

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -u + \iint_{\Omega} w(x,y,q,s) f(u - th)\, dq\, ds - \beta a + I \\
\tau \frac{\partial a}{\partial t} &= Cu - a
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -u + \iint_{\Omega} w(x,y,q,s) f(u - a - th)\, dq\, ds + I \\
\tau \frac{\partial a}{\partial t} &= Cu - a,
\end{aligned}
$$

where $a(x,y,t)$ is an “adaptation" or “recovery" variable.

Target Patterns

(I.) Use the PDE approach to transform

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= -u + \iint_\Omega w(x,y,q,s) f(u - a - th)\, dq\, ds \\
\tau \frac{\partial a}{\partial t} &= Cu - a.
\end{aligned}
$$

into

$$
\begin{aligned}
(\nabla^4 + 2M\nabla^2 + B + M^2)(u + u_t) &= Af(u - a - th) \\
\tau a_t &= Cu - a.
\end{aligned}
$$

(II.) Find time independent axially symmetric solutions of

$$\begin{cases} Lu = Af(u - a - th), \\ 0 = Cu - a, \\ u'(0) = u'''(0) = 0, \text{ and } \lim_{r\to\infty}(u, u', u'', u''') = (0,0,0,0). \end{cases}$$

where

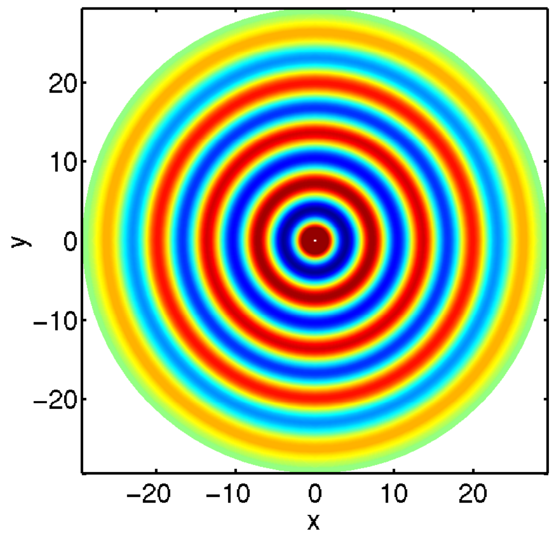
$$L \equiv \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} + 2M\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + B + M^2$$

Use Auto97 to find a solution $(u_C(r), a_C(r))$ for $C \geq 0$.

(III.) For each $C \geq 0$ solve

$$\begin{aligned}(\nabla^4 + 2M\nabla^2 + B + M^2)(u_t + u) &= Af(u - a - th) \\ \tau a_t &= Cu - a,\end{aligned}$$

$$(u(r,0), a(r,o)) = (u_C(r), a_C(r)) + \text{ small perturbation.}$$



Target Pattern.

Rotating Waves

Substitute $(u,a) = (u(r,\phi), a(r,\phi)),\ \ \phi = \theta - \omega t$ into

$$
\begin{aligned}
(\nabla^4 + 2M\nabla^2 + B + M^2)(u + u_t) &= Af(u - a - th) \\
\tau a_t &= Cu - a
\end{aligned}
$$

and obtain

$$
\begin{aligned}
(\nabla^4 + 2M\nabla^2 + B + M^2)(u - \omega\frac{\partial u}{\partial \phi}) &= Af(u - a - th) \\
-\omega\tau\frac{\partial a}{\partial \phi} &= Cu - a.
\end{aligned}
$$

Limiting Case: $\omega = -1,\ C = 0,\ a \equiv 0,\ f = H(u - th).$

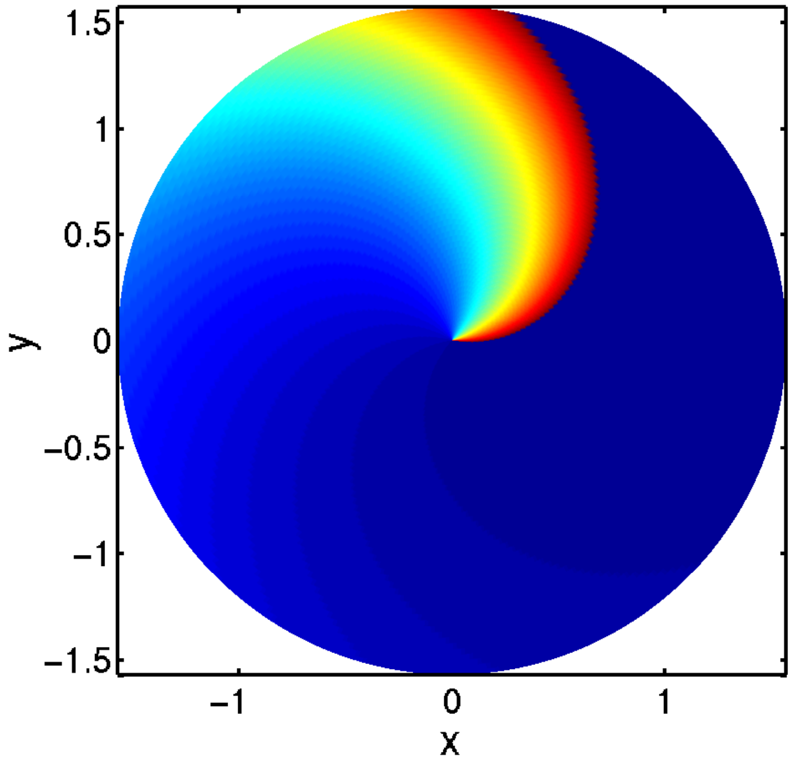
$$\begin{aligned}(\nabla^4 + 2M\nabla^2 + B + M^2)(u - \omega\frac{\partial u}{\partial \phi}) &= Af(u - a - th) \\ -\omega\tau\frac{\partial a}{\partial \phi} &= Cu - a.\end{aligned}$$

reduces to

$$(\nabla^4 + 2M\nabla^2 + B + M^2)(u + \frac{\partial u}{\partial \phi}) = AH(u - th)$$

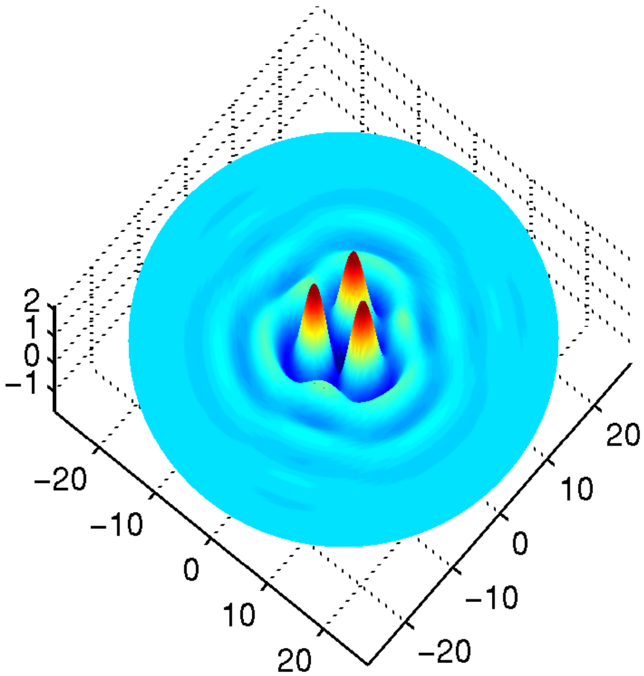
$$u(r,\phi) = \left(\frac{A}{B + M^2} + (th - \frac{A}{B + M^2})e^{r-\phi})\right) H(\phi - r)$$

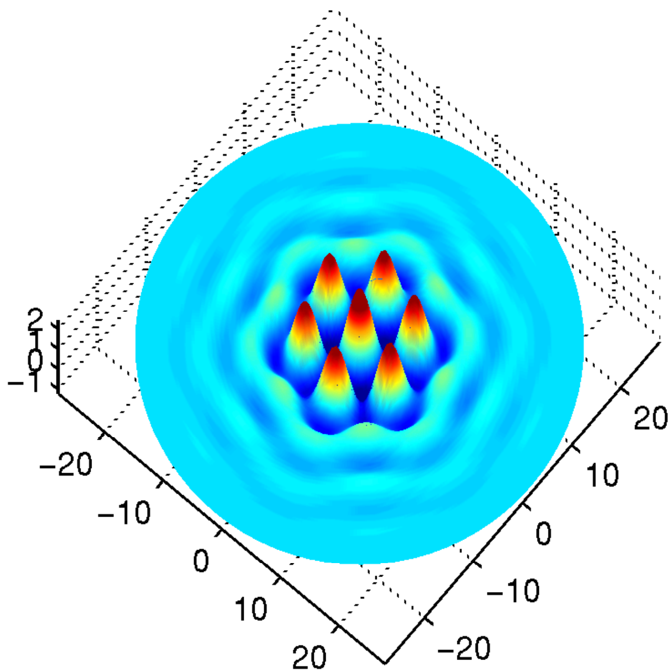
$$C = 0: \quad u(r,\phi) = \left(\frac{A}{B+M^2} + (th - \frac{A}{B+M^2})e^{r-\phi} \right) H(\phi - r)$$



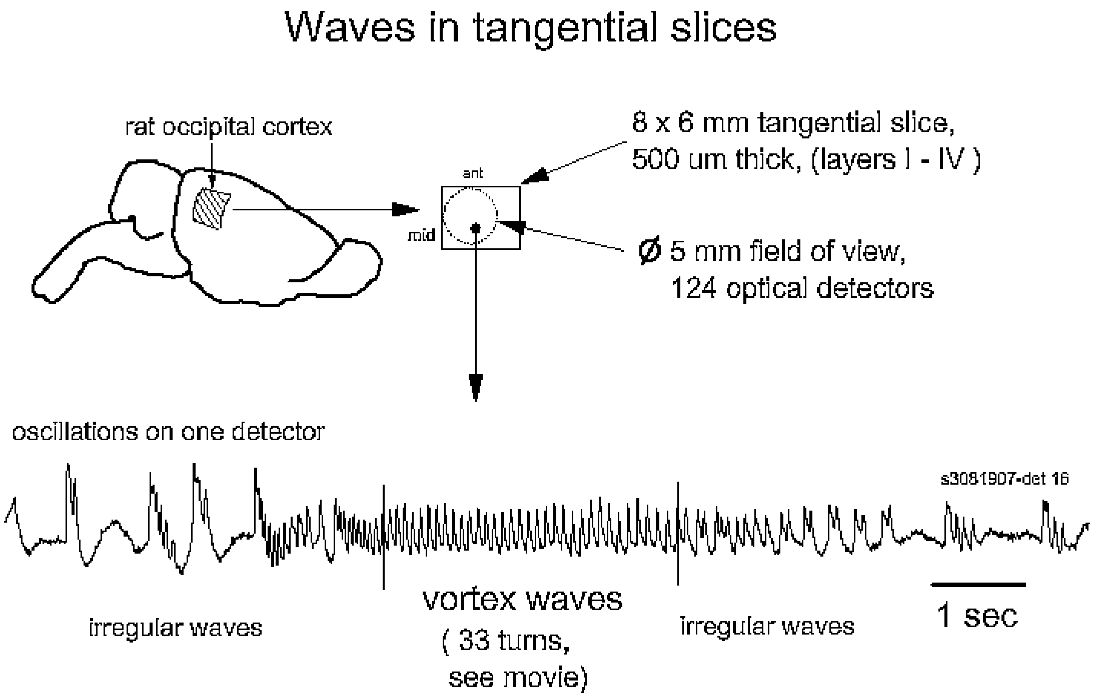
Spiral for C=0. Spiral for C>0.

3-Bump and 7-Bump Rotators





Experiment: J. Y. Wu (August 2003)



Rotating Wave