Chaos, Black Holes and Quantum Mechanics

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Regular motion

- Newtonian view of the world:
- Specify initial configuration, then evolve by dynamical laws.

- Future is easily predictable.
Chaos

- Not typical.
- It is usually hard to predict the future.

- It is hard to be a good pool player!
"Sensitive dependence on initial conditions"

A small change in initial impact parameter makes a large change in final trajectory.
Sensitive dependence on initial conditions $\leftrightarrow$ strong chaos.

- The weather.
- The "butterfly effect."
- A fundamental source of uncertainty, even in classical physics.
Sensitive dependence on initial conditions does not mean that nothing is predictable.

The summer is hotter than the winter.

Dumping large amounts of carbon dioxide into the atmosphere warms the planet.

The detailed size and time course of these effects is hard to predict.
Chaos and thermal behavior

- Chaos leads to simplicity.

- Atoms spread out uniformly – constant density.
- Energy becomes uniformly shared – constant energy density.
- Hallmarks of thermal behavior.
- Uniformity $\leftrightarrow$ thermal equilibrium.
- Kinetic energy per particle $= \frac{3}{2} k_B T$
- Thermal behavior in such systems is a consequence of chaos.
Quantifying chaos

- How sensitive to initial conditions?

- Trajectories diverge **exponentially** in time.
  - 1 inch apart, 2 inches, 4, 8, $\rightarrow$ 85 feet in 10 seconds!
  - distance grows like $e^{\lambda_L t}$
  - $\lambda_L$ is a Lyapunov exponent
  - Large $\lambda_L$ means trajectories diverge more quickly – stronger chaos.
  - Weather: one of the $\lambda_L$ gives a divergence time of $\sim$ 5 days.
Quantifying chaos, contd.

- Another way – “run the movie backwards.”
- Fine tune initial conditions to end up in a special configuration after a time $t$.

How much can you disturb the initial conditions and still end up in the special configuration? An exponentially small amount at late time, $e^{-\lambda t}$.

- Can arrange such a fine tuned situation in a black hole.
Black holes

- A striking consequence of General Relativity.
- A profound distortion of space-time geometry.
- Classically nothing, including light, can leave a black hole.
Black holes are hard to visualize:

- Horizon: once inside you cannot get out, no matter how much energy you use. You inevitably hit the singularity.
- The horizon size grows with mass.
- From just outside the horizon you can escape if you use enormous energy.
Set up a fine tuned initial condition to end up in a special configuration:

- A photon (particle of light) starting just outside the horizon, tuned to stay close for a long time $t$ and finally depart.
- Requires starting exponentially close to the horizon, because the distance increases exponentially in time.
- Requires an exponentially large amount of energy, which “redshifts” away.
Chaos in black holes, contd.

- Now perturb this initial condition a bit by dropping a particle into the black hole [t Hooft]

- The black hole becomes more massive, the horizon grows and swallows the photon.
The photon is trapped behind the horizon so it inevitably hits the singularity, a dramatically different outcome than before.

If $t$ is large the photon starts exponentially close to the horizon, so an exponentially small mass increase is enough to cause this.

This exponential sensitivity is a signal of Lyapunov behavior.
What we have just described is a kind of **classical** chaos present in classical general relativity. What about quantum mechanics?

“Gauge-gravity duality” maps this to quantum chaos in a “dual” system.
Gauge/gravity duality

- Gauge/gravity duality: [Maldacena]
- A precise mapping (a duality) between an ordinary quantum mechanical system without gravity (like QCD) and a quantum gravitational one.
- The duality provides a dictionary to translate between the two systems.
The grey shell is the boundary of space.
The ordinary quantum system lives on the boundary (the “boundary theory”).
Quantum gravity lives inside the shell (the “bulk” theory).
The quantum black hole is thermal, at the Hawking temperature, so the boundary theory is thermal too.
When the boundary system has many degrees of freedom, the gravitational system – the black hole – behaves classically, while the boundary system remains strongly quantum mechanical. So classical gravity can give results, via the duality dictionary, about certain quantum systems with many degrees of freedom.

The classical Lyapunov behavior of black holes we discussed above can be used to determine the quantum Lyapunov behavior of a boundary system with many degrees of freedom – a calculation of the quantum butterfly effect in a strongly interacting many-body system.

The quantum Lyapunov exponent is given by

\[ \lambda_L = \frac{2\pi k_B T}{\hbar} \]
This calculation has taught us some lessons about quantum chaos in many-body physics.

For example, it motivated the derivation of "The chaos bound." [Maldacena–SS–Stanford]

The quantum Lyapunov exponent of any quantum system with many degrees of freedom is bounded by the black hole value.

$$\lambda_L \leq \frac{2\pi k_B T}{\hbar}$$

with reasonable physical assumptions

"Black holes are the fastest scramblers in nature.” [Sekino-Susskind]
These results are satisfying, but the gravitational physics involved, the classical behavior of black holes in general relativity, is not new.

Can we extend beyond the limit where gravity is classical and use quantum chaos to learn new things about quantum gravity?
Quantized energy levels

- A defining feature of quantum systems (whose motion is bounded) is that the energies can only assume discrete values.
- The energy is “quantized.”
- Discrete energy levels correspond to independent quantum states.
- The Hydrogen atom is an example:
Quantized energy levels in quantum chaotic systems follow a rather different pattern:

- In chaotic systems neighboring energy levels “repel”: “short range level repulsion.”
- The “gas” of energy levels is hard to compress: “long range spectral rigidity.”
Energy levels in quantum chaotic systems

- Chaos gives rise to simplicity in the energy level pattern.
- A remarkable conjecture: quantum chaotic energy levels obey random matrix statistics. [Wigner, Dyson, Mehta, Bohigas-Giannoni-Schmit, Berry, ...]
- A universal pattern
The dynamical rules of a quantum system determine a matrix, the Hamiltonian matrix, which can be quite complicated:

$$\begin{pmatrix}
1.7 & 2.14 & -3.56 & .71 \\
2.14 & 5.9 & .04 & 95 \\
-3.56 & .04 & -3.8 & 66 \\
.71 & 95 & 66 & 7.2
\end{pmatrix}$$

“Diagonalize” this matrix to find its “eigenvalues.” These are the energy levels (Matrix mechanics).

A quantum chaotic system has energy levels corresponding to the eigenvalues of a random matrix

$$\begin{pmatrix}
+1 & -1 & -1 & +1 \\
-1 & +1 & +1 & +1 \\
-1 & +1 & -1 & -1 \\
+1 & +1 & -1 & -1
\end{pmatrix}$$
Random matrix eigenvalues

- Easy to compute properties of eigenvalues of a random matrix (chaos leads to simplicity, again...)

Quantum chaos: Spectral statistics

- Long range spectral rigidity and short range level repulsion are clearly visible.
Black holes have energy levels

- Black holes in quantum gravity are bound quantum systems.
- They should have quantized energy levels.
- How many?
- In thermal systems (like a black hole) the logarithm of the number of energy levels (in an energy band) is called the “entropy.”
The entropy of quantum black holes

• The entropy of a quantum black hole: [Bekenstein, Hawking]

\[ S_{\text{BH}} = \frac{1}{4} \frac{\text{Horizon area}}{\text{Planck area}} \]

The Planck area is the Planck length squared, where the Planck length is \( \sim 10^{-33} \) cm.

• One “qubit” per Planck area
For a solar mass black hole the horizon radius is $\sim 3\text{km}$. Entropy $\sim 10^{78}$ qubits, a large number.

But the number of energy levels is $10^{10^{78}}$, a very large number.

Black holes are chaotic systems so the levels should look like the eigenvalues of a very large random matrix.

In the limit of classical gravity this quantization will go away and the energies will become continuous.

Discreteness and random matrix behavior are distinctly quantum gravitational phenomena.
A simple model

- These ideas have been extremely difficult to study until recently.
- The Hamiltonian of the boundary theory is complicated, and the matrices very large.
- But recently a simple model of a quantum black hole has been introduced, enabling us to make progress in understanding these questions. [Sachdev-Ye, Kitaev]
The Sachdev-Ye-Kitaev model

- Make a model of “pure quantum information.”

- $N$ qubits coupled in groups of 4, with random couplings. The Sachdev-Ye-Kitaev (SYK) model.
There is a spirited competition going on to determine the most practical kind of qubit for quantum computation.

Santa Barbara is a hotbed of this:
Google/UCSB (Martinis group) – superconducting qubits;
Microsoft Station Q/UCSB – Majorana fermions.

The SYK model uses Majorana fermions. [Kitaev]
Good things about the Sachdev-Ye-Kitaev model

- The SYK model is maximally chaotic, like a black hole. [Kitaev]
- It has a sector in its bulk dual that behaves like gravity, in 1 space and 1 time dimension, so lessons can be learned about quantum gravity.
- For $N$ Majorana fermions the number of energy levels is $2^{N/2}$. $N = 34$ has $\sim 128,000$ levels, large enough to be interesting, small enough to study on modern (classical) computers.
The energy levels of SYK display long range spectral rigidity and short range eigenvalue repulsion, like random matrix levels.

- We need a more quantitative test.
The spectral form factor: a quantitative diagnostic of long range spectral rigidity and short range energy level repulsion.

\[ \sum_{n,m} e^{i(E_n - E_m)t} \]

The Fourier transform of the energy difference distribution.

Try it out on random matrices:

The “ramp” is a signature of long range spectral rigidity.

The “plateau” is a signature of short range energy level repulsion.
Spectral form factor

- Compute the spectral form factor for the SYK model [Jordan Cotler, Guy Gur-Ari, Masanori Hanada, Joe Polchinski, Phil Saad, Stephen Shenker, Douglas Stanford, Alex Streicher, Masaki Tezuka].

![Graphs showing the spectral form factor for SYK and Random models]

- Quantitative agreement with random matrix statistics.
- A reasonable conjecture that this is the case for general black holes.
- Should have an explanation in quantum gravity.
In **quantum** gravity the space-time geometry is not definite, because of the uncertainty principle.

The contribution of multiple geometries must be included.

The ramp can be explained by the contribution of a different geometry than that of the original black hole – the “double cone.”

[Phil Saad-SS-Douglas Stanford].

- Represents a one dimensional universe (a line) evolving in time.
- Has a natural generalization to higher dimensional black holes.
The plateau as a D-brane

- The plateau is the “smoking gun” for quantized energy levels. It is a much more subtle phenomenon than the ramp.
- We believe its explanation in quantum gravity requires effects beyond geometry [Phil Saad-SS-Douglas Stanford, in progress].
- Arbitrary numbers of “baby universes” ending on a D-brane [Polchinski].

We will keep you posted...
Thank You!