KITP Santa Barbara, November 16 2004 ‘A Tale of Two Towers’

MHV Rules for Amplitudes with Higgs and many Gluons

Valya Khoze
University of Durham

based on

- Lance Dixon, Nigel Glover, VVK hep-th/0411092

—

- George Georgiou, VVK hep-th/0404072 JHEP 0405 (2004) 070
- George Georgiou, Nigel Glover, VVK hep-th/0407027 JHEP 0407 (2004) 048

An (incomplete) list of References

- E. Witten, hep-th/0312171.
- F. Cachazo, P. Svrček and E. Witten, hep-th/0403047, 0406177, 0409245.
- S.J. Parke and T.R. Taylor, Phys. Rev. Lett. **56**, 2459 (1986).
- F.A. Berends and W.T. Giele, Nucl. Phys. B **306**, 759 (1988).
- V. P. Nair, Phys. Lett. B **214** (1988) 215.
- R. Roiban, M. Spradlin and A. Volovich, hep-th/0402016, 0403190.
- C.J. Zhu + J. B. Wu and C. J. Zhu, hep-th/0403115, hep-th/0406085.
- I. Bena, Z. Bern and D.A. Kosower + Kosower hep-th/0406133, 0406175.
- Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, hep-ph/9403226, 9409265.
- A. Brandhuber, B. Spence and G. Travaglini + J. Bedford hep-th/0407214, 0410280.
- C. Quigley and M. Rozali, hep-th/0410278.
- I. Bena, Z. Bern, D.A. Kosower and R. Roiban, hep-th/0410054.
- F. Cachazo, hep-th/0410077.
- R. Britto, F. Cachazo and B. Feng, hep-th/0410179.
- Z. Bern, V. Del Duca, L.J. Dixon and D.A. Kosower, hep-th/0410224.
- S.J. Bidder, N.E.J. Bjerrum-Bohr, L.J. Dixon and D.C. Dunbar, hep-th/0410296.

- Perturbative amplitudes in conformal $\mathcal{N} = 4$ SYM for $N_c \to \infty$ can be interpreted as D-instanton contributions in a topological string theory in twistor space $CP^{3|4}$ -Witten.

- In hep-th/0403047 Cachazo, Svrcek and Witten (CSW) proposed a new approach for calculating scattering amplitudes of n gluons in SYM and in QCD.

- Amplitudes in gauge theory are found by summing new scalar diagrams. Use scalar propagators, $1/q^2$, and effective scalar vertices. Gives results for purely kinematic parts of the colour-ordered amplitudes.

- The effective vertices are off-shell continuations of the maximally helicity-violating (MHV) n-gluon scattering amplitudes of Parke and Taylor.

  Suppressing the overall momentum conservation factor, MHV amplitude is

$$A_n^{\text{MHV}}(1^+, \ldots, r^-, \ldots, s^-, \ldots, n^+) = \frac{\langle r\, s\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n-1,\, n\rangle \langle n\, 1\rangle}$$

  Here r and s are the only gluons with negative helicity.

- These MHV vertices, with a suitable definition for $\langle i\, j\rangle$ when the momenta k_i or k_j are off shell, are then connected with scalar propagators.

In the spinor helicity formalism an on-shell momentum of a massless particle, $p_\mu p^\mu = 0$, is represented as

$$p_{a\dot{a}} \equiv p_\mu \sigma^\mu_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

where λ_a and $\tilde{\lambda}_{\dot{a}}$ are two commuting spinors of positive and negative chirality.

Spinor inner products are defined by

$$\langle \lambda_i, \lambda_j \rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b \equiv \langle i\; j \rangle \;, \qquad [\tilde{\lambda}_i, \tilde{\lambda}_j] = -\epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}} \equiv [i\; j]$$

and a scalar product of two null vectors, $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ and $q_{a\dot{a}} = \lambda'_a \tilde{\lambda}'_{\dot{a}}$, becomes

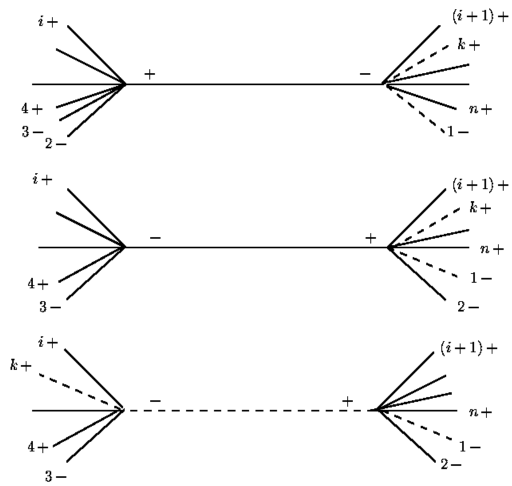
$$p_\mu q^\mu = \frac{1}{2} \langle \lambda, \lambda' \rangle [\tilde{\lambda}', \tilde{\lambda}] \equiv \frac{1}{2} \langle p\; q \rangle [q\; p]$$

A_n^{MHV} is composed entirely of the 'holomorphic' products $\langle i\, j\rangle$ rather than their anti-holomorphic partners $[i\, j]$,

$$A_n^{\text{MHV}}(1^+, \ldots, p^-, \ldots, q^-, \ldots, n^+) = i g^{n-2} \delta^4(\sum k_i) \, \frac{\langle p\, q\rangle^4}{\langle 1\, 2\rangle \langle 2\, 3\rangle \cdots \langle n-1,\, n\rangle \langle n\, 1\rangle}$$

In twistor space, where the anti-holomorphic spinors $\tilde{\lambda}_{i,\dot{\alpha}}$ are traded for their Fourier-transform variables $\mu_i^{\dot{\alpha}} = -i\partial/\partial\tilde{\lambda}_{i,\dot{\alpha}}$, each MHV vertex is localized on a line. The lines are connected through the off-shell propagators.

Tree diagrams with MHV vertices contributing to the $---+++\ldots++$ amplitude with 2 fermions and $n-2$ gluons.



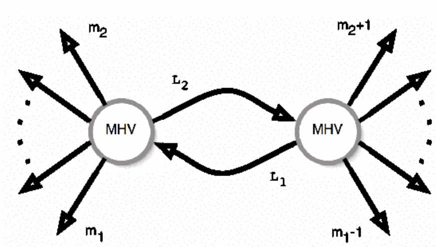
CSW off-shell continuation: pick an arbitrary spinor $\xi^{\dot{a}}_{\mathrm{Rcf}}$ and define λ_a for any internal line carrying momentum $q_{a\dot{a}}$ by

$$\lambda_a = q_{a\dot{a}}\,\xi^{\dot{a}}_{\mathrm{Rcf}}$$

External lines in a diagram remain on-shell, and for them λ is defined in the usual way.

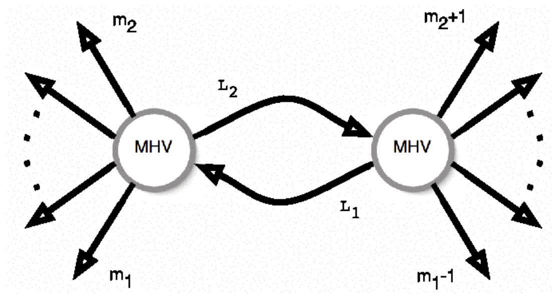
Dependence on $\xi^{\dot{a}}_{\mathrm{Rcf}}$ will disappear from final results for the amplitudes.

- Multi-particle MHV amplitudes used as effective vertices lead to dramatic savings on a number of permutations in Feynman diagrams.
- Tree-level and also loop diagrams in SYM posses a tractable geometric structure when transformed from Minkowski to twistor space.
- The CSW approach has been extended to amplitudes with fermions. New tree-level gauge-theory results were obtained in this approach for non-MHV amplitudes involving gluons, fermions and scalars. At tree-level supersymmetry is not necessary for the approach to work.
- The MHV rules work at 1-loop level in supersymmetric Yang-Mills theories. Brandhuber, Spence and Travaglini used MHV rules to reproduce the series of one-loop n-gluon MHV amplitudes in $\mathcal{N}=4$ SYM, previously computed in BDDK'94 via unitarity cuts.



- This approach was shown to work also for the same amplitudes in $\mathcal{N}=1$ SYM.

- Furthermore, the 'holomorphic anomaly' approach makes it possible to evaluate 1-loop amplitudes in supersymmetric gauge theories without doing loop integrals. The anomaly for a unitarity cut freezes the phase-space integration, making its evaluation simple.
  Cachazo, Bena-Bern-Kosower-Roiban, Cachazo-Svrcek-Witten
  Britto-Cachazo-Feng ↔ Bern-Del Duca-Dixon-Kosower



Impressive progress of MHV rules at one loop in supersymmetric theories is related to the fact that such theories are 'cut-constructible' — at one loop, intermediate states can be assigned four-dimensional helicities, even though the loop-momentum integral must be regulated dimensionally, with $D = 4 - 2\epsilon$. (Bern-Dixon-Dunbar-Kosower '94)

Application of MHV rules to loop amplitudes in *non-supersymmetric* theories seems to be a different matter. Consider n-gluon one-loop amplitudes for which all gluons (or all but one) have the same helicity

$$\mathcal{A}_n^{1-\text{loop}}(1^\pm, 2^+, 3^+, \ldots, n^+)$$

Such amplitudes vanish in the supersymmetric case, but are nonzero for nonsupersymmetric combinations of massless gluons, fermions or scalars circulating in the loop. They are finite as $\epsilon \to 0$, and in this limit they become cut-free, rational functions of the kinematic invariants.

Compare the sequence of pure QCD 1-loop amplitudes

$$A_{n;1}^{1-\mathrm{loop}}(1^+,2^+,\ldots,n^+) \propto \frac{\sum_{1\le i_1<i_2<i_3<i_4\le n} \langle i_1\, i_2\rangle\, [i_2\, i_3]\, \langle i_3\, i_4\rangle\, [i_4\, i_1]}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n-1,\, n\rangle\, \langle n\, 1\rangle}\,.$$

and the Higgs plus n-gluon 1-loop amplitudes (in the limit $2m_t \gg M_H$)

$$A_n(H,1^+,2^+,\ldots,n^+) \propto \frac{m_H^4}{\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n-1,\, n\rangle\, \langle n\, 1\rangle}\,,$$

- Both sets of amplitudes are generated first at one loop.
- They are both rational functions of the kinematic variables and contain no branch cuts.
- Their collinear and multi-particle factorization properties are very similar, as reflected in the factors in their denominators, $\langle 1\, 2\rangle\, \langle 2\, 3\rangle \cdots \langle n\, 1\rangle$.
- The numerator factors are also quite similar, both are bi-linear in the anti-holomorphic spinor products $[i\, j]$.

$$m_H^4 = (\textstyle\sum_{1\le i<j\le n} \langle i\, j\rangle\, [j\, i])^2$$

The quasi-local nature of the $(\pm{+}{+}\cdots{+})$ amplitudes suggests that some of them should become new fundamental vertices, like the tree-level MHV vertices (?)

CSW-2 made a well-motivated but unsuccessful attempt to generate the 1-loop 5-point QCD amplitude

$$A_{5;1}^{1-\mathrm{loop}}(-{+}{+}{+}{+})$$

from an off-shell continuation of the one-loop 4-point amplitude $A_4^{1-\mathrm{loop}}(++++)$ plus the off-shell tree-level MHV vertex $A_3^{\mathrm{MHV}}(--+)$.

In the Higgs case, we have made an analogous attempt to generate the Higgs plus three gluon amplitude

$$A_3(H,-{+}{+}) \;<=\; A_2(H,{+}{+}) \;\oplus\; A_3^{\mathrm{MHV}}(--+)$$

Our attempt failed; it led to results which depended on the choice of the 'reference spinor' in the CSW construction, and thus could not be correct.

For the Higgs case, we shall resolve this problem in a different way, yet still using an MHV-type perturbation theory.

Motivation for studying Higgs plus Gluons amplitudes

1. Important to extend the range of processes for which twistor-inspired MHV rules can be used. MHV rules here will involve a non-trivial extension of the CSW rules.

2. Generally need MHV rules for massive particles. Higgs momentum is not null, $p_H^2 = m_H^2$.

3. Higgs plus gluons amplitudes come from 1-loop effects in nonsupersymmetric Standard Model. Can be useful for developing MHV rules for non-susy loops.

4. Production of the Standard Model Higgs at hadron colliders – dominated by gluon-gluon fusion, $gg \to H$, through a one-loop diagram containing the top quark in the loop. Precision electroweak data, interpreted within the Standard Model, indicate

$$2m_t \approx 360 \text{ GeV} > m_H > 260 \text{ GeV}$$

Can integrate out the heavy top quark, summarizing its effects via the dimension-five operator $H \,\mathrm{tr}\, G_{\mu\nu}G^{\mu\nu}$.

This operator can then be 'dressed' by standard QCD vertices in order to generate Higgs plus multi-parton amplitudes. We will provide a set of MHV rules for such amplitudes.

The Higgs Model

In the Standard Model the Higgs boson couples to gluons through a fermion loop. The dominant contribution is from the top quark. For large m_t, the top quark can be integrated out leading to the effective interaction

$$\mathcal{L}_H^{\rm int} = \frac{C}{2} H \,\mathrm{tr}\, G_{\mu\nu}G^{\mu\nu} \ , \qquad C = \alpha_s/(6\pi v) := \ 1$$

Introduce a complex field $\phi = \frac{1}{2}(H + iA)$ and divide $\mathcal{L}_H^{\rm int}$ into two terms, containing purely selfdual (SD) and purely anti-selfdual (ASD) gluon field strengths

$$\mathcal{L}_{H,A}^{\rm int} = \ \frac{1}{2}\Big[H \,\mathrm{tr}\, G_{\mu\nu}G^{\mu\nu} + iA \,\mathrm{tr}\, G_{\mu\nu}\, {}^*G^{\mu\nu}\Big] \ = \ \phi \,\mathrm{tr}\, G_{SD\,\mu\nu}G_{SD}^{\mu\nu} + \phi^\dagger \,\mathrm{tr}\, G_{ASD\,\mu\nu}G_{ASD}^{\mu\nu}$$

The key point is that, due to selfduality, the amplitudes for ϕ plus n gluons, and those for $\phi^\dagger$ plus n gluons, separately have a simpler structure than the amplitudes for H.

$$G_{SD}^{\mu\nu} = \tfrac{1}{2}(G^{\mu\nu} + {}^*G^{\mu\nu}) \ , \quad G_{ASD}^{\mu\nu} = \tfrac{1}{2}(G^{\mu\nu} - {}^*G^{\mu\nu}) \ , \quad {}^*G^{\mu\nu} \equiv \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}G_{\rho\sigma} \ .$$

This interaction can be embedded into a supersymmetric effective Lagrangian,

$$\mathcal{L}^{\rm int}_{\rm SUSY} = -\int d^2\theta\ \Phi\, {\rm tr}\, W^\alpha W_\alpha - \int d^2\bar\theta\ \Phi^\dagger\, {\rm tr}\, \overline{W}_{\dot\alpha} \overline{W}^{\dot\alpha} .$$

$G^{\mu\nu}_{SD}$ is the bosonic component of the chiral superfield W_α, and ϕ is the lowest component of the chiral superfield Φ. The helicity assignment is:

$$W_\alpha = \{g^-, \lambda^-\} , \qquad \Phi = \{\phi, \psi^-\} ,$$
$$\overline{W}^{\dot\alpha} = \{g^+, \lambda^+\} , \qquad \Phi^\dagger = \{\phi^\dagger, \psi^+\}$$

$g^\pm$ correspond to gluons with $h = \pm 1$ helicities, $\lambda^\pm$ are gluinos with $h = \pm 1/2$, ϕ and $\phi^\dagger$ are complex scalar fields, and $\psi^\pm$ are their fermionic superpartners.

It can be shown now that

$$A_n(\phi, 1^\pm, 2^+, 3^+, \ldots, n^+) = 0 , \quad A_n(\phi^\dagger, 1^\pm, 2^+, 3^+, \ldots, n^+) \neq 0$$

The ϕ-MHV amplitudes, with precisely two negative helicities, $\phi\, g^- g^+ \ldots g^+ g^- g^+ \ldots g^+$, are the first non-vanishing ϕ-amplitudes.

Furthermore, the first few known ϕ amplitudes have precisely the same form as the QCD case — except for the implicit momentum carried out of the process by the Higgs boson.

$$A_2(\phi, 1^-, 2^-) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,1\rangle} = -\langle 1\,2\rangle^2,$$
$$A_3(\phi, 1^-, 2^-, 3^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,1\rangle} = \frac{\langle 1\,2\rangle^3}{\langle 2\,3\rangle \langle 3\,1\rangle}$$
$$A_4(\phi, 1^-, 2^-, 3^+, 4^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,4\rangle \langle 4\,1\rangle}$$

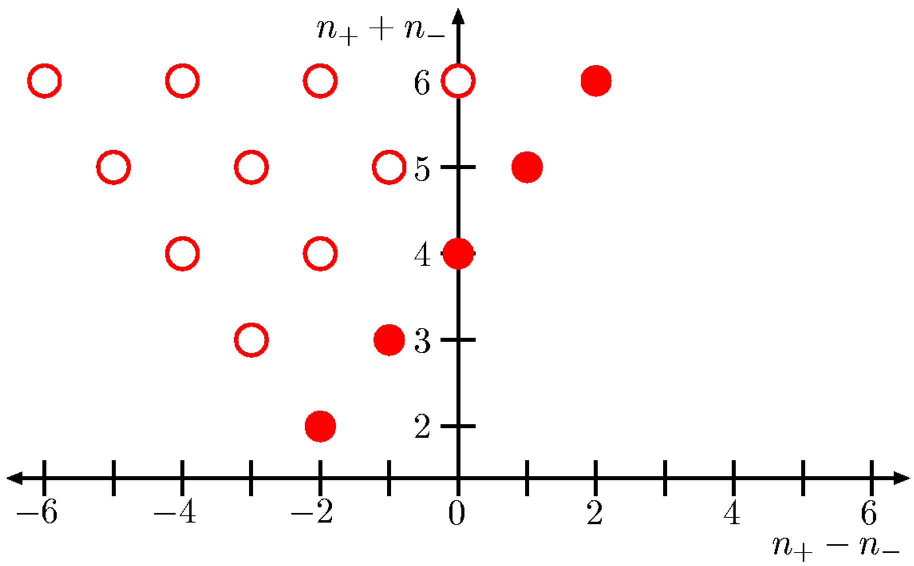
This leads to the obvious assertion for *all* 'ϕ-MHV' amplitudes,

$$A_n(\phi, 1^+, 2^+, \ldots, p^-, \ldots, q^-, \ldots, n^+) = \frac{\langle p\,q\rangle^4}{\langle 1\,2\rangle \langle 2\,3\rangle \cdots \langle n-1,\, n\rangle \langle n\,1\rangle}$$

where only legs p and q have negative helicity.

Besides the correct collinear and multi-particle factorization behavior, these amplitudes also correctly reduce to pure QCD amplitudes as the ϕ-momentum approaches zero.

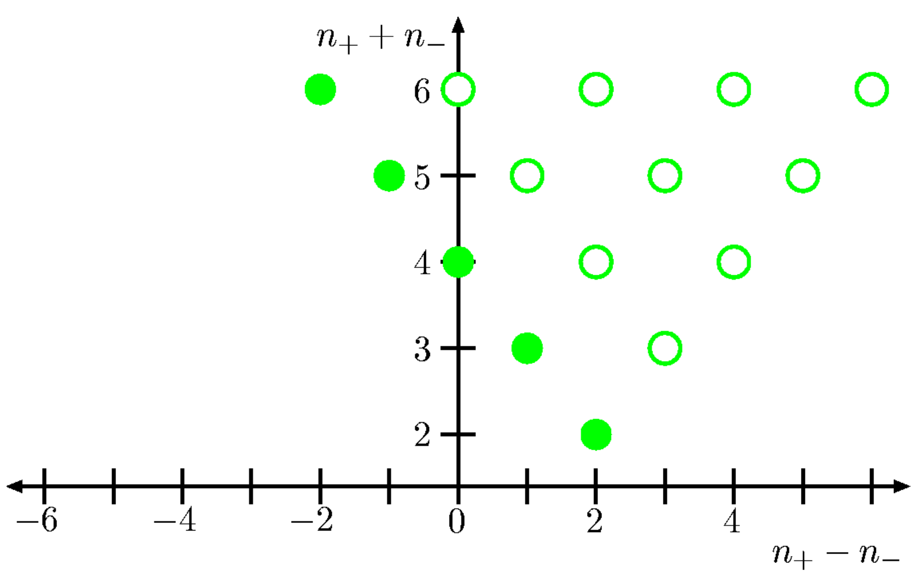
MHV structure of Non-MHV ϕ plus multi-gluon amplitudes



The number of positive (negative) helicity gluons is n_+ (n_-).

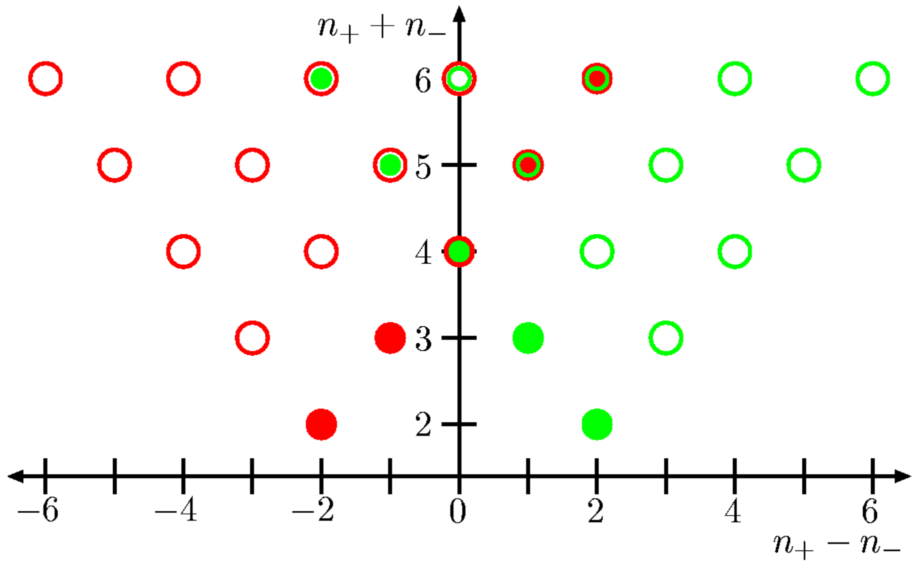
Solid red dots represent fundamental ϕ-MHV vertices. Open circles are composite ϕ amplitudes, which are built from the ϕ-MHV vertices plus pure-gauge-theory MHV vertices.

Anti-MHV structure of $\phi^\dagger$ plus multi-gluon amplitudes



Solid green dots represent fundamental $\phi^\dagger$-anti-MHV vertices, which coincide with the $\phi^\dagger g^+ g^+ g^- \ldots g^-$ amplitudes. Open green circles are composite $\phi^\dagger$ amplitudes built from the $\phi^\dagger$-anti-MHV vertices plus pure-gauge-theory anti-MHV vertices.

The structure of Higgs plus multi-gluon amplitudes

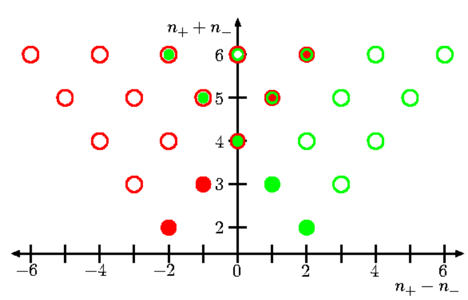


The MHV tower is shown in red. The anti-MHV tower is shown in green. Amplitudes for the scalar Higgs are obtained by adding the ϕ and $\phi^\dagger$ amplitudes.

Green and red towers are related by parity, which exchanges $\langle i\, j\rangle \leftrightarrow [j\, i]$ and reflects points across the vertical axis, $n_+ - n_- \to -(n_+ - n_-)$.

The new MHV rules for computing Higgs plus n-gluon scattering amplitudes can be summarized as follows:

1. For the ϕ couplings, everything is exactly like the CSW MHV rules (except for the momentum carried by ϕ).
2. For $\phi^\dagger$, we just apply parity. That is, we compute with ϕ, and reverse the helicities of every gluon. Then we let $\langle i\, j\rangle \leftrightarrow [j\, i]$ to get the desired $\phi^\dagger$ amplitude.
3. For H, we add the ϕ and $\phi^\dagger$ amplitudes.



These rules can easily be used to reproduce all of the available analytic formulae for Higgs + n-gluon scattering ($n \leq 5$) at tree level and derive new expressions for $n \geq 6$. In some cases they generate considerably shorter expressions.

Twistor space interpretation

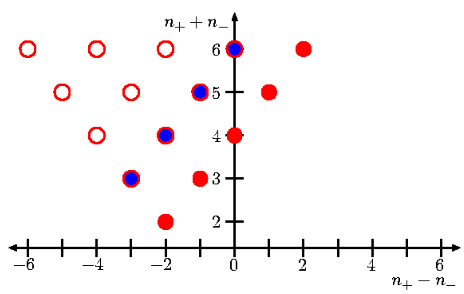
- For the ϕ plus n-gluon amplitudes, we can consider a twistor space $(\lambda_1, \lambda_2, \mu^{\dot{1}}, \mu^{\dot{2}})$, for each of the n gluons, by replacing the anti-holomorphic spinor coordinates $\tilde{\lambda}_{i,\dot{\alpha}}$ by their Fourier transforms, $\mu_i^{\dot{\alpha}} = -i\partial/\partial\tilde{\lambda}_{i,\dot{\alpha}}$.
- The momentum of the massive scalar is left untouched.
- Then => ϕ-MHV amplitudes are localized on a line in twistor space.

$$\begin{aligned} A(\lambda_i, \mu_i) &= \int \prod_{i=1}^{n} d\tilde{\lambda}_i \exp(i\mu_i\tilde{\lambda}_i) A(\lambda_i) \delta\Big(k_\phi + \sum_{i=1}^{n} k_i\Big) \\ &= \int d^4x A(\lambda_i) \int \prod_{i=1}^{n} d\tilde{\lambda}_i \exp(i\mu_i\tilde{\lambda}_i) \exp\Big[ix^{\dot{\alpha}\alpha}\Big((k_\phi)_{\alpha\dot{\alpha}} + \sum_{i=1}^{n} \lambda_{i,\alpha}\tilde{\lambda}_{i,\dot{\alpha}}\Big)\Big] \\ &= \int d^4x \exp(ix\cdot k_\phi) A(\lambda_i) \prod_{i=1}^{n} \delta(\mu_i + x\lambda_i) \end{aligned}$$

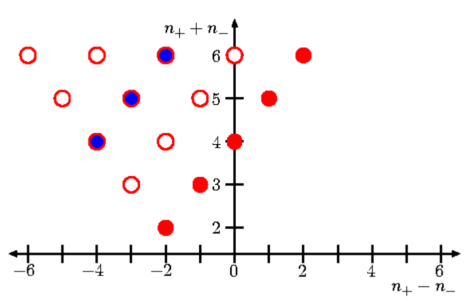
- The ϕ amplitudes with n_- negative-helicity gluons are similarly localized on networks of $(n_- - 1)$ intersecting lines in twistor space.
- $\phi^\dagger$-anti-MHV amplitudes are localized on a line in anti-twistor space.

We will derive expressions for

1. NMHV amplitudes $\phi \to - - - + \ldots +$:



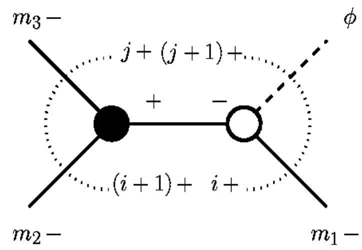
2. NNMHV amplitudes $\phi \to - - - - + \ldots +$:

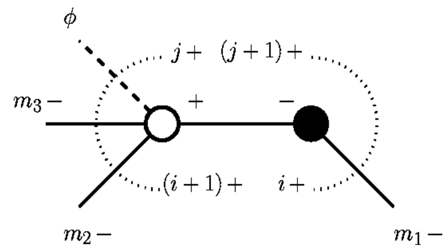


3. Arbitrary minus-only amplitudes $\phi \to - - - \ldots -$

1. NMHV amplitudes $A_n(\phi, \ldots, m_1^-, \ldots, m_2^-, \ldots, m_3^-, \ldots)$

$$A_n(\phi, m_1^-, m_2^-, m_3^-) = \frac{1}{\prod_{l=1}^n \langle l,\, l+1\rangle} \sum_{i=1}^{2} \sum_{C(m_1,m_2,m_3)} A_n^{(i)}(m_1, m_2, m_3)$$





$$A_n^{(1)}(m_1, m_2, m_3) = \sum_{i=m_1}^{m_2-1} \sum_{j=m_3}^{m_1-1} \frac{\langle m_2\, m_3\rangle^4 \, \langle m_1^- | \not{q}_{i+1,j} | \xi^-\rangle^4}{D(i, j, q_{i+1,j})},$$

$$A_n^{(2)}(m_1, m_2, m_3) = \sum_{i=m_1}^{m_2-1} \sum_{j=m_3}^{m_1-1} \frac{\langle m_2\, m_3\rangle^4 \, \langle m_1^- | \not{q}_{j+1,i} | \xi^-\rangle^4}{D(i, j, q_{j+1,i})}$$

$$D(i,j,q) := \langle i^- | \not{q} | \xi^-\rangle \langle (j+1)^- | \not{q} | \xi^-\rangle \langle (i+1)^- | \not{q} | \xi^-\rangle \langle j^- | \not{q} | \xi^-\rangle \frac{q^2}{\langle i,\, i+1\rangle \, \langle j,\, j+1\rangle}$$

We have checked, with a help of a symbolic manipulator, that our NMHV results:

Are ξ-independent (gauge invariant) and

Agree numerically with the known analytic formulae in the simplest cases

Dawson and Kauffman 1992

- $H \to ---$

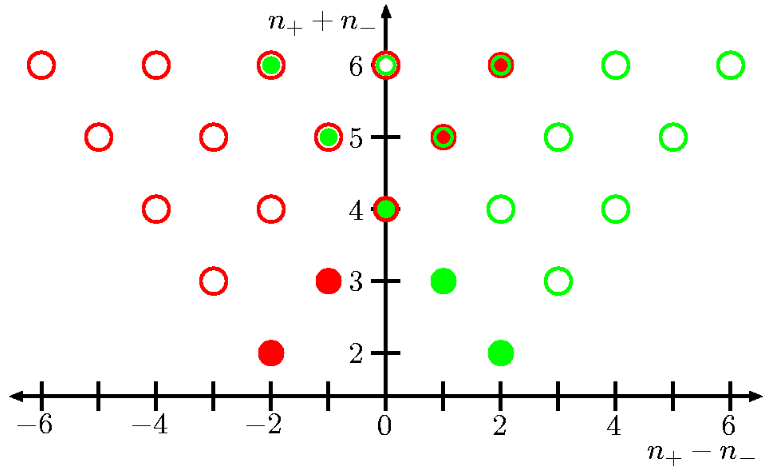
$$A_3(H, 1^-, 2^-, 3^-) = -\frac{m_H^4}{[1\,2]\,[2\,3]\,[3\,1]}$$

- $H \to +---$

$$A_4(H, 1^+, 2^-, 3^-, 4^-) = \frac{\langle 3^- | \not{k}_H | 1^-\rangle^2 \langle 2\,4\rangle^2}{s_{124} s_{12} s_{14}} + \frac{\langle 4^- | \not{k}_H | 1^-\rangle^2 \langle 2\,3\rangle^2}{s_{123} s_{12} s_{23}} + \frac{\langle 2^- | \not{k}_H | 1^-\rangle^2 \langle 3\,4\rangle^2}{s_{134} s_{14} s_{34}} - \frac{\langle 2\,4\rangle}{\langle 1\,2\rangle\,[2\,3]\,[3\,4]\,\langle 4\,1\rangle} \left(s_{23} \frac{\langle 2^- | \not{k}_H | 1^-\rangle}{[4\,1]} + s_{34} \frac{\langle 4^- | \not{k}_H | 1^-\rangle}{[1\,2]} - s_{234} \langle 2\,4\rangle \right)$$

where $k_H = k_\phi$.

Amplitudes in the overlap of two towers



- $H \to ++--$

- $H \to ++---$

- $H \to ++--$

This is the simplest amplitude which receives contributions from both the MHV and the anti-MHV towers.

$$A_4(H, 1^+, 2^+, 3^-, 4^-) = \frac{\langle 3\,4\rangle^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\,\langle 3\,4\rangle\,\langle 4\,1\rangle} + \frac{[1\,2]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,1]},$$

which is the correct result.

- $H \to ++---$

The contribution from the ϕ-MHV tower is an NMHV amplitude $A_5(\phi, 1^+, 2^+, 3^-, 4^-, 5^-)$.

The contribution from the $\phi^\dagger$-anti-MHV tower is the simple anti-MHV diagram

$$A_5(\phi^\dagger, 1^+, 2^+, 3^-, 4^-, 5^-) = -\frac{[1\,2]^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,5]\,[5\,1]}.$$

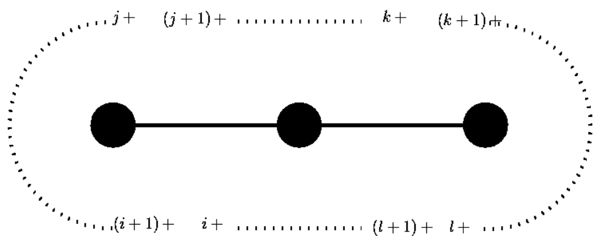
For the ϕ-MHV tower there are 7 contributions of type $A_5^{(1)}$ and 4 of type $A_5^{(2)}$. The final result,

$$A_5(H, 1^+, 2^+, 3^-, 4^-, 5^-) = A_5(\phi, 1^+, 2^+, 3^-, 4^-, 5^-) + A_5(\phi^\dagger, 1^+, 2^+, 3^-, 4^-, 5^-),$$

is gauge invariant. It agrees numerically with eq. (B.3) of Del Duca, Frizzo, Maltoni hep-ph/0404013. Our result is 12 simple terms, it is a considerably shorter expression.

2. NNMHV amplitudes $A_n(\phi, \ldots, m_1^-, \ldots, m_2^-, \ldots, m_3^-, \ldots, m_4^-, \ldots)$

The Next-to-Next-to-MHV amplitudes follow from diagrams with three MHV vertices.

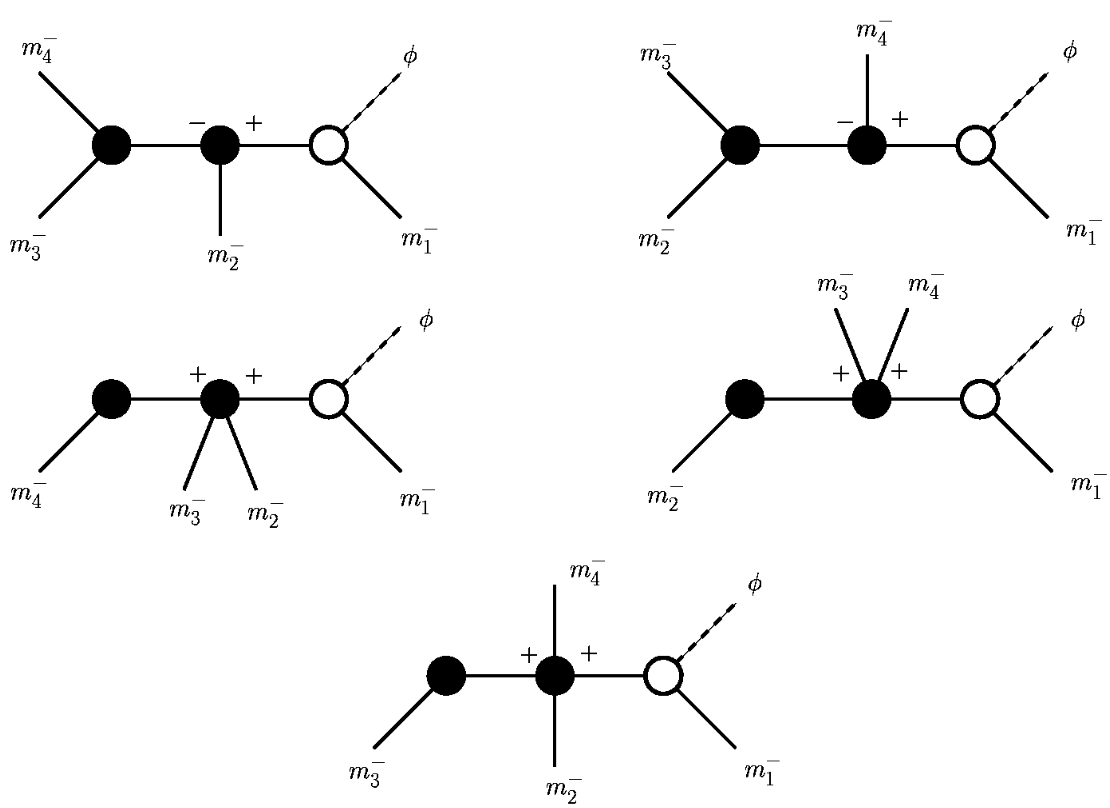


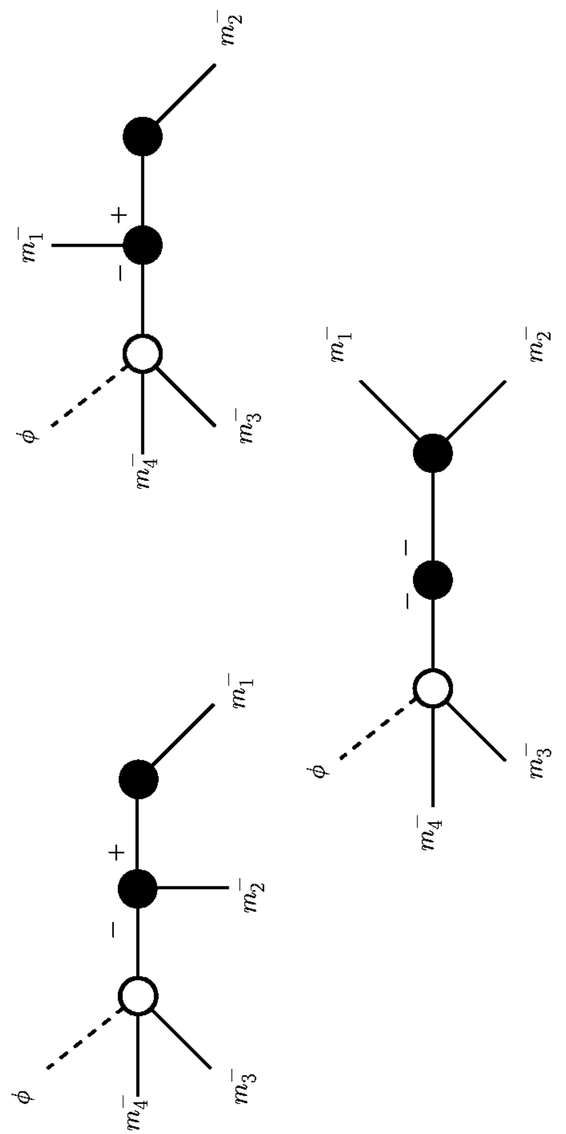
There are 13 topologically distinct diagrams in this case.

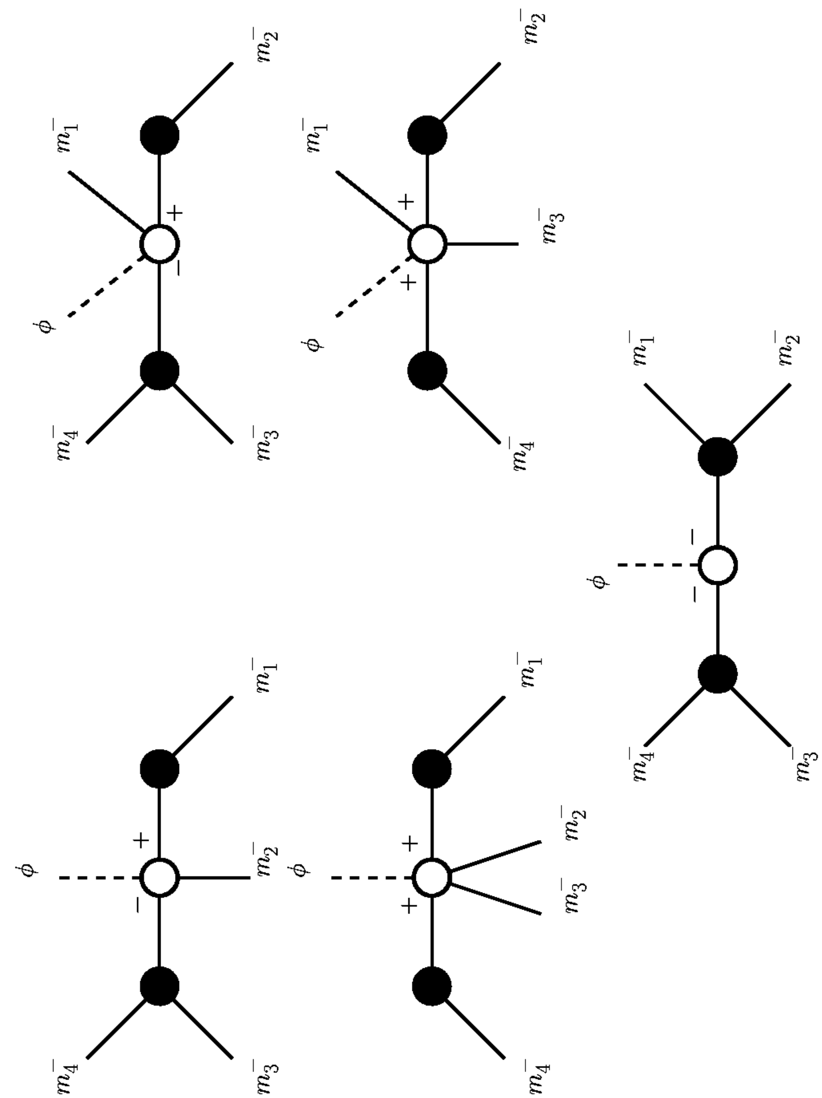
The resulting total amplitude is given by

$$A_n(\phi, m_1^-, m_2^-, m_3^-, m_4^-) = \frac{1}{\prod_{l=1}^{n} \langle l, \, l+1 \rangle} \sum_{i=1}^{13} \sum_{C(m_1,m_2,m_3,m_4)} A_n^{(i)}(m_1, m_2, m_3, m_4)$$

Here we sum over all cyclic permutations, $C(m_1, m_2, m_3, m_4)$.







The contributions of the first five diagrams are

$$\begin{aligned}
A_n^{(1)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_3\, m_4\rangle^4 \langle m_2^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(2)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1}\sum_{j=m_3}^{m_4-1}\sum_{i=m_1}^{m_2-1}\sum_{l=m_1}^{i} \frac{\langle m_2\, m_3\rangle^4 \langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(3)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_3}^{m_4-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_2\, m_3\rangle^4 \langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(4)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1}\sum_{j=m_2}^{m_3-1}\sum_{i=m_1}^{m_2-1}\sum_{l=m_1}^{i} \frac{\langle m_3\, m_4\rangle^4 \langle m_2^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,, \\
A_n^{(5)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1}\sum_{j=m_3}^{m_4-1}\sum_{i=m_2}^{m_3-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_2\, m_4\rangle^4 \langle m_3^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{l+1,k}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{l+1,k})}\,.
\end{aligned}$$

The contributions of diagrams 6, 7 and 8 read

$$\begin{aligned}
A_n^{(6)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_3\, m_4\rangle^4 \langle m_2^-|\not{q}_{j+1,i}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(k,l,q_{k+1,l},i,j,q_{j+1,i})}\,, \\
A_n^{(7)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_1}^{m_2-1}\sum_{j=m_4}^{m_1-1}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \langle m_1^-|\not{q}_{j+1,i}|\xi^-\rangle^4 \langle m_2^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(k,l,q_{k+1,l},i,j,q_{j+1,i})}\,, \\
A_n^{(8)}(m_1,m_2,m_3,m_4) &= \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \langle \xi^+|\not{q}_{j+1,i}\,\not{q}_{k+1,l}|\xi^-\rangle^4 \langle m_1\, m_2\rangle^4}{DD(k,l,q_{k+1,l},i,j,q_{j+1,i})}\,.
\end{aligned}$$

The effective propagator DD is defined by

$$DD(i,j,q_1,k,l,q_2) := \chi(j,k,q_1,q_2)\chi(l,i,q_2,q_1)\, D(i,j,q_1)\, D(k,l,q_2)$$

where D is gthe same as before, and where

$$\begin{aligned}
\chi(j,k,q_1,q_2) &= 1 \qquad \text{if } j \neq k, \\
&= \frac{\langle j,\, j+1\rangle \langle \xi^+|\not{q}_1\not{q}_2|\xi^-\rangle}{\langle (j+1)^-|\not{q}_1|\xi^-\rangle\langle j^-|\not{q}_2|\xi^-\rangle} \qquad \text{if } j = k
\end{aligned}$$

Finally, the last five diagrams give

$$A_n^{(9)}(m_1,m_2,m_3,m_4) = \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_3\, m_4\rangle^4 \,\langle m_2^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(10)}(m_1,m_2,m_3,m_4) = \sum_{k=m_1}^{m_2-1}\sum_{j=m_4}^{m_1-1}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \,\langle m_1^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_2^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(11)}(m_1,m_2,m_3,m_4) = \sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_3}^{m_4-1}\sum_{l=m_1}^{m_2-1} \frac{\langle m_2\, m_3\rangle^4 \,\langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_1^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(12)}(m_1,m_2,m_3,m_4) = \frac{1}{2}\sum_{k=m_1}^{m_2-1}\sum_{j=m_4}^{m_1-1}\sum_{i=m_3}^{m_4-1}\sum_{l=m_2}^{m_3-1} \frac{\langle m_1\, m_3\rangle^4 \,\langle m_4^-|\not{q}_{i+1,j}|\xi^-\rangle^4 \langle m_2^-|\not{q}_{k+1,l}|\xi^-\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})},$$

$$A_n^{(13)}(m_1,m_2,m_3,m_4) = \frac{1}{2}\sum_{k=m_4}^{m_1-1}\sum_{j=m_4}^{k}\sum_{i=m_2}^{m_3-1}\sum_{l=m_2}^{i} \frac{\langle m_3\, m_4\rangle^4 \,\langle \xi^+|\not{q}_{i+1,j}\,\not{q}_{k+1,l}|\xi^-\rangle^4 \,\langle m_1\, m_2\rangle^4}{DD(i,j,q_{i+1,j},k,l,q_{k+1,l})}.$$

• $H \to ----$

Our general NNMHV expressions can be applied to the simple case with no positive-helicity gluons, $n_+ = 0$. Only diagrams 1, 2 and 13 survive. We checked numerically that our result is gauge invariant and agrees with the known expression,

$$A_4(H,1^-,2^-,3^-,4^-) = A_4(\phi,1^-,2^-,3^-,4^-) = \frac{m_H^4}{[1\,2]\,[2\,3]\,[3\,4]\,[4\,1]}.$$

• $H \to +----$

The amplitude for $n_+ = 1$,

$$A_5(H,1^+,2^-,3^-,4^-,5^-) = A_5(\phi,1^+,2^-,3^-,4^-,5^-),$$

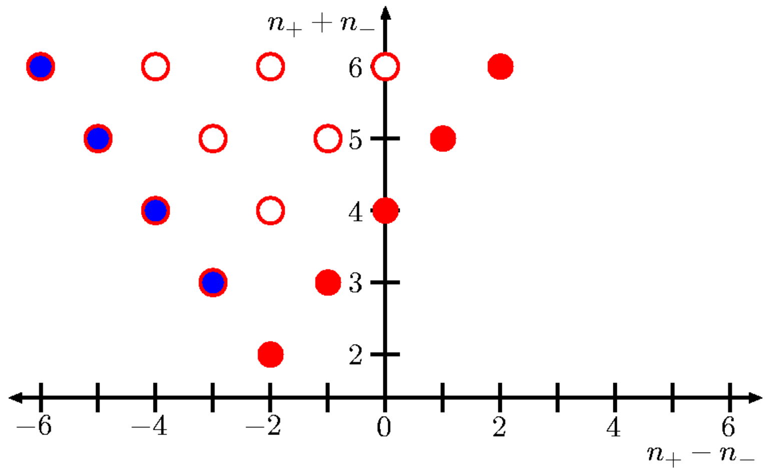
can be obtained by setting $m_1 = 2$, $m_2 = 3$, $m_3 = 4$ and $m_4 = 5$.

The result is again gauge invariant, and it agrees numerically with eq. (B.2) in

Del Duca, Frizzo, Maltoni hep-ph/0404013

.

3. Arbitrary minus-only amplitudes $\phi \to - - - \ldots -$

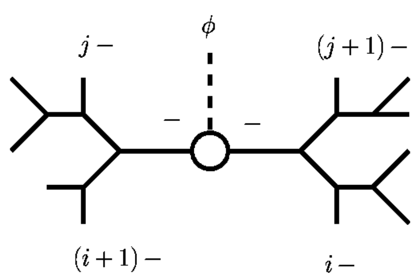


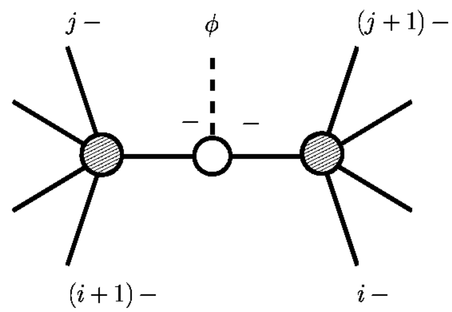
Amplitudes $\phi^\dagger \to + + + \ldots +$ can be obtained from the ϕ amplitudes by parity, which exchanges $\langle i\, j\rangle \leftrightarrow [j\, i]$ and reflects points across the vertical axis, $n_+ - n_- \to -(n_+ - n_-)$.

In the MHV-rules approach, the 'minus-only' amplitudes,

$$A_n(H, 1^-, 2^-, 3^-, \ldots, n^-) = A_n(\phi, 1^-, 2^-, 3^-, \ldots, n^-)$$

are constructed by attaching $n-2$ of the tree-level three-point MHV vertices, $A_3(+--)$, to the ϕ-MHV two-gluon vertex, $A_2(\phi, --)$, in all possible ways.





Each of the two showers of three-point MHV vertices can be represented by an off-shell effective vertex

$$V_n(g_1^{+*}, g_2^-, g_3^-, \ldots, g_n^-) = \frac{p_1^2}{[\xi\, 2]\, [n\, \xi]} \frac{(-1)^n}{[2\, 3]\, [3\, 4] \cdots [n-1,\, n]}$$

where only the g^+ leg is off-shell and hence $p_1^2 \neq 0$. (Zhu hep-th/0403115).

Attaching the off-shell V-vertices on both sides of the amplitude $A_2(\phi,-,-)$, we get

$$A_n(\phi,1^-,2^-,3^-,\ldots,n^-) = \frac{(-1)^n}{[1\,2]\,[2\,3]\cdots[n\,1]}\,\mathcal{A}\,,$$

where $\mathcal{A}$ is the sum

$$\mathcal{A} = -\sum_{1\le i<j\le n} \frac{[i,\,i+1]\,[j,\,j+1]}{[i\,\xi]\,[\xi,\,i+1]\,[j\,\xi]\,[\xi,\,j+1]}\,\langle\lambda_{i+1,j}\,\lambda_{j+1,i}\rangle^2 \;=\; m_H^4$$

We conclude that

$$A_n(\phi,1^-,2^-,3^-,\ldots,n^-) = \frac{(-1)^n\,m_H^4}{[1\,2]\,[2\,3]\cdots[n\,1]}$$

and, similarly, the parity conjugate amplitude is

$$A_n(\phi^\dagger,1^+,2^+,3^+,\ldots,n^+) = \frac{m_H^4}{\langle 1\,2\rangle\,\langle 2\,3\rangle\cdots\langle n\,1\rangle}$$

Conclusions and outlook

1. We have constructed and tested a novel set of MHV rules for calculating scattering amplitudes of the massive Higgs boson plus an arbitrary number of gluons.
2. The model is the tree-level pure gauge theory plus an effective interaction $HG_{\mu\nu}G^{\mu\nu}$. This effective interaction is generated in the heavy top quark limit from the non-supersymmetric Standard Model by integrating out the heavy top-quark loop.
3. We split the interaction into selfdual and anti-selfdual pieces to derive the MHV rules.
4. The MHV rules lead to compact formulae for the Higgs plus multi-parton amplitudes induced at leading order in QCD in the large m_t limit.
5. This structure may also be useful for going to the next order in QCD.
6. Apart from being interesting on its own right, we believe that this model gives important insights into the structure of the MHV rules in generic nonsupersymmetric theories at the loop level.

- Our MHV-rules construction was designed to address effective interactions at tree level.
- From the perspective of the microscopic theory, our approach enables us to address only massive loops in a non-supersymmetric theory.
- These MHV rules amount to more than adding a new class of vertices to the tree-level rules of CSW, in particular, the two towers of MHV and anti-MHV diagrams are crucial for the construction to work.
- In the soft-Higgs limit, $k_H \to 0$, each tower becomes proportional to the pure-gauge-theory tower, but the constant of proportionality depends on the helicity content of the amplitude.
- We expect that our findings will be useful in constructing MHV rules also for massless loops in nonsupersymmetric theories.