Universal Quantum Simulation for Open-System Dynamics

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Problem statement

Construct an efficient algorithm for designing an efficient quantum algorithm, implemented with a small single-qubit universal gate set, that accurately simulates any completely-positive (CP) trace-preserving (TP) single-qubit mapping for any input state within pre-specified tolerance ϵ in terms of distance between states.

Strategy for open-system quantum simulation

- Dilate CPTP map to unitary operation U.
- Decompose U as ordered operator exponential
  - Solovay-Kitaev-Dawson-Nielsen algorithm
  - Explicitly construct lookup table for this algorithm
- Decompose CPTP map into product of channel types:
  - Unitary
  - Pauli
  - Amplitude attenuation/amplification
- Produce efficient algorithm for constructing efficient circuit.

Dilate CPTP map to unitary evolution

Operator sum decomposition

CP $\mathscr{E}:\mathcal{T}(\mathscr{H})\mapsto\mathcal{T}(\mathscr{H}):\rho\mapsto\sum_i K_i\rho K_i^\dagger,\ \sum_i K_i^\dagger K_i\leq\mathbb{1}$.

- Trace-preserving (TP): $\sum_i K_i^\dagger K_i=\mathbb{1}$.
- Introduce environment with $\perp$ basis $\{|i\rangle\}$ spanning $\tilde{\mathscr{H}}$.
- Dilated Hilbert space: $\mathscr{H}'=\mathscr{H}\otimes\tilde{\mathscr{H}}$.
- Construct $U:\mathscr{H}'\to\mathscr{H}'$ s.t. each $K_i=\langle i|U|0\rangle$.
- U is *minimal* if $\dim\tilde{\mathscr{H}}=(\dim\mathscr{H})^2=d^2\implies\dim\mathscr{H}=d^3$.
- Kraus operator K_i are realized via measurement on ancillæ.
- For N-qubit open system, $2N$ qubit ancillæ are needed.
- Reduce cost: recycle ancillæ $+$ use random classical bit(s).

Simulation accuracy condition

Definition

The inaccuracy for simulating q channel $\mathscr{E}_1$ by approximate channel $\mathscr{E}_2$ is $\|\mathscr{E}_1 - \mathscr{E}_2\|_{1\to 1} := \sup_\rho \|\mathscr{E}_1(\rho) - \mathscr{E}_2(\rho)\|_1$ for $\|\bullet\|_1 := \mathrm{tr}\sqrt{\bullet^\dagger \bullet}$ the Schatten 1-norm.

Proposition

$\forall \epsilon \in \mathbb{R}^+$, $\rho \in \mathcal{T}(\mathscr{H})$ and CPTP maps $\mathscr{E}_1, \mathscr{E}_2 : \mathcal{T}(\mathscr{H}) \to \mathcal{T}(\mathscr{H})$ with respective minimal dilations $U_1, U_2 : \mathscr{H}' \to \mathscr{H}'$, then $\|U_1 - U_2\| \leq \epsilon/2 \implies \|\mathscr{E}_1 - \mathscr{E}_2\|_{1\to 1} \leq \epsilon$ with $\|\bullet\|$ the operator norm[a].

[a] $\|U_1 - U_2\| := \sup_{|\psi\rangle} \|(U_1 - U_2)|\psi\rangle\| < \epsilon/2$.

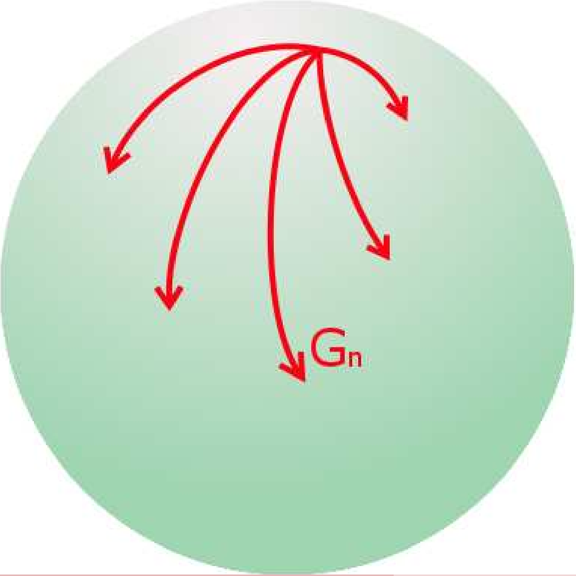
Efficient decomposition of unitary gate

- The Solovay-Kitaev theorem guarantees any N-qubit unitary gate can be decomposed into a gate sequence obtained from a finite instruction set with length $O\left(\log^c(\frac{1}{\epsilon})\right)$.
- We employ the universal gate set
  - $H = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,
  - $T \equiv Z^{1/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$,
  - $CNOT = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & X \end{pmatrix}$.

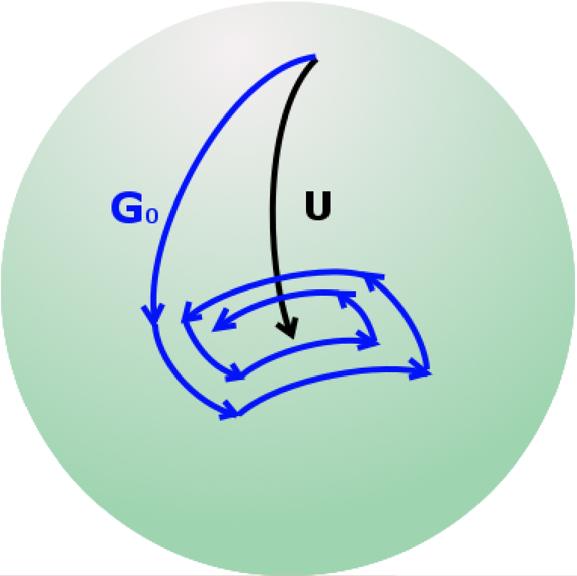
Available decomposition approaches

- Kitaev-Shen-Vyalyi phase-kickback algorithm [Toffoli gates].
- **SKDN algorithm** [proven to be efficient].
- Depth-optimal algorithm: shortest run-time [perhaps counted as # non-Clifford gates] through parallelized circuit
  - Fowler: QIC **11**, 867–873 (2011).
  - Bocharov-Svore: arXiv.org:1206.3233
  - Trung-Meter-Horsman: arXiv.org:1209.4139
  - Kiluchnikov-Maslov-Mosca: arXiv.org:1206.5236
  - Amy-Maslov-Mosca-Roetteler: arXiv.org:1206.0758

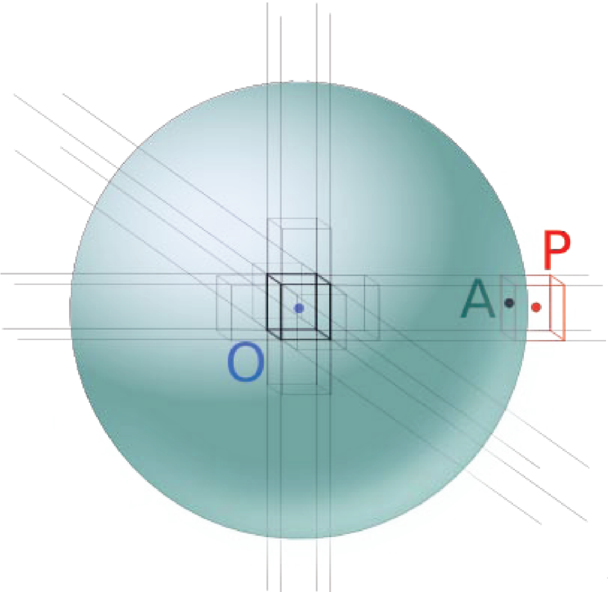
SKDN algorithm: Elements of database



SKDN algorithm: "Balanced group commutator"



Constructing look-up database for the SKDN algorithm



Decomposing the single-qubit CPTP channel

Geometric representation of single-qubit CPTP channel

For single-qubit state representation $\rho \mapsto \frac{1}{2}(I + \mathbf{b} \cdot \boldsymbol{\sigma})$ and single-qubit CPTP mapping representation, $\mathscr{E} \mapsto \mathbb{T} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{t} & \mathrm{T} \end{pmatrix}$ s.t. $\mathbb{T}_{ij} \mapsto \frac{1}{2}\mathrm{tr}\left(\sigma_i \mathscr{E}(\sigma_j)\right)$, $\mathscr{E}(\rho) \mapsto \frac{1}{2}(I + \mathbf{b}' \cdot \boldsymbol{\sigma})$ is the affine map for $\mathbf{b}' = \mathrm{T}\mathbf{b} + \mathbf{t}$, and $\mathscr{E}$ is CPTP iff the Choi matrix $\mathbb{C} \geq 0$ for $(i,k|\mathbb{T}|j,l) := (i,j|\mathbb{C}|k,l)$ with $|k,l) := |k\rangle\langle l|$.

Single-qubit CPTP channel decomposition

- Implement T by Pauli channel $\rho \mapsto \sum_{i=0}^{3} c_i \sigma_i \rho \sigma_i$ $-\boxed{\mathscr{E}_\mathrm{P}}-$
- Implement shift **t** by rotated amplitude-attenuation/amplification channel

$$-\boxed{\mathscr{E}_\mathrm{R}}- \;=\; -\boxed{R_m(\theta)}-\boxed{\mathscr{E}_\mathrm{A}}-\boxed{R_n(\phi)}-$$

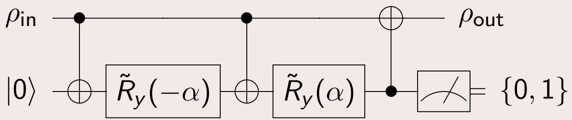
Proposition

Any single-qubit CPTP channel $\mathcal{E}$ can be decomposed into the 12-parameter gate sequence

$$-\boxed{\mathscr{E}}- \;=\; -\boxed{V}-\boxed{\mathscr{E}_\mathrm{P}}-\boxed{R_m(\theta)}-\boxed{\mathscr{E}_\mathrm{A}}-\boxed{R_n(\phi)}-\boxed{V'}-$$

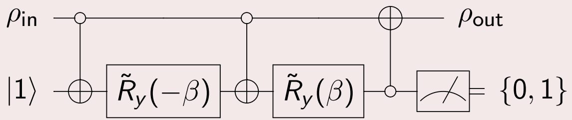
Amplitude attenuation/amplification channel simulation

Amplitude attenuation channel $\mathscr{E}_A^+$ simulator



$$\mathcal{E}^+(\rho) = K_0 \rho K_0^\dagger + K_1 \rho K_1^\dagger,\ K_0 = \begin{pmatrix} 0 & \sin\alpha \\ 0 & 0 \end{pmatrix},\ K_1 = \mathrm{diag}(1, \cos\alpha)$$

Amplitude amplification channel $\mathscr{E}_A^-$ simulator



$$\mathcal{E}_A^-(\rho) = K_2 \rho K_2^\dagger + K_3 \rho K_3^\dagger,\ K_2 = \begin{pmatrix} 0 & 0 \\ \sin\beta & 0 \end{pmatrix},\ K_3 = \mathrm{diag}(\cos\beta, 1)$$

Decomposing the single-qubit CPTP channel

$$\mathscr{E} = V \; \mathscr{E}_{\mathrm{P}} \; R_m(\theta) \; \mathscr{E}_{\mathrm{A}} \; R_n(\phi) \; V'$$

$$\tilde{\mathscr{E}} = \tilde{V} \; \tilde{\mathscr{E}}_{\mathrm{P}} \; \tilde{R}_m(\theta) \; \tilde{\mathscr{E}}_{\mathrm{A}} \; \tilde{R}_n(\phi) \; \tilde{V}'$$

- Resources
  - cbits + recyclable ancillary qubit for $\tilde{\mathscr{E}}_{\mathrm{A}}$
  - one classical bit for $\tilde{\mathscr{E}}_{\mathrm{P}}$.
- Complexity (follows SKDN algorithm complexity):
  - # gates in q circuit $\in O(\log^{3.97}(1/\epsilon))$;
  - CPU time on classical computer $\in O(\log^{2.71}(1/\epsilon))$.

Results

- CPTP channel simulation error is bounded by the inaccuracy of simulating corresponding minimally dilated unitary map.
- Derived exact decomposition of arbitrary single-qubit CPTP channel as a product of unitary channels and one Pauli and one rotated amplitude attenuation/amplification channel
- Simplified single-qubit CPTP channel simulation from two qubit ancillæ to one recyclable ancilla plus classical bits
- Developed an explicit geometric look-up database for use in the Solovay-Kitaev-Dawson-Nielsen algorithm
- Classical algorithm and quantum gate length polylog$(1/\epsilon)$