Entanglement Negativity in Quantum Field Theory

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Outline

- Bipartite entanglement in pure states
- Path integral approach and correlators of twist operators in QFT
- Entanglement in mixed states and negativity
- Results in 1+1-dimensional CFT
- Higher dimensions

Work largely carried out with Pasquale Calabrese (Pisa) and Erik Tonni (Trieste)
Bipartite Entanglement in Pure States

Quantum system in a pure state $|\Psi\rangle$, density matrix $\rho = |\Psi\rangle\langle\Psi|$.

$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

Schmidt decomposition:

$$|\Psi\rangle = \sum_j c_j |\psi_j\rangle_A \otimes |\psi_j\rangle_B$$

with $c_j \geq 0$, $\sum_j c_j^2 = 1$, and $|\psi_j\rangle_A, |\psi_j\rangle_B$ orthonormal.

One quantifier of the amount of entanglement is the entropy

$$S_A \equiv -\sum_j |c_j|^2 \log |c_j|^2 = S_B$$
Equivalently, in terms of A’s reduced density matrix

\[ \rho_A \equiv \text{Tr}_B |\psi\rangle \langle \psi| \]

\[ S_A = -\text{Tr}_A \rho_A \log \rho_A = S_B \]

Similar information is contained in the Rényi entropies

\[ S_A^{(n)} = (1 - n)^{-1} \log \text{Tr}_A \rho_A^n \]

\[ S_A = \lim_{n \to 1} S_A^{(n)} \]
Other measures of bipartite entanglement exist, but **entropy** has several nice properties: additivity, convexity, . . .

It is monotonic under Local Operations and Classical Communication (LOCC)

It gives the amount of classical information required to specify $\rho_A$ (important for numerical computations)

It gives a basis-independent way of identifying and characterising quantum phase transitions

In a relativistic QFT the entanglement in the vacuum encodes all the data of the theory (spectrum, anomalous dimensions, . . .)
Rényi entropies from the path integral

\[ \Psi(\{a\}, \{b\}) = Z_1^{-1/2} \int_{a(0)=a, b(0)=b} [da(\tau)][db(\tau)] e^{-(1/\hbar)S[\{a(\tau)\}, \{b(\tau)\}]} \]

where \( S = \int_{-\infty}^{0} L(a(\tau), b(\tau)) d\tau \)

Similarly, \( \Psi^*(\{a\}, \{b\}) \) is given by the path integral from \( \tau = 0 \) to \( +\infty \).
\[ \rho_A(a_1, a_2) = \int \, db \, \psi(a_1, b) \psi^*(a_2, b) \]

This is given by the path integral over \( \mathbb{R}^2 \) cut open along \( A \cap \{ \tau = 0 \} \), divided by \( Z_1 \):
Rényi entropies

\[ \text{Tr}_A \rho_A^n \] is given by the partition function on \( n \) sheets sewn together cyclically along \( A \cap \{\tau = 0\} \), forming a conifold \( \mathcal{R}_n \), with opening angles \( 2\pi n \) at each conical singularity.

\[ \text{Tr}_A \rho_A^n = \frac{Z(\mathcal{R}_n)}{Z_1^n} \]

– equivalently, \( n \) copies of the CFT within the fields cyclically identified across \( A \cap \{\tau = 0\} \):

\[ a_j(0-) = a_{j+1}(0+) \mod n \]
If space is 1d and $A$ is an interval $(r_1, r_2)$ (and $B$ is the complement) then $Z(\mathcal{R}_n)$ can be thought of as the insertion of twist operators into $n$ copies of the CFT:

$$Z(\mathcal{R}_n)/Z_1^n = \langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2) \rangle_{(CFT)^n}$$

These have similar properties to other local operators e.g. in a massless QFT (a CFT)

$$\langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2) \rangle \sim |r_1 - r_2|^{-2\Delta_n}$$

Main result for $d = 1$ [Holzhey et al., CC]:

$$\Delta_n = (c/12)(n - 1/n)$$

where $c$ is the central charge of the UV CFT
Two intervals

\[ Z(R_n)/Z^n = \langle P_n^{-1}(r_1)P_n(r_2)P_n^{-1}(r_3)P_n(r_4) \rangle \]

In general there is no simple result but for \( r_{12}, r_{34} \ll r_{23}, r_{14} \) we can use an operator product expansion [Headrick, CCT]

\[ P_n^{-1}(r_1) \cdot P_n(r_2) = \sum_{\{k_j\}} C_{\{k_j\}}(r_1 - r_2) \prod_{j=1}^{n} \Phi_{k_j}(\frac{1}{2}(r_1 + r_2)_j) \]

in terms of a complete set of local operators \( \Phi_{k_j} \).

This shows that the mutual information \( S_{A_1 \cup A_2} - S_{A_1} - S_{A_2} \) is more related to correlations between \( A_1 \) and \( A_2 \) and not their quantum entanglement.
Twist operators correspond to a cyclic permutation $P_n$ of the replicas as we go around the conical singularity.

More generally we could consider

$$\langle P_n^{(1)}(r_1)P_n^{(2)}(r_2)P_n^{(3)}(r_3)P_n^{(4)}(r_4) \rangle$$

where the $P_n^{(k)}$ are more general permutations of $n$ objects (with $\prod_k P_n^{(k)} = 1$.)

These are related to new measures of the mixed state entanglement between $A_1$ and $A_2$. 
In particular

\[
\langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\mathcal{P}_n(r_3)\mathcal{P}_n^{-1}(r_4) \rangle
\]

gives

\[
\text{Tr} \left( \rho_{\overline{T}_2 A_1 \cup A_2} \right)^n
\]

where \( \rho_{\overline{T}_2 A_1 \cup A_2} \) is the partial transpose

\[
\rho_{\overline{T}_2 A_1 \cup A_2} (a_1, a_2; a_1', a_2') = \rho_{A_1 \cup A_2} (a_1, a_2'; a_1', a_2)
\]
This is related to *negativity* [Vidal-Werner 2002].

Although $\text{Tr} \rho^{T_2} = 1$, it may have negative eigenvalues $\lambda_j$, and this will happen if

$$
\mathcal{E} \equiv \log \text{Tr} |\rho^{T_2}| = \log \sum_j |\lambda_j| > 0
$$

$\mathcal{E}$ has nice quantum information properties, e.g. monotonicity under LOCC.
Negativity in 1+1 dimensional CFT

Note that

$$\text{Tr} \left( \rho^T_2 \right)^n = \sum_j \lambda_j^n = \sum_j |\lambda_j|^n \quad \text{for } n \text{ even}$$

so if the continuations to $n = 1$ from even and odd $n$ are different, we can have negativity.

$$\langle \mathcal{P}_n^{-1}(r_1)\mathcal{P}_n(r_2)\mathcal{P}_n(r_3)\mathcal{P}_n^{-1}(r_4) \rangle$$

This can happen if $r_{23} \ll r_{12}, r_{34}$, because of the OPE

$$\mathcal{P}_n \cdot \mathcal{P}_n \ \cong \ \mathcal{P}_n \quad \text{for } n \text{ odd}$$

$$\cong \mathcal{P}_{n/2} \otimes \mathcal{P}_{n/2} \quad \text{for } n \text{ even}$$

This has scaling dimension $2(c/12)(n/2 - 2/n) \to -c/4$ as $n \to 1$. 
In this limit we get

\[ \mathcal{E} \sim \left( \frac{c}{4} \right) \log\left( \frac{r_{12} r_{34}}{r_{23} r_{14}} \right) \]

This has been confirmed numerically for uncompactified free boson and for the Ising model.

\[ \langle \mathcal{P}^{-1}(r_1) \mathcal{P}(r_2) \mathcal{P}(r_3) \mathcal{P}^{-1}(r_4) \rangle \]

In the opposite limit \( r_{12}, r_{34} \ll r_{23}, r_{14} \) we can use the short interval expansion, analytically continued from the usual ordering of the arguments: as \( n \to 1 \) every term in the OPE vanishes!

But numerically we find [Markovitch et al., CCT]

\[ \mathcal{E} \propto \exp \left( -C \frac{r_{23} r_{14}}{r_{12} r_{34}} \right) \]

Non-perturbative terms in the OPE!
\[ y = \frac{r_{12}r_{34}}{r_{13}r_{24}} \]
Other results

In general $\text{Tr} \left( \rho_{T_2}^{A_1 \cup A_2} \right)^n$ is given by the partition function on a surface of the same genus as that for $\text{Tr} \left( \rho_{A_1 \cup A_2} \right)^n$, but on a different section of the moduli space.

However for $n = 2$

\[
\text{Tr} \left( \rho_{T_2}^{A_1 \cup A_2} \right)^2 = \text{Tr} \left( \rho_{A_1 \cup A_2} \right)^2 \propto Z_{\text{torus}}
\]

but

\[
\text{Tr} \left( \rho_{T_2}^{A_1 \cup A_2} \cdot \rho_{A_1 \cup A_2} \right) \propto Z_{\text{Klein bottle}}
\]

Correlators of products of twist operators corresponding to general permutations are given by CFT partition functions on non-orientable surfaces.
For $d > 1$ for 2 large regions a finite distance apart

$N(A_1, A_2) \propto \text{Area of common boundary between } A_1 \text{ and } A_2$

- universal corrections to this ’area law’?
- if $A_1$ and $A_2$ are far apart, a generalisation of small interval expansion again gives vanishing negativity to all order vanishes to all orders – non-perturbative corrections??
Summary

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...when gravity fails and negativity won’t pull you through...