Entanglement Entropy and Geometry

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Three hints that entanglement entropy and geometry are deeply related:

1. EE basics and some quantum information (review)

2. Calculations in 2d CFT

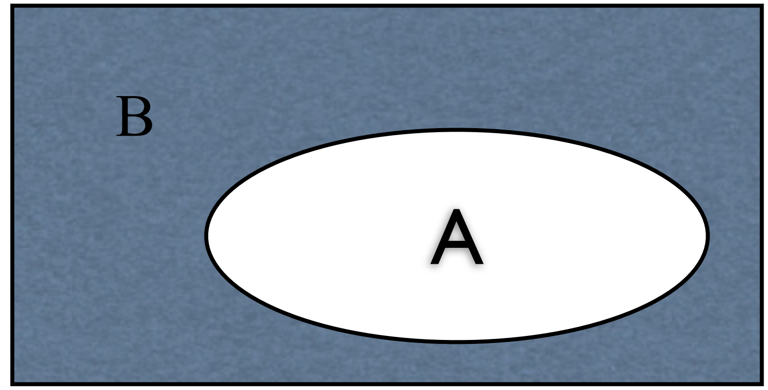
3. Reconstruction of bulk geometry

Entanglement entropy

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\rho_A = \mathrm{tr}_B\ \rho$$

$$S_A = -\mathrm{tr}\ \rho_A \log \rho_A$$

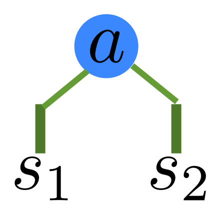


Example: thermal entropy

A=system, B=bath, state=equilibrium

Example: 2 spins

$$|\Psi\rangle = \sum_{s_1, s_2} a_{s_1 s_2} |s_1\rangle \otimes |s_2\rangle$$



$$= \pm 1$$

Bell state: $S_1 = \log 2$

Entanglement entropy encodes the organization of quantum information.

Ex: 1+1d spins

In general,

$$|\Psi\rangle = \sum_{\{s_i\}} a_{s_1 \dots s_k} |s_1\rangle \cdots |s_k\rangle \quad \sim 2^k \text{ complex numbers}$$

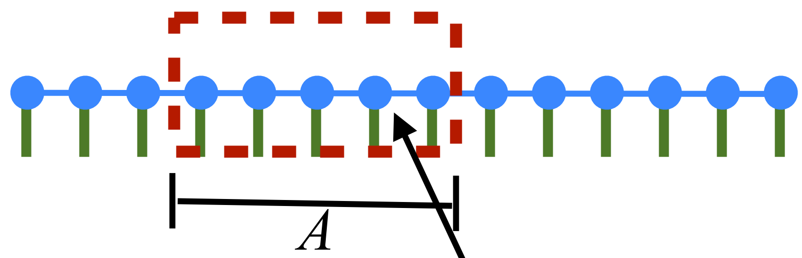
$$S_\ell \sim \ell \quad \text{“Volume law”}$$

Groundstates of local Hamiltonians are very special, occupying only a tiny corner of this enormous Hilbert space.

For a gapped Hamiltonian,

$$S_\ell \sim \text{ constant} \quad \text{“Area law”}$$

This implies



“Matrix product states”

$$P \times P \text{ matrices}$$

$$a_{s_1 \cdots s_k} \approx M_{s_1} \cdot M_{s_2} \cdots M_{s_k}$$

$$kP^2 \text{ complex numbers } \ll 2^k$$

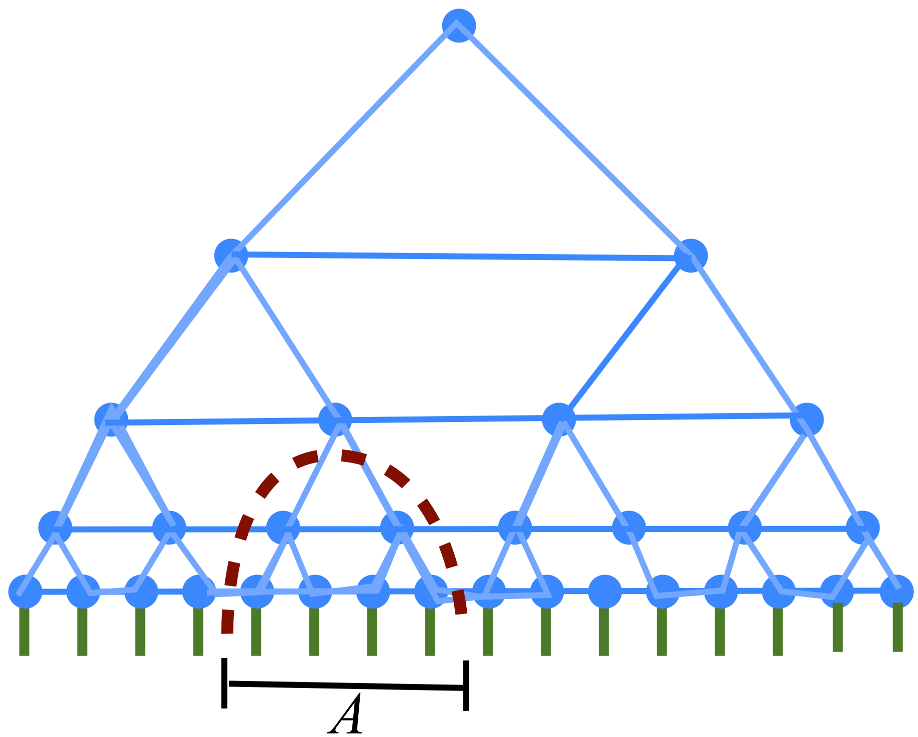
To see if this is a good representation of the wavefunction, we ask how much entanglement it can encode:

$$S_\ell \lesssim \log P \sim \text{const.}$$

At a critical point,

$$S_\ell = \frac{c}{3}\log\ell$$

This suggests an efficient tensor network representation as a tree:



“Tensor networks”
“MERA”

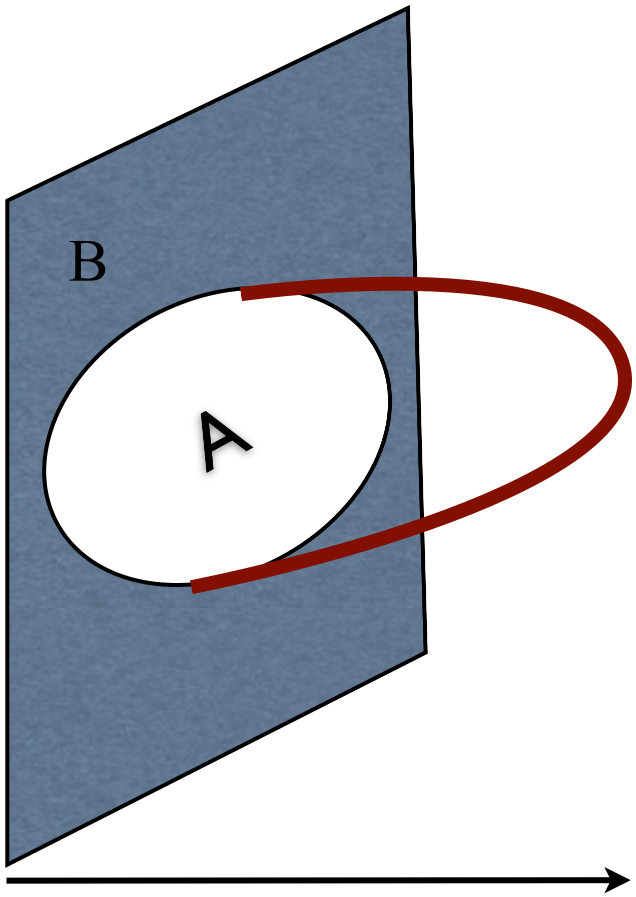
Vidal ’06

How much entanglement can the tree encode?

$$S_\ell \lesssim \log\ell$$

Connection to holography

The entanglement entropy of a holographic CFT can be computed by the Ryu-Takayanagi formula:



holographic radial direction

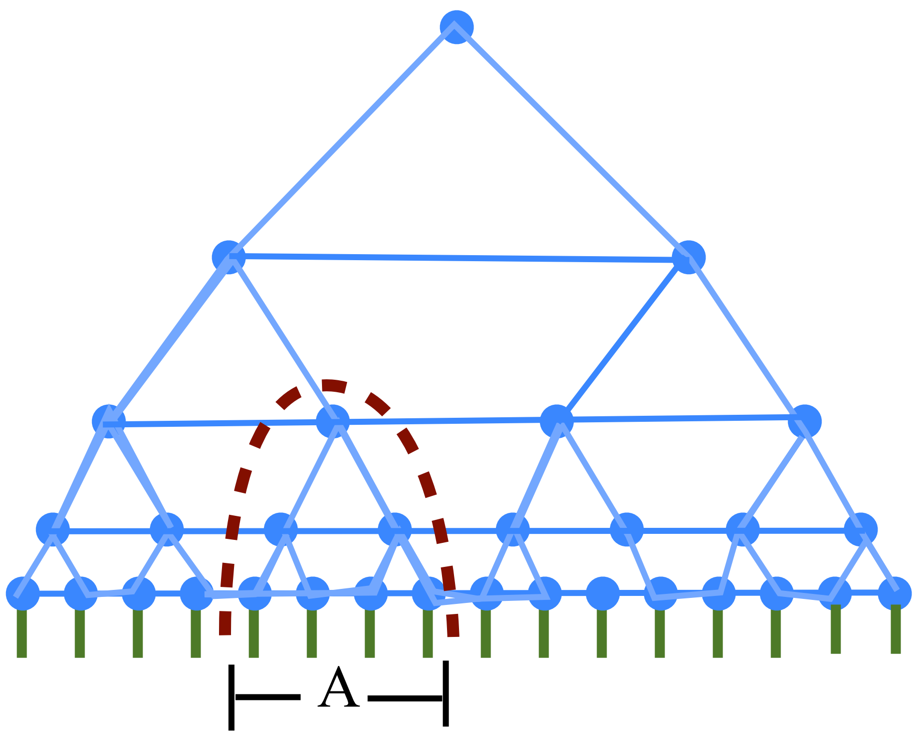
$$S_A = \frac{\text{Area(minimal surface)}}{4G_N}$$

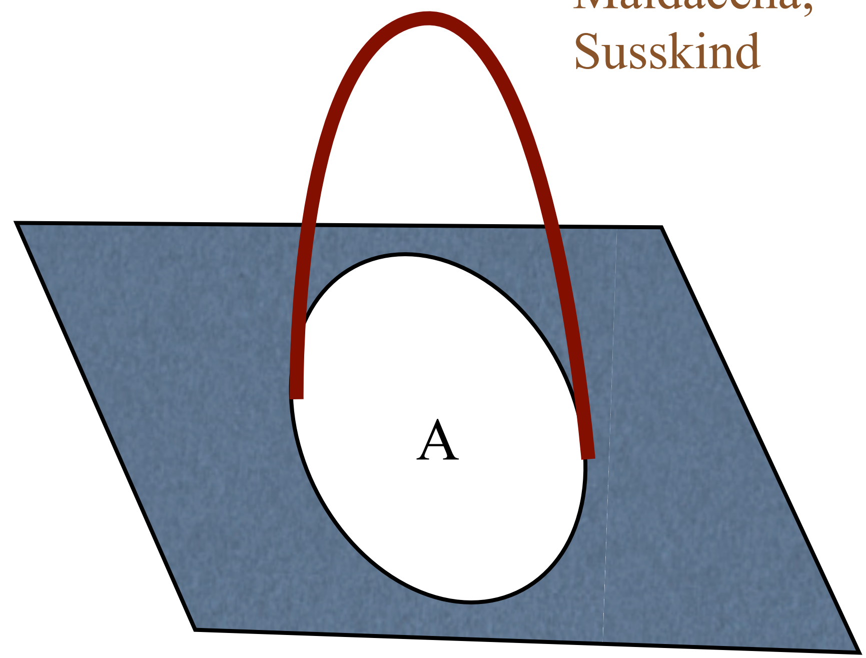
This generalizes the Bekenstein-Hawking entropy to other types of surfaces, including Rindler horizons.

Ryu & Takayanagi '06
Lewkowycz & Maldacena '13

"Cutting" the tensor network to compute entanglement entropy resembles the Ryu-Takayanagi minimal surface:

Swingle;
Van Raamsdonk;
Maldacena;
Susskind

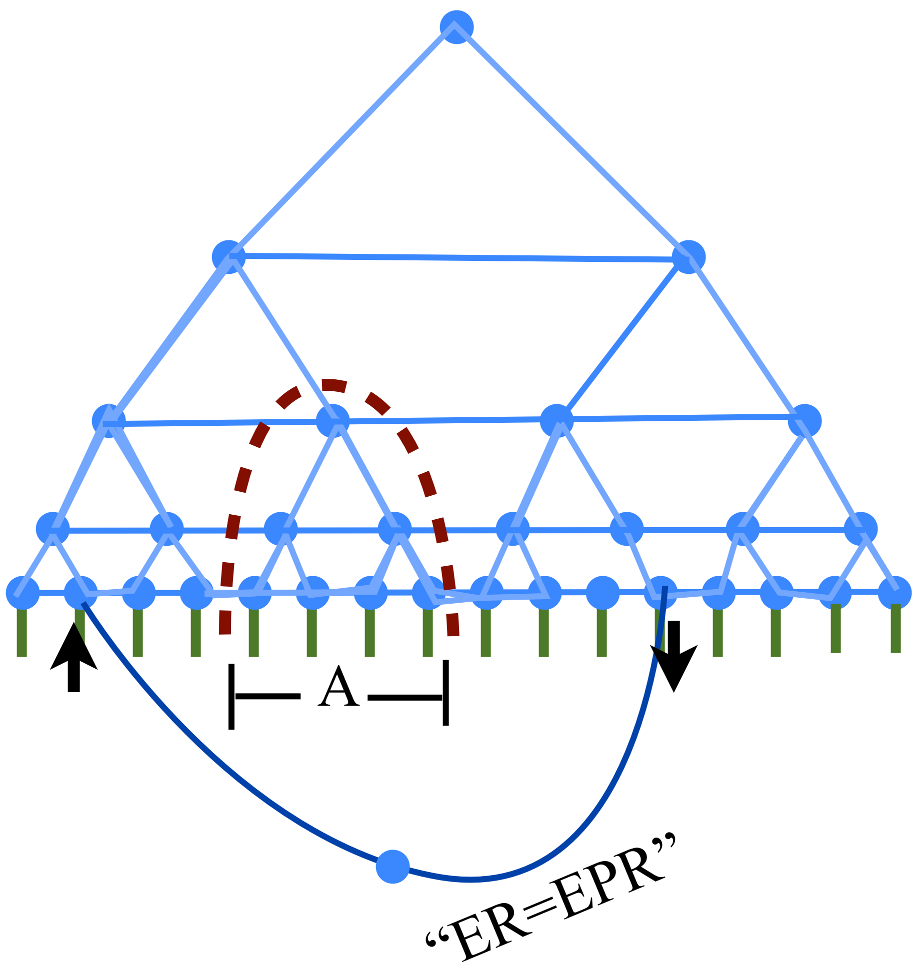


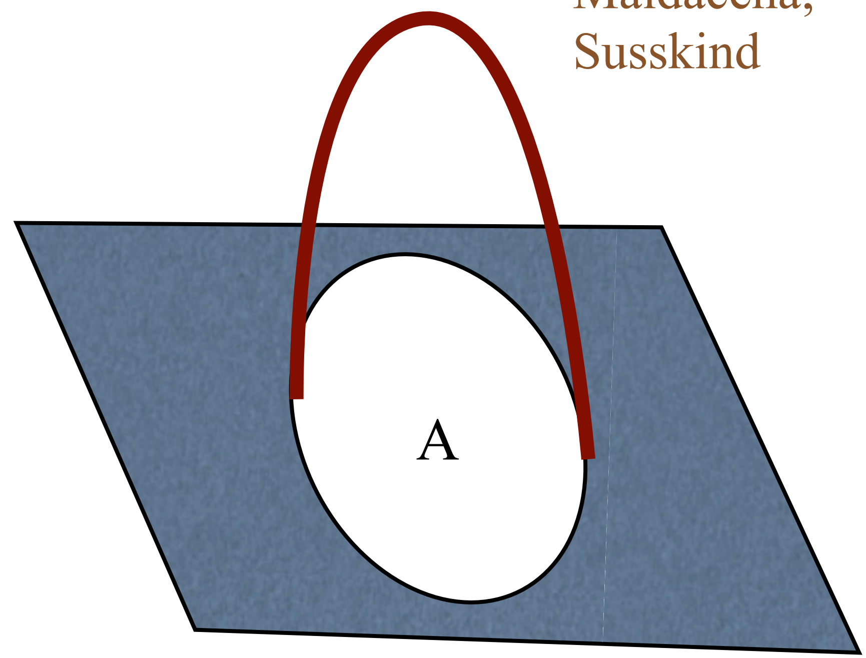


- The MERA relation is *qualitative*. Each tensor is "AdS-radius-sized".
- It provides another perspective on the statement that the radial direction is RG flow.
- It suggests that entanglement entropy plays a fundamental role in understanding how the bulk geometry emerges.

“Cutting” the tensor network to compute entanglement entropy resembles the Ryu-Takayanagi minimal surface:

Swingle; Van Raamsdonk; Maldacena; Susskind





Extra entanglement requires extra “wormhole” lines connecting distant points

Outline

1. EE basics and some quantum information

2. Calculations in 2d CFT

3. Reconstruction of bulk geometry

In 1+1 dimensions:

Space is a line, so **A** consists of one or more intervals:

The Replica Method

We want to compute the entanglement entropy

$$S_A = -\mathrm{Tr}\ \rho_A \log \rho_A$$

First compute the Renyi/replica partition functions for n=2,3,...

$$Z^{(n)} = \mathrm{Tr}\ \rho_A^n$$

and use

$$S_A = -\partial_n Z^{(n)}|_{n=1}$$

This is useful because $Z^{(n)}$ can be computed by a Euclidean path integral (for the groundstate and some other special states).

Calabrese & Cardy '04

These partition functions can be computed analytically in CFT in (at least) 4 situations:

- "A" is a connected region (single interval)

Holzhey, Larsen, Wilczek '94
Calabrese & Cardy '04

$$S_\ell = \frac{c}{3} \log\left(\frac{\ell}{\epsilon_{UV}}\right)$$

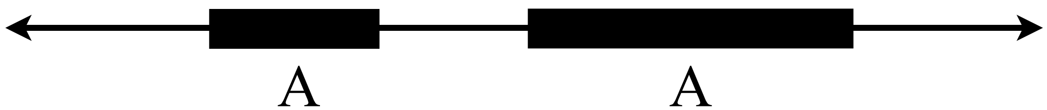
universal!

- Multiple intervals:
  - free field theory
  - In a small-interval expansion
  - In a limit of large central charge

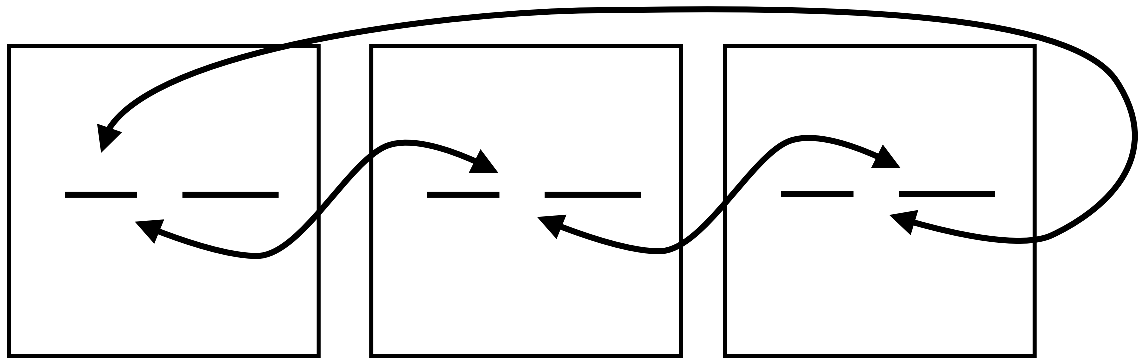
Multiple Intervals

Casini, Fosco, Huerta;
Cardy, Calabrese, Tonni

Example where A is 2 intervals, replica number=3:



$$\mathrm{Tr}\ \rho_A^3 = Z\Big($$

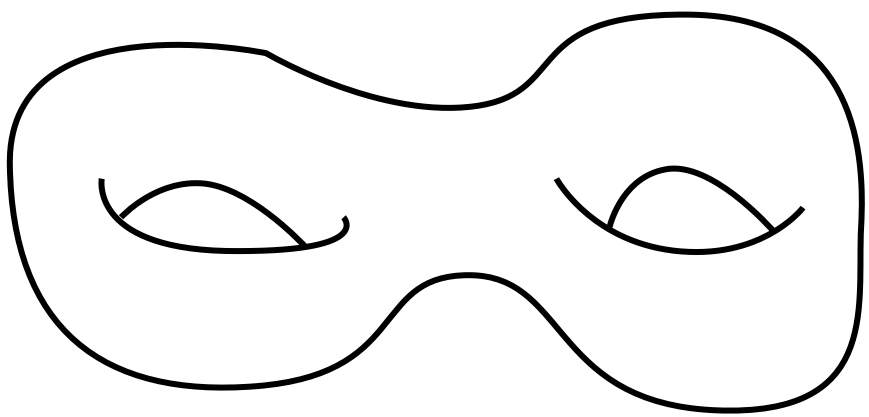


$$\Big)$$

This is a Riemann surface with nontrivial topology.

This example (2 slits, 3 replicas) has genus 2:

$$= Z\Big($$



$$\Big)$$

Free Fields

This partition function has been evaluated exactly in free field theory.

- Free fermion
- Free compact boson

Casini, Fosco, Huerta;
Cardy, Calabrese, Tonni

- Replica partition function is known, but the analytic continuation to n=1 is in general unsolved.

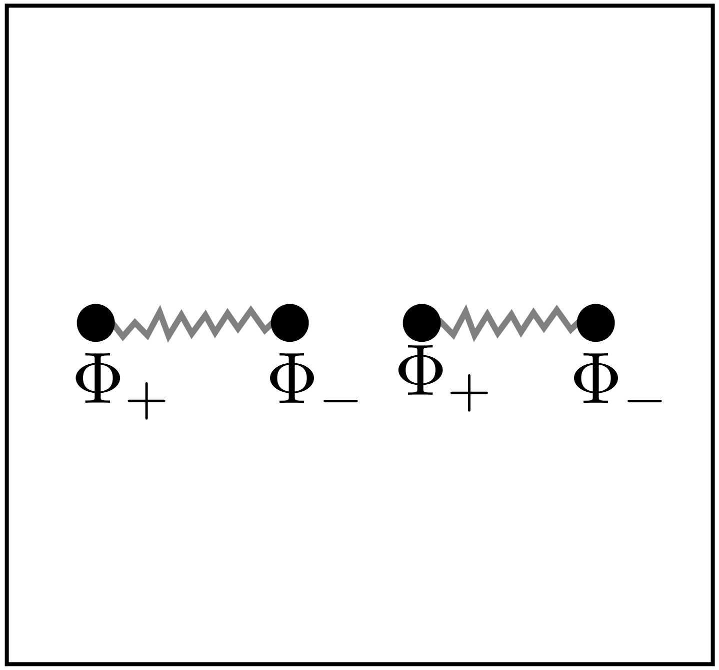
Two interacting cases where we can calculate analytically

1. Small-interval expansion
2. Large-*c* expansion

Headrick; Cardy, Calabrese, Tonni; T.H.

Twist operators

The replica partition function can be viewed as a correlation function of "twist operators" that glue the sheets together.

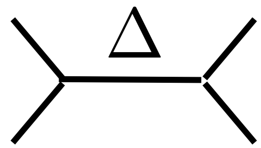


$$\text{Tr}\ \rho_A^n = \langle \Phi_+ \Phi_- \Phi_+ \Phi_- \rangle_{CFT^n}$$

Dixon, Friedan, Martinec & Shenker '87
Calabrese, Cardy, Tonni '04 - '13

2pt functions are fixed by conformal invariance (single interval).

4pt functions are not fixed, but are constrained to have the form

$$\langle \Phi_+ \Phi_- \Phi_+ \Phi_- \rangle = \sum_{\Delta}$$ 

$$= \sum_{\Delta} c_{\Delta}^2 \mathcal{F}(\Delta, H_n, z) \mathcal{F}(\bar{\Delta}, H_n, \bar{z})$$

Virasoro Conformal Blocks

OPE coefficient

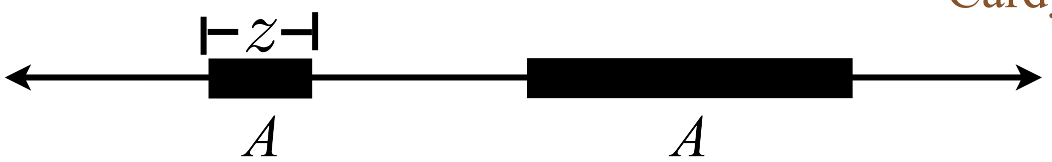
$$H_n = \frac{c}{24}(n - 1/n) = \text{ dimension of twist operator}$$

First applied in this context by
Headrick '10

Two ways to proceed:

1. Expand in z: Small-interval expansion

Headrick; Cardy, Calabrese, Tonni; Cardy



2. Expand in 1/c (c = central charge) T.H.

- Also assume: small number of low-dimension operators
- This is the class of theories that could plausibly have a (semiclassical) holographic dual.
    - In holographic CFTs, c >> 1 is the AdS radius in Planck units.
    - *Not* assuming AdS/CFT.

Outline of the large-c calculation

Virasoro blocks have a nice form at large central charge: Zamolodchikov '87

$$\mathcal{F}(\Delta, H_n, z) \approx e^{-cf(\frac{\Delta}{c}, \frac{H_n}{c}, z)}$$

From this we can evaluate the 4pt function of heavy operators to leading order in 1/c:

$$\mathrm{Tr}\ \rho_A^n \approx e^{-2cf(0, \frac{H_n}{c}, z)}$$

Comments:

- This contribution is universal (independent of CFT details)
- Valid at leading order in 1/c (but all orders in OPE!)
- Also assumed low operator multiplicities
- It is the Virasoro block for the vacuum rep, which includes the operators

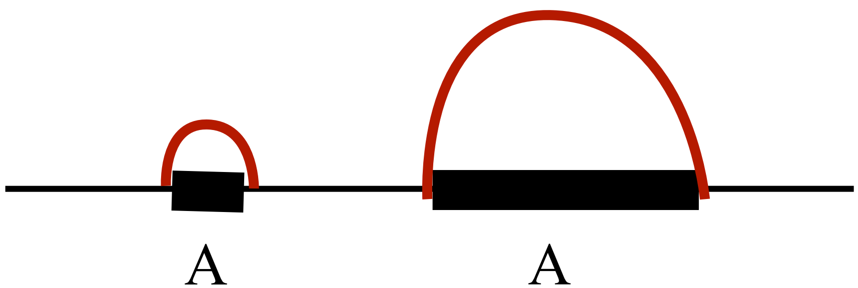
$$1\ , T\ , \partial T\ , T^2\ , T\partial T\ , \cdots$$

- Heavy correlators are exponentially dominated by exchange of operators built from the stress tensor. (Dual: 3d graviton)

The "semiclassical block" f_0 can be computed in Liouville CFT using a null-state decoupling equation. In general this can be solved numerically; in the limit n-->1, it is easy to solve analytically.

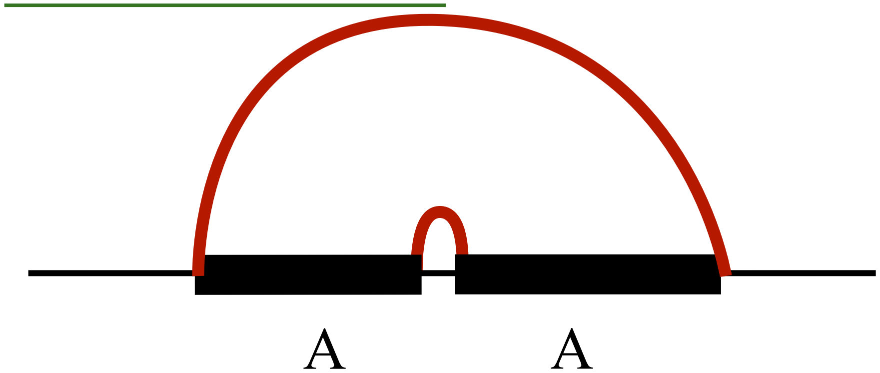
$$S_A = -\partial_n \mathrm{Tr}\ \rho_A^n|_{n=1}$$

<u>s-channel OPE:</u>



$$S_A = \frac{c}{3}\log(L_1) + \frac{c}{3}\log(L_2)$$

<u>t-channel OPE:</u>



$$S_A = \frac{c}{3}\log(L_3) + \frac{c}{3}\log(L_4)$$

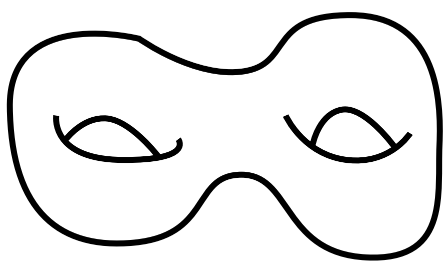
Agrees with holographic Ryu-Takayanagi formula

(assuming no other non-perturbative contributions, ie non-geometric saddles)

The replica partition functions also agree with AdS/CFT: Solving the Liouville-like equation that appears in Zamolodchikov's computation of f_0 is *identical, step by step* to constructing a 3d geometry satisfying the Einstein equations.

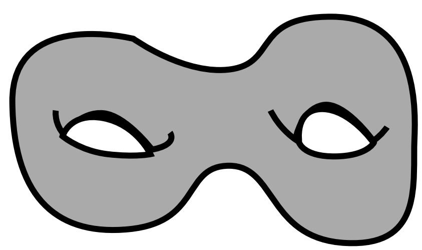
- [*quick explanation*: Zamoldchikov computes *f* by constructing a particular *SL(2,C)* connection on the Riemann surface. 3d gravity is classically equivalent to *SL(2,C)* Chern-Simons theory, and Zamoldchikov's construction can be interpreted as a smooth geometry.]

The large-c vacuum block corresponding to the replica manifold



is the Einstein action of the "filled in" 3-manifold

$$2cf_0 = S_{Einstein}\Big(\quad \Big)$$



Faulkner '13
T.H. '13

Overview

1. EE basics and some quantum information

2. Calculations in 2d CFT

3. Reconstruction of bulk geometry

- Hint #1: RT formula; MERA geometry and Hyperbolic geometry
- Hint #2: In the 2d CFT calculation, the 3d geometries pop out automatically from the CFT calculation of Renyi entropy at large c.
- Now I will work in d dimensions and describe a direct relation between fluctuations in geometry and fluctuations in entanglement entropy.

Blanco, Casini, Hung, Myers ’13
Lashkari, McDermott, Van Raamsdonk ’13
Faulkner, Guica, TH, Myers, Van Raamsdonk ’13

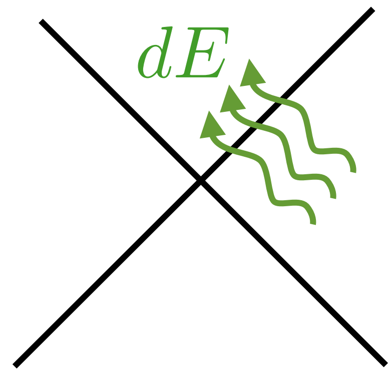
Jacobson

Jacobson '95
Guedens, Jacobson, Sarkar '11

Apply the 1st law of thermodynamics to local Rindler horizons:

$$TdS = dE$$

$$S = \text{ area}$$



This 1st law implies the Einstein equation.

- This suggests the Einstein equations are "thermodynamic" in nature
- However, microscopic definitions of T, S, E are not clear

The aim (only partly successful) is to make this precise in AdS/CFT.

- If geometry comes from entanglement, then the Einstein equations should govern fluctuations of entanglement.

Fluctuations in Entanglement Entropy

Define the "modular Hamiltonian" H by

$$\rho_A = \frac{e^{-H_A}}{\mathrm{tr}\ e^{-H_A}}$$

H is a state-dependent operator.

Varying both sides gives

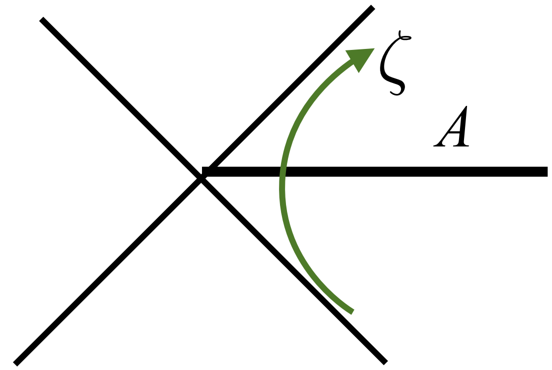
$$\delta S_A = \delta \langle H_A \rangle$$

This is the "first law" of entanglement entropy.

- In general, this is a tautology
- However, in certain contexts H_A is a conserved charge that can be defined independently; then it is useful.
- Example: Thermofield double --> ordinary 1st law

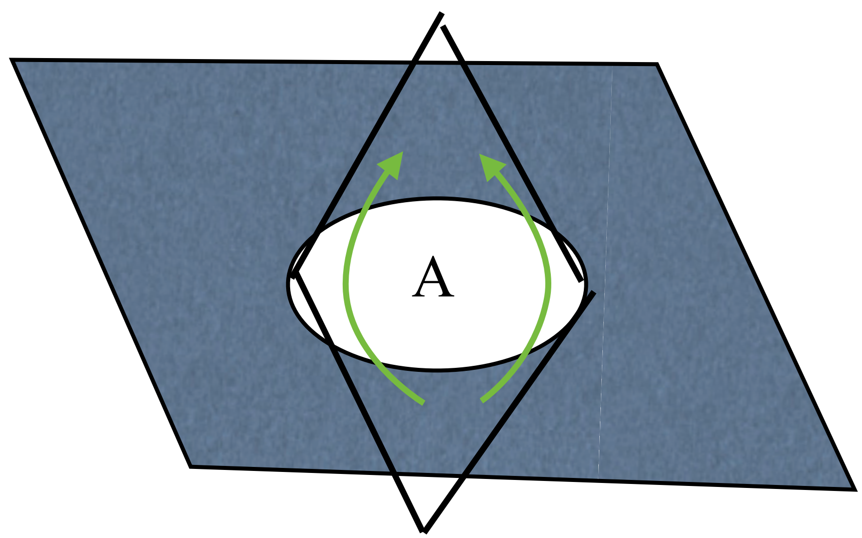
For a half-plane in Lorentz-invariant QFT,
H is the charge associated to boosts:

$$H_{groundstate} = \int d\Sigma^\mu \, T_{\mu\nu}\zeta^\nu$$



In a CFT, this can be conformally mapped to a ball-shaped region:

$$H_{groundstate} = \int d\Sigma^\mu \, T_{\mu\nu}\zeta^\nu$$
$$= \int \frac{R^2 - r^2}{2R} T_{tt}$$



Consider perturbations above the groundstate in a CFT.

The 1st law of entanglement entropy,

$$\delta S(R, \vec{x}_0) = \delta \langle H(R, \vec{x}_0) \rangle$$

gives an equation for every ball (centered at $\vec{x}_0$ with radius R)

So far this discussion is general. Now consider a holographic CFT.

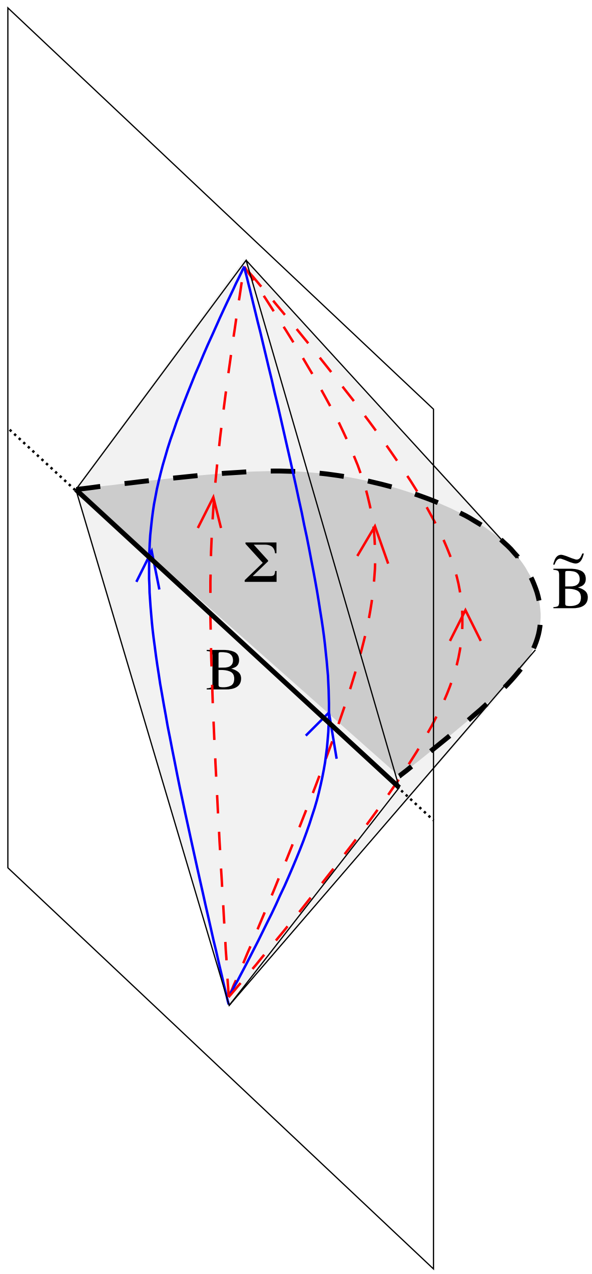
Claim:

In a holographic CFT, this infinite set of relations is (an integral transform of) the linearized Einstein equation,

$$\delta EOM(r, x) = 0$$

and similarly in an arbitrary higher-derivative theory of gravity.

A ball-shaped region in CFT is associated to an AdS-Rindler horizon:



Fluctuations in
entanglement entropy
=
Fluctuations in the area
of the AdS-Rindler horizon

On the gravity side, the 1st law

$$\delta S(R, \vec{x}_0) = \delta\langle H(R, \vec{x}_0)\rangle$$

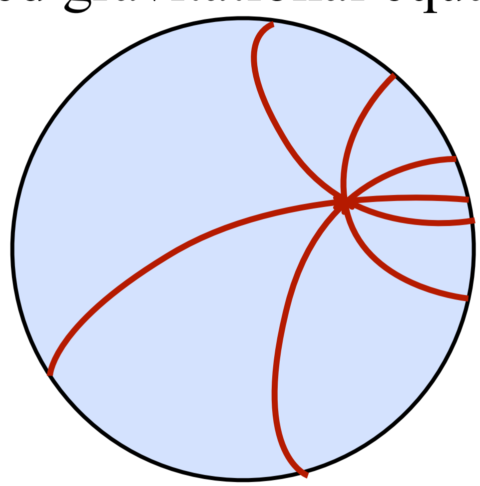
is an equation for the linearized field perturbation of the form:

$$\int_{RT surface} s(\delta g) = \int_{bdry} T(\delta g)$$

where s is entropy density and T is modular energy density.

This infinite set of integral equations (one for every R , $\vec{x}_0$) can be transformed to the linearized gravitational equations of motion.

Hyperbolic space:



“Set of all horizons through a point gives local info.”

Comparison to Jacobson

- Jacobson's 1st law
  - Local Rindler horizons
  - Full nonlinear equations
  - Microscopic definitions of *S, E, T* not clear

- 1st law of entanglement entropy
  - Global AdS-Rindler horizons
  - Every quantity involved has a precise microscopic definition in the dual
  - However, we get only the equations linearized about the vacuum.
  - The obstacle is that the "1st law" is (apparently) not useful applied to excited states.

Recap & Some Open Questions

Three hints for a deep relationship between entanglement entropy and geometry:

1. Tensor networks and the Ryu-Takayanagi formula

    - Can this be made quantitative?

2. Large-c calculations in 2d CFT automatically "produce" 3d geometries

    - How far can we push the $1/c$ expansion?

3. The 1st law of entanglement entropy allows us to reconstruct the bulk geometry in AdS/CFT, to linearized level about the vacuum

    - Nonlinear gravity?