Localization: an Overview

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Introduction

- In recent years there has been extensive progress on exact computations of partition functions and correlation functions in supersymmetric quantum field theories.
- These exact results are made possible by the technique of supersymmetric localization, reducing the infinite dimensional path integral to a finite dimensional integral.
- In this talk we will review the history and essential ideas behind localization, summarize the key computations that have been done, and survey some of the wide ranging applications to quantum field theory.

Outline

Earlier Examples of Partition Functions (1/2)

- Witten index
  - Can be defined as $I = \mathrm{Tr}(-1)^F e^{\beta H}$ for SUSY quantum mechanics. Result is an integer counting ground states, $N_b - N_f$.
  - Typically invariant under suitably nice continuous deformations of the theory $\rightarrow$ go to weak coupling limit.
  - Important example: partition function on T^d $\rightarrow$ when spectrum is discrete (eg, mass gap), counts ground states on T^{d-1} and tests SUSY breaking.
- Topological Quantum Field Theories
  - Schwarz type - action is explicitly independent of metric (eg, Chern-Simons theory)
  - Witten/Cohomological Type - exists a scalar supercharge Q such that deformations of metric are Q-exact
  - Eg, Donaldson-Witten theory, a topological twist of $\mathcal{N} = 2$ SYM, localizes onto instanton configurations.

Earlier Examples of Partition Functions (2/2)

- Instanton Partition Function
  - [Nekrasov] considered this twisted $\mathcal{N}=2$ theory on $\mathbb{R}^4$, and, using a certain combinatation of supercharges, $Q_{\epsilon_1,\epsilon_2}$,showed the computation localizes to point-like instantons at the origin of $\mathbb{R}^4$:

$$Z = \exp\left(\frac{\mathcal{F}^{inst}(a;q)}{\epsilon_1\epsilon_2} + ...\right)$$

    where $\mathcal{F}^{inst}$ is related to the Seiberg-Witten prepotential, $q=e^{2\pi i\tau}$, and a,ϵ_i are equivariant parameters.
  - One can similarly define a "vortex partition function" in $2d$ $\mathcal{N}=(2,2)$ gauge theories which count pointlike vortex configurations.
  - These are also relevant for localization on compact manifolds, as we will see.

- Examples we consider in this talk will be theories on compact curved manifolds which are not topological, following the computation of [Pestun] on S^4.
- These include partition functions on S^d, $S^d\times S^1$, and several other examples.

Basic Localization Argument

- Assume we are given a SUSY theory on a manifold M with a supercharge Q. We want to compute:

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]}$$

- Suppose one finds a functional V such that $\{Q, V\}$ is positive semi-definite and Q-invariant. Then:

$$Z_t = \int \mathcal{D}\Phi e^{-(S+t\{Q,V\})}$$

$$\Rightarrow \quad \frac{d}{dt} Z_t = \int \mathcal{D}\Phi e^{-(S+t\{Q,V\})}\{Q, V\} = \int \mathcal{D}\Phi \{Q, e^{-(S+tQV)} V\} = 0$$

- This is independent of t, and for large t the path integral only gets contributions near $QV_{bos} = 0$.
- Can also insert Q-invariant operators (lines, surfaces, defects, etc.).

Expanding on the Localization Argument (1/2)

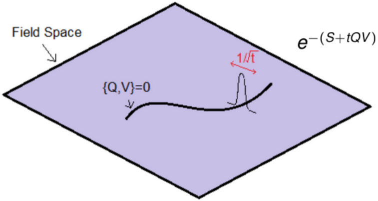
- Why should such a V exist?
- In most cases, we can take:

$$V = \int_M \sqrt{g} d^d x \sum_\Psi (Q\Psi)^\dagger \Psi$$

$$\Rightarrow QV_{bos} = \int_M \sqrt{g} d^d x \sum_\Psi (Q\Psi)^\dagger Q\Psi \Rightarrow \text{ localize to } Q\Psi = 0$$

- Q^2 is a bosonic symmetry that must annihilate V (eg, $Q^2 = 0$, $Q^2 = \mathcal{L}_v + [\Phi, \cdot]$, etc.)
- Important that Q is off-shell closed.

Expanding on the Localization Argument (2/2)



- Write fields near saddle point as $\Phi = \Phi_o + \frac{1}{t}\Phi'$

$$S + tQV = S[\Phi_o] + QV[\Phi_o]_{quad}(\Phi') + O(t^{-1/2})$$

- Classical contribution: $e^{-S[\Phi_o]}$.
- Quadratic Determinant of fluctuations: $Z_{1-loop}[\Phi_o] = \mathrm{sdet}(QV[\Phi_o]_{quad})$

$$Z = \int d\Phi_o e^{-S[\Phi_o]} Z_{1-loop}[\Phi_o]$$

Survey of Results for Partition Functions

Flat space SUSY

- A flat space theory must have a certain amount of supersymmetry in order to couple to curvature and localize.
- Most examples we consider will require 4 supercharges in flat space, eg, $\mathcal{N}=1$ in 4d, with a $U(1)$ R-symmetry:

$$\text{gauge multiplet: } A_\mu, D, \lambda, \quad \text{chiral multiplet } \phi, \psi, F$$

- The superpotential W should be quasi-homogeneous to preserve a $U(1)_R$ symmetry ($R_W = 2$).
- $\mathcal{N}=2$ in 3d and $\mathcal{N}=(2,2)$ in 2d related by dimensional reduction with minor modifications.
  - Gauge field A_μ components along extra dimensions become scalars $\rightarrow$ $A_4 = \sigma$ in 3d, also $A_3 = \eta$ in 2d.
  - SUSY Chern-Simons term available in 3d.
  - Twisted superpotential in 2d.
- Some require more SUSY ($\mathcal{N}=2$ in 4d and $\mathcal{N}=1$ in 5d) and with less ($\mathcal{N}=(2,0)$ in 2d).

Sphere Partition Functions

Motivation

- Have important connections to renormalization group flow and entanglement entropy $\rightarrow$ eg, F-theorem of [Jafferis,Klebanov,Pufu,Safdi], EE connection of [Casini,Huerta,Myers]
- Spheres are conformally flat $\rightarrow$ we can conformally map a flat space correlation function to a computation on the sphere.

History

- First computation performed on S^4 by [Pestun] for $\mathcal{N} = 2$ theories. Recently generalized to squashed S^4 by [Nosaka,Terashima].
- S^3 computation performed for round sphere by [Kapustin, BW, Yaakov], [Jafferis] ,[Hasa, Hosomichi, Lee], and to S^3_b by [HHL], [Imamura, Yokoyama].
- S^2 computation performed for $\mathcal{N} = (2, 2)$ theories by [Benini, Cremonesi] [Doroud, Gomis, Le Floch, Lee].
- Computation on S^5 for $\mathcal{N} = 1$ theories performed by [Hosomichi, Seong, Terashima],[Källén,Qiu,Zabzine] for zero-instanton sector, and later instanton contributions computed by [Kim,Kim,Lee].

Finding the Superconformal Fixed Point

- S^d is conformally flat $\Rightarrow$ canonical way to couple a CFT_d to curvature. For SCFT, one can preserve full superconformal group. For a non-conformal theory, there is no canonical couping.
- Recall the choice of $U(1)_R$ symmetry of a theory is non-unique, and can be shifted by a $U(1)$ flavor symmetry:

$$R = R_o + \sum_a c_a F_a$$

- The coupling to curvature of the UV theory, and resulting localized partition function, will depend on the choice of R-symmetry of the theory (ie, on the c_a).
- Claim: when this trial R-symmetry agrees with the superconformal R-symmetry of the IR fixed point, the sphere partition function agrees with that of the conformally mapped IR superconformal fixed point.
- [Jafferis] argued that the correct superconformal R-symmetry can be found by extremizing $|Z|$ as a function of the trial R-charge. Later clarified by [Closset, Dumitrescu, Festuccia , Komargodski, Seiberg].

Sample Computation - S_b^3 (1/4)

- As an example, consider the localization of $\mathcal{N}=2$ gauge theories on S_b^3.
- Geometry: unit sphere in $\mathbb{R}^4=\mathbb{C}^2$ with metric (writing $z_1=\cos\chi e^{i\phi}$, $z_2=\sin\chi e^{i\theta}$):

$$ds^2=r^2(f(\chi)^2d\chi^2+b^2\sin^2\chi d\theta^2+b^{-2}\cos^2\chi d\phi^2)$$

  where $f(\chi)$ approaches b, b^{-1} at endpoints (shown by [Alday, Martelli, Richmond, Sparks] that any such $f(\chi)$ will give the same answer).
- Flat space SUSY action and transformations are modified as (eg, for a chiral multiplet of R-charge R):

$$Q\phi=i\bar{\epsilon}\psi,\quad QF=\epsilon(-\gamma^\mu D_\mu\psi+\sigma\psi+\lambda\phi)+\frac{i}{2rf(\chi)}(2R-1)\epsilon\psi$$

$$Q\psi=-\gamma^\mu\epsilon D_\mu\phi-\epsilon\sigma\phi+i\bar{\epsilon}F-\frac{iR}{rf(\chi)}\epsilon\phi,\qquad \text{where:}\quad D_\mu\epsilon=\frac{i}{2rf(\chi)}\gamma_\mu\epsilon$$

$$S_{kin}^{chiral}=\int d^3x\sqrt{g}\Big(D_\mu\bar{\phi}D^\mu\phi+\bar{\phi}\sigma^2\phi+\frac{i(2R-1)}{rf(\chi)}\bar{\phi}\sigma\phi+\frac{R(2-R)}{(rf(\chi))^2}\bar{\phi}\phi+$$
$$+i\bar{\phi}D\phi+\bar{F}F-i\bar{\psi}\gamma^\mu D_\mu\psi-\frac{2R-1}{2rf(\chi)}\bar{\psi}\psi+i\bar{\psi}\lambda\phi-i\bar{\phi}\bar{\lambda}\psi\Big)$$

Sample Computation - S_b^3 (2/4)

- One computes:

$$Q^2 = i\mathcal{L}_v + [i\sigma - v^\mu A_\mu, \cdot] + \frac{1}{2r}(b+b^{-1})R$$

$$S_{kin}^{chiral} = Q\frac{1}{2}\int d^3x\sqrt{g}((Q\psi)^\dagger\psi + (Q\bar{\psi})^\dagger\bar{\psi}), \quad S_{YM} = Q\frac{1}{2}\int d^3x\sqrt{g}((Q\lambda)^\dagger\lambda + (Q\bar{\lambda})^\dagger\bar{\lambda})$$

- Localize onto Φ_o solving $QV[\Phi_o] = 0$, which gives here (assuming real D):

$$\text{chiral: all fields vanish,} \quad \text{vector:} \quad F_{\mu\nu} = 0, \quad D_\mu\sigma = 0, \quad D = -\frac{\sigma}{irf(\chi)}$$

- Classical contribution:

$$S_{CS} = i\mathrm{Tr}_{CS}\int\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A + \sqrt{g}d^3x(2\sigma D - \bar{\lambda}\lambda)\right), \qquad W_R = \mathrm{tr}_R \exp\left(\oint_{\gamma^\pm}(A + \sigma d|x|)\right)$$

$$\Rightarrow e^{-S[\Phi_o]} = e^{-i\pi Tr_{CS}\sigma^2}\mathrm{tr}_R e^{2\pi b^{\pm 1}\sigma}$$

Sample Computation - S_b^3 (3/4)

- One-loop determinant for chiral operator given by, eg:

$$Z_{1-loop}^{chiral} = \prod_{\rho \in R} \frac{\det(\mathcal{D}^\rho_{fer})}{\det(\mathcal{D}^\rho_{bos})}$$

- Many bosonic and fermionic modes cancel due to supersymmetry $\rightarrow$ can see directly using (equivariant) Atiyah-Singer index theorem ([Drukker, Okuda, Passerini]). Final result for general b ($Q = b + b^{-1}$):

$$Z_{1-loop}^{chi} = \prod_{m,n \geq 0} \frac{mb + nb^{-1} + \frac{Q}{2} + i\sigma + \frac{Q}{2}(1-R)}{mb + nb^{-1} + \frac{Q}{2} - i\sigma - \frac{Q}{2}(1-R)} = \Gamma_h(\frac{iQR}{2} + \sigma)$$

- Similarly, for the vector multiplet one computes:

$$Z_{1-loop}^{vec} = \prod_{\alpha \in Ad^+} (2\sinh \pi b\alpha(\sigma))(2\sinh \pi b^{-1}\alpha(\sigma))$$

Sample Computation - S_b^3 (4/4)

- Putting this together, the partition function is given by:

$$Z = \int_{\mathfrak{t}} d\sigma e^{-\pi i Tr_{CS}\sigma^2} \prod_{\rho \in R} \Gamma_h\left(\frac{iQ}{2}R + \rho(\sigma)\right) \prod_{\alpha \in Ad(g)^+} (2\sinh \pi b\alpha(\sigma))(2\sinh \pi b^{-1}\alpha(\sigma))$$

- In addition to scalars in dynamical gauge multiplets, which are integrated over, one can couple to classical, background scalars to global symmetries of the theory. Similar to real mass terms in flat space.
- To summarize, S_b^3 partition function depends on the following data:
  - Geometry of S_b^3, but only through parameter b.
  - Field content and symmetries of the action (eg, gauge representation, R-charges) $\rightarrow$ $D-$ terms are irrelevant, and F-terms only enter by restricting the symmetries.
  - Background vector multiplet scalars (real masses).
  - Chern-Simons terms.

Results

- The Z_{S^d} computation for gauge theories on spheres localizes onto constant values of a scalar in the gauge multiplet. Thus is it given by an integral over the "Coulomb branch" (but see notes):

$$Z_{S^d} = \int_{\mathfrak{h}} d^{r_G}\phi \ \det_R Z^{matter}_{1-loop}(\phi) \ \det_{Ad} Z^{vector}_{1-loop}(\phi) \ Z_{classial}(\phi)$$

where:

S^d	Z^{matter}_{1-loop}	Z^{vector}_{1-loop}	$Z_{classical}$	Notes
S^2	$\frac{\Gamma(\frac{q}{2}-ir\sigma-\frac{m}{2})}{\Gamma(1-\frac{q}{2}+ir\sigma-\frac{m}{2})}$	$\frac{\vert m\vert}{2}+i\sigma$	$e^{8\pi i Re\tilde{W}(\frac{\sigma}{r}+\frac{im}{2r})}$	saddle pts also have flux m for A_μ
S^3_b	$\Gamma_h(\frac{iQ}{2}R+\sigma)$	$4\sinh\pi b\sigma$ $\times\sinh\pi b^{-1}\sigma$	$e^{i\pi Tr_{CS}\sigma^2}$	
S^4	$G(1\pm ir\phi)^{-1}$	$G(1\pm ir\phi)$	$e^{-\frac{8\pi^2 r^2}{g_{YM}^2}tr\phi^2}$	also $\vert Z_{inst}(r,r^{-1})\vert^2$
S^5	$\cos(i\pi\phi)e^{-f(\frac{1}{2}\pm i\phi)}$	$\sin(i\pi\phi)e^{f(i\phi/2)}$	$e^{-\frac{8\pi^3 r}{g_{YM}^2}tr\phi^2}$	$\mathbb{CP}^2$ instantons

- $\mathcal{N}=4$ in $4d$ is Gaussian $\rightarrow$ confirms conjecture of [Erickson, Semenoff, Zarembo],[Drukker, Gross]

Higgs Branch Localization and Factorization (1/2)

- Alternative "Higgs" localization for S^2 partition function in certain theories by adding an additional term [Benini, Cremonesi] [Doroud, Gomis, Le Floch, Lee].:

$$QV_{higgs} \sim QTr(\epsilon^\dagger \lambda - \lambda^\dagger \epsilon)(\phi\phi^\dagger - \chi \mathbf{1}) \Rightarrow \text{ imposes } (\sigma + M)\phi = \phi\phi^\dagger - \chi\mathbf{1} = 0$$

- For $\chi \to \pm\infty$ (depending on matter content), computation localizes onto pointlike vortex configurations on north/south pole:

$$Z = \sum_{vacua} e^{-4\pi\xi \sum_j \sigma_j} Z_{1-loop} Z_{vortex} Z_{\bar{vortex}}$$

Higgs Branch Localization and Factorization (2/2)

- This was motivated by a related result in three dimensions [Pasquetti], where the S^3 partition function factorizes as:

$$\sum_{vacua} Z_{1-loop} Z^K_{vortex}(q = e^{2\pi i b}) Z^K_{\overline{vortex}}(\tilde{q} = e^{2\pi i b^2})$$

  where Z^K_{vortex} is the K-theoretic vortex partition function, reducing to the ordinary VPF as $q \to 1$.
- Later developed and extended to $S^2 \times S^1$ by [Beem,Pasquetti,Dimofte] - both partition functions arise from gluing “holomorphic blocks”, homeomorphic to $\mathbb{R}^2 \times S^1$, in different ways.
- Recently this factorization derived directly by localization [Fujitsuka, Honda, Yoshida] [Benini, Peelaers]. One adds an additional Q-exact term similar to the S^2 case and localizes to vortex string configurations.
- Not known if factorization extends to higher dimensions.

Supersymmetric Indices

- Superconformal index of $d+1$ dimensional theories first studied by [Kinney, Maldacena, Minwalla, Raju], [Bhattacharyya, Bhattacharyya, Minwalla, Raju], defined as a kind of Witten index over states on S^d:

$$\mathcal{I}(\mu_i) = \mathrm{Tr}_{S^d}(-1)^F \prod_i {\mu_i}^{Q_i} e^{\beta\{Q,S\}}$$

- Constructed to only recieve contributions from short multiplets that cannot pair up $\Rightarrow$ insensitive to continuous deformations of the theory.
- Equivalently, counts local BPS operators in flat space.
- This can be given the interpretation of a partition function on $S^d \times S^1$, where one couples flat background connections along S^1 to the global and R-symmetries.
- Later applied to non-conformal $4d$ theories by [Romelsberger]; to make this argument precise one should use localization.

Supersymmetric Indices

- Path integral localizes onto flat connections with holonomy along the S^1, so partition function is given by an integral over these holonomies (again with some exceptions):

$$I_{S^d \times S^1} = \int_T \prod_{i=1}^{r_G} \frac{dz_i}{z_i} \prod_{\rho \in R} Z_{1-loop}^{gauge}(z^\rho) \prod_{\alpha \in Ad(G)} Z_{1-loop}^{matter}(z) Z_{classical}(z)$$

- For example, for $S^3 \times S^1$, one has [Romelsberger]:

$$I_{S^3 \times S^1}(p, q, \mu_a) = \mathrm{tr}(-1)^F p^{j_1 + j_2 - \frac{R}{2}} q^{j_1 - j_2 - \frac{R}{2}} \prod_a {\mu_a}^{F_a}$$

$$Z_{1-loop}^{chiral}(z) = \Gamma_e((pq)^{R/2} z), \quad Z_{1-loop}^{vector}(Z) = \Gamma_e(z; p, q)^{-1}$$

where:

$$\Gamma_e(z; p, q) = \prod_{j,k \geq 0} \frac{1 - p^{j+1} q^{k+1} z^{-1}}{1 - p^j q^k z}$$

$S^d \times S^1$ Index in Various Dimensions

- $d = 1$
  - Computed for $\mathcal{N} = (2,2)$ and $\mathcal{N} = (0,2)$ theories by localization by [Benini, Eager, Hori, Tachikawa] (RR) and using representation theory by [Gadde, Gukov]
  - This is just the elliptic genus, computed long ago in many examples, but only recently for gauge theories using localization.
- $d = 2$
  - First computed by localization by [Kim] for ABJM theory $S^2 \times S^1$, later generalized to arbitrary $\mathcal{N} = 2$ theories by [Imamura, Yokoyama]
  - The computation also includes a sum over monopoles configurations, as on S^2. These correspond to local monopole operators that exist in gauge theories on $\mathbb{R}^3$.
- Computed for $d = 4$ by [Kim,Kim,Lee],[Terashima]; one also has contributions from point like instantons sitting at the poles of S^4, wrapping the S^1.
- Additionally one may think of the S^5 partition function of $\mathcal{N} = 2$ SYM as an index of 6d $(2,0)$ theory; very non-trivial that one obtains an expansion in fugacities with integer coefficients [Kim,Kim].

Partition Functions on Other Manifolds

- [Benini,Nishioka,Yamazaki] derived the partition function on $S^3/\mathbb{Z}_p \times S^1$ by perfmorming a projection on the $S^3 \times S^1$ index.
- [Alday,Fluder,Sparks] localized the $S^3/\mathbb{Z}_p$ partition function.
- [Källén],[Ohta,Yoshida] localized Chern-Simons-matter theories on arbitrary Seifert manifolds (S^1 fibration of Riemann surface).
- Also recent examples of localization on manifolds with boundaries, namely hemispheres for $d = 2$ [Honda,Okuda], [Hori,Romo], [Sugishita,Terashima], and $d = 3$.
- [Closset,Shamir] initiated study of partition function on $S^2 \times T^2$ by computing chiral multiplet contribution.

Applications

Holography

- AdS/CFT maps strong coupling on one side to weak coupling on the other $\Rightarrow$ typically hard to test. Now have exact results on the CFT side for all coupling, many non-trivial tests can be done.
- One can typically solve these matrix models in the 't Hooft limit by saddle point approximation:

$$Z \sim \int d^N\lambda e^{-F(\lambda)} \rightarrow \text{ take } \rho(\lambda) = \frac{1}{N}\sum_i \delta(\lambda - \lambda_i), \text{ extremize } F[\rho]$$

- Basic example: one can compute the free energy using Z_{S^d} and match with supergravity prediction, $Z_{CFT}(S^d) \approx e^{-I_{SUGRA}(AdS_{d+1})}$ Some examples:

d	Theory	F	Ref
3	ABJM	$\frac{4\pi^3\sqrt{2}}{3}\frac{N^{3/2}}{k^{3/2}}$	[Drukker, Marino, Putrov]
4	$\mathcal{N}=4$ SYM	$\frac{N^2}{2}\log\lambda$	[Russo,Zarembo]
5	$\mathcal{N}=1 USp(2N)+N_f hypers$	$-\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}}$	[Jafferis,Pufu]
$5\rightarrow 6$	$\mathcal{N}=2$ $SYM \sim 6d(2,0)$	$-\frac{27}{512}\frac{{g_{YM}}^2 N^3}{\pi r}$	[Kim,Kim] [Källén, Minahan, Nedelin, Zabzine]

- One can also compare Wilson loop correlation functions to minimal area surface computation on SUGRA side.

Compactifications of $6D$ $(2,0)$ Theory

- Take $6d$ $(2,0)$ superconformal theory on $M_d \times M_{6-d}$ - partially twisting along M_{6-d} one may preserve some SUSY on M_d. Then $Z_{M_d} = Z_{M_{6-d}}$ gives non-trivial correspondence.

M_{6-d}	$T[M_{6-d}]$	M_d	$T[M_d]$	Ref
$\Sigma_{g,h}$	Class $\mathcal{S}$	S^4	Liouville/Toda CFT	[Alday, Gaiotto, Tachikawa], [Wyllard]
$\Sigma_{g,h}$	Class $\mathcal{S}$ (eg, $SU(N)$ quivers)	$S^3 \times S^1$	$2d$ TQFT (q-YM, etc.)	[Gadde, Rastelli, Razamat, Yan]
M_3	Class $\mathcal{R}$ (eg, abelian CSM)	$S^3, S^2 \times S^1$	$SL(2,\mathbb{C})$ Chern-Simons	[Dimofte,Gaitto,Gukov]
M_4	Class $\mathcal{H}$ (eg, heterotic)	T^2	Vafa-Witten Twist of $\mathcal{N}=4$ SYM	[Gadde,Gukov,Putrov]

Renormalization Group Flow and Entanglement Entropy

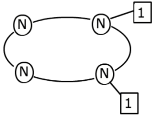
- Studies of Z_{S^3} in several examples led [Jafferis,Klebanov,Pufu,Safdi] to propose the F-theorem, that F is decreasing along RG flow for any (even non-SUSY) theory.
- Tested in many SUSY and non-SUSY examples [Klebanov,Pufu,Safdi].
- Generalizes $c-$ and $a-$theorems, but not related to anomalies. Shown to fit into general picture of entanglement entropy c-theorems [Myers,Sinha] by conformally mapping S^d to $H^{d-1} \times S^1$ [Casini,Huerta,Myers], and F-theorem proven by [Casini,Huerta].
- Supersymmetric version of Reyni entropy can also be computed by localization [Nishioka,Yaakov].

$2d$ and $3d$ Mirror Symmetry

- Exact results allow one to test dualities, where one or both sides may be strongly coupled, eg:
- $2d$ mirror symmetry- One may used S^2 partition function to compute exact Kahler potential for CY_3 sigma models and compute Gromov-Witten invariants :

$$e^{-K} = Z_{S^2}$$

- Conjectured by [Jockers, Kumar,Lapan, Morrison, Romo] and shown by [Gomis,Lee] by find squashing of two sphere which does not affect partition function (Q-exact) and reduces to tt^* geometry [Cecotti, Vafa].
- $3d$ mirror symmetry [Intriligator,Seiberg],[de Boer, Hori, Ooguri, Oz] $\rightarrow$ relates, eg, $U(N)$ quivers theories:



- Tested by [Kapustin, BW, Yaakov], [Krattenthaler, Spiridonov, Vartanov], [Dey],....

Seiberg-like Duality

Seiberg-like dualities:

- $4d\ \mathcal{N}=1$ Seiberg duality. Eg:

$$SU(N_c)+N_f \text{ hypers } \leftrightarrow SU(N_f-N_c)+N_f \text{ hypers } \ +W=Mq\tilde{q}$$

Tested on $S^3\times S^1$ by [Romelsberger],[Dolan,Osborn] based on identities of [Rains]:

$$Z_{N_f,N_c}(\mu_a)=\frac{(p,p)^{N_c-1}(q,q)^{N_c-1}}{N_c!}\int\prod_i\frac{dz_i}{z_i}\frac{\prod_i\prod_a\Gamma_e((pq)^{R/2}\mu_a z_i^{\pm 1};p,q)}{\prod_{i<j}\Gamma_e(z_i^{\pm 1}z_j^{\mp 1};p,q)}$$

- $3d\ \mathcal{N}=2$ Aharony duality, involving monopole-creating defect operators.

$$U(N_c)+N_f \text{ hypers } \leftrightarrow U(N_f-N_c)+N_f \text{ hypers + mesons + singlets } V_\pm$$

$$\text{with } W=Mq\tilde{q}+V_\pm\tilde{V}_\mp$$

Tested on S^3 by [BW,Yaakov],[Benini,Closset,Cremonesi] based on identities of [van de Bult]. Also related Chern-Simons matter dualities.

- $2d\ \mathcal{N}=(2,2)$ Hori-Tong duality. Eg, $SU(N_c)+N_f$ fundamental chirals $\leftrightarrow SU(N_f-N_c)+N_f$ fundamental chirals
Tested on S^2 by [Doroud,Gomis,Le Floch,Lee]

Dimensional Reduction and Duality (1/3)

- The apparent similarity of some of these dualities suggests they may be related, but not by naive dimensional reduction, but there are subtleties [Aharony, Seiberg, Razamat, BW].
- This reduction can be performed at the level of supersymmetric partition functions [Gadde, Yan],[Dolan, Spiridonov, Vartanov], [Imamura]:

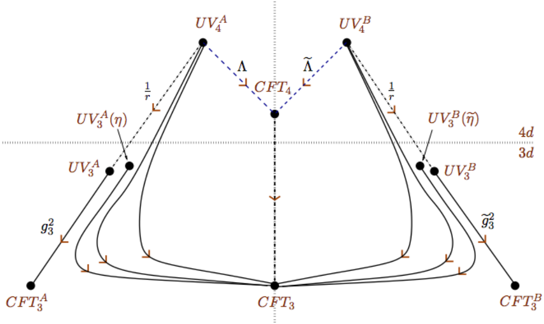
$$\lim_{r\to 0} I_{S^d\times S^1}(\mu_a = e^{rm_a}) = f(r) Z_{S^d}(m_a)$$

- Thus if $I^A_{S^3\times S^1}(\mu_a) = I^B_{S^3\times S^1}(\mu_a)$, by reducing we find $Z^{A_{red}}_{S^3}(m_a) = Z^{B_{red}}_{S^3}(m_a)$. This suggests that dualities might be preserved by naive dimensional reduction, but this is not true $\rightarrow$ one must be careful with anomalies.

Dimensional Reduction and Duality (2/3)

- Operations of flowing to the IR fixed point and dimensionally reducing do not commute.
- However, by carefully studying the low energy on $\mathbb{R}^3 \times S^1$, we find that the reduction of the 4d IR fixed point can be reached in 3d if we add a certain superpotential.

$W = \eta Y \rightarrow$



- This superpotential exactly breaks the symmetries which don't match in Z_{S^3}, and so is consistent with the partition function results.

Dimensional Reduction and Duality (3/3)

- One can perform further flows to recover known dualities (eg, Aharony duality), and new ones, eg, a 3d version of $SU(N)$ Seiberg duality where:

$$SU(N_f) + N_f \text{ flavors} \leftrightarrow$$

$$U(N_f - N_f) + N_f \text{ flavors} + \text{chirals: } Y \text{ (singlet) and } \; b, \tilde{b} \; U(1)$$

$$\text{with } W = Mq\tilde{q} + b\tilde{b}Y$$

Global Aspects of 4d Gauge theories

- [Aharony,Seiberg,Tachikawa] recently described a "discrete theta angle" one can introduce in a 4d gauge theory with G non-simply connected, eg, for $G = SU(N)/\mathbb{Z}_N$):

$$Z(s \in \mathbb{Z}_N) = \sum_{P \to G-bundle} e^{\pi i s w_2(P)/N} Z_P$$

- $S^3 \times S^1$ and S^4 partition functions are only sensitive to Lie algebra, however, $S^3/\mathbb{Z}_r \times S^1$ supports non-trivial G bundles.
- In [Razamat, BW], we compared lens space indices for $\mathcal{N} = 1$ Seiberg and $\mathcal{N} = 4$ S dualities, eg:

$$Z_{Spin(N_c)_{N_f}} = \tilde{Z}_{SO_-(N_f+4-N_c)_{N_f}},$$

$$Z_{SO_+(N_c)_{N_f}} = \tilde{Z}_{SO_+(N_f+4-N_c)_{N_f}},$$

$$Z_{SO_-(N_c)_{N_f}} = \tilde{Z}_{Spin(N_f+4-N_c)_{N_f}}$$

Summary

- We have seen that exact results are available on a wide variety of manifolds in many dimensions.
- These have applications to holography, supersymmetric dualities, and have taught us about renormalization group flow in general QFTs
- Future Directions:
  - Understanding more deeply and extending factorization and the Coulomb/Higgs localizations. More generally, localizing in different ways may give non-trivial correspondences.
  - Localization onto a submanifold $\rightarrow$ relating QFTs in different dimensions (eg, [Giombi,Pestun]).
  - Localize on more of the manifolds that have been studied by [Seiberg,Festuccia],[Closset, Dumitrescu, Festuccia, Komargodski].