Twistor and ambitwistor string theories

From twistor strings to quantum gravity?

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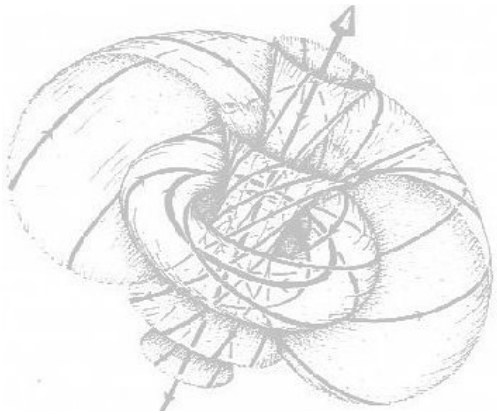
Quantum Gravity at KITP

With David Skinner. arxiv:1311.2564, and collaborations with Tim Adamo, Eduardo Casali, Yvonne Geyer, Arthur Lipstein, Ricardo Monteiro & Kai Roehrig, Piotr Tourkine, 1312.3828, 1404.6219, 1405.5122, 1406.1462, etc..

[Cf. also Cachazo, He, Yuan arxiv:1306.2962, 1306.6575, 1307.2199, 1309.0885, 1412.3479]

Ambitwistors

Ambitwistor spaces: spaces of complex null geodesics.



- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991] Baston & M. [1987] .

Ambitwistor Strings:

- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 1311.2564]
- Models for Einstein-YM, DBI, BI, NLS, etc. CGMMRS 150?.?.
- Related to $\mathscr{I}$, null geodesic scattering and the BMS group
- New form of maximal supergravity loop integrand.

Provide string theories at $\alpha' = 0$ for field theory amplitudes.

Chiral bosonic strings at $\alpha' = 0$

Bosonic ambitwistor string action:

- Σ Riemann surface, coordinate $\sigma \in \mathbb{C}$
- Complexify space-time (M, g), coords $X \in \mathbb{C}^d$, g hol.
- $(X, P) : \Sigma \to T^*M, \quad P \in K$, holomorphic 1-forms on Σ.

$$S_B = \int_\Sigma P_\mu \bar{\partial} X^\mu - e\, P^2/2\,.$$

Underlying geometry:

- e enforces $P^2 = 0$,
- P^2 generates gauge freedom: $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$.

So target is

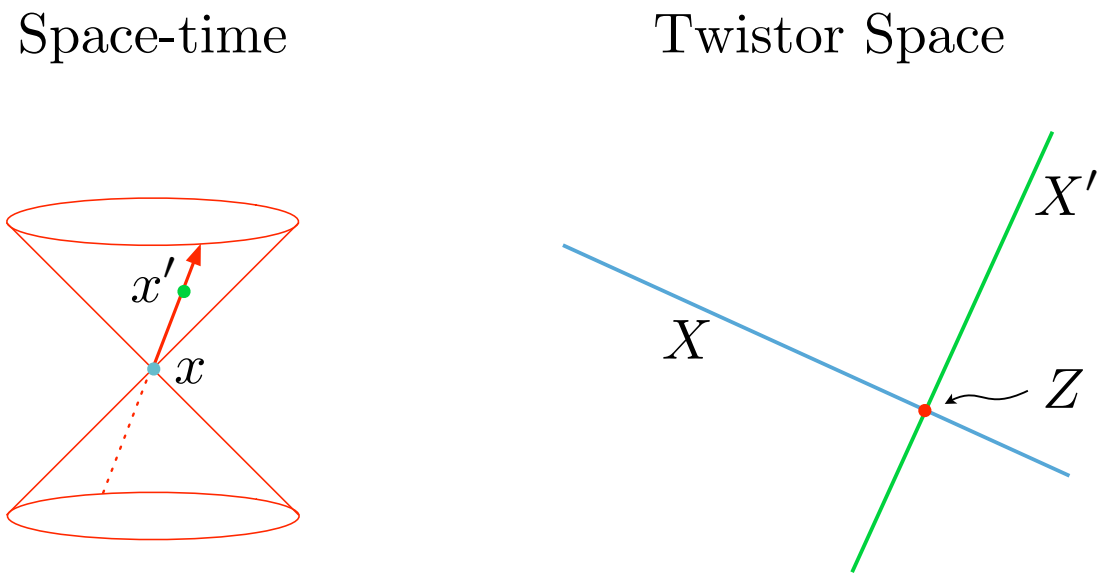
$$\mathbb{A} = T^*M|_{P^2=0}/\{\text{gauge}\}.$$

This is *Ambitwistor space*, space of complexified light rays.

Space of light rays as primary geometric arena

Ambitwistor space $\mathbb{A}$ is space of complexified light rays.

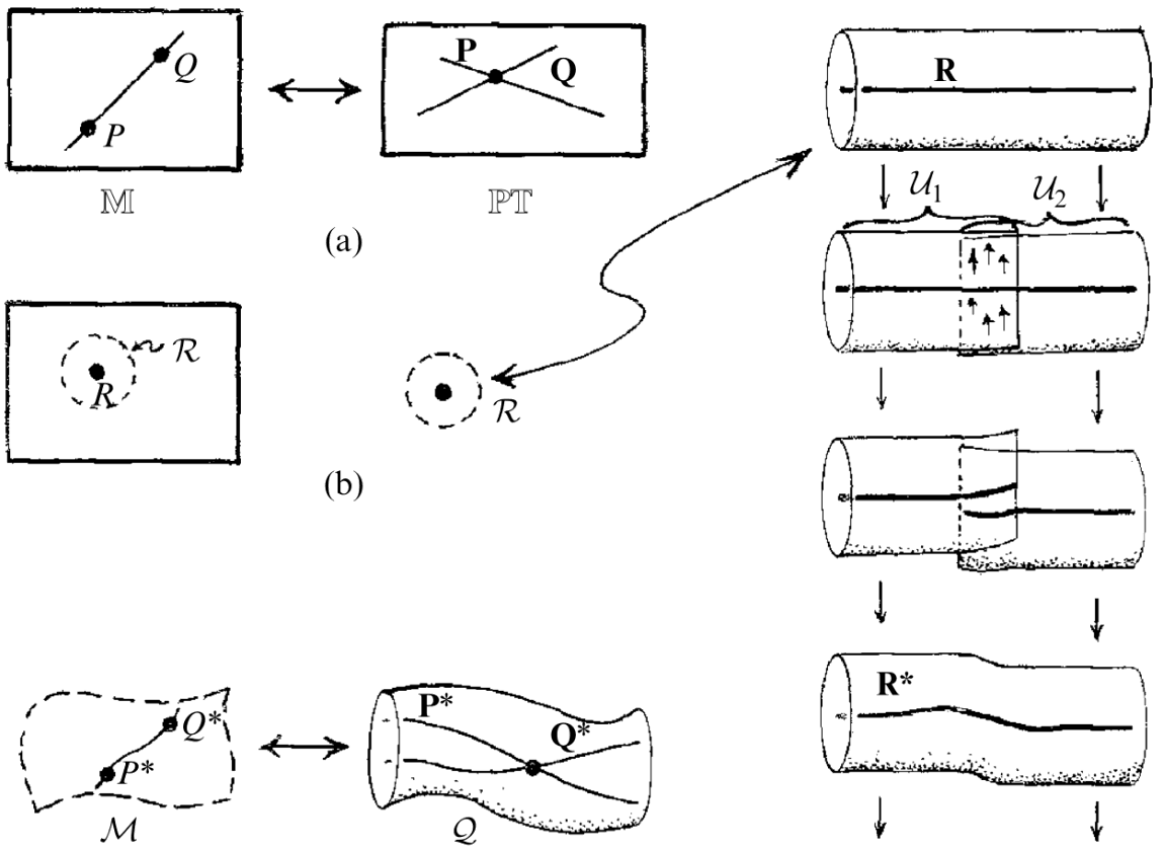
- Light rays primary, events determined by lightcones $X \subset \mathbb{A}$ of light rays incident with x.
- Space-time $M =$ space of such $X \subset \mathbb{A}$.



Space-time geometry is encoded in complex structure of $\mathbb{A}$.

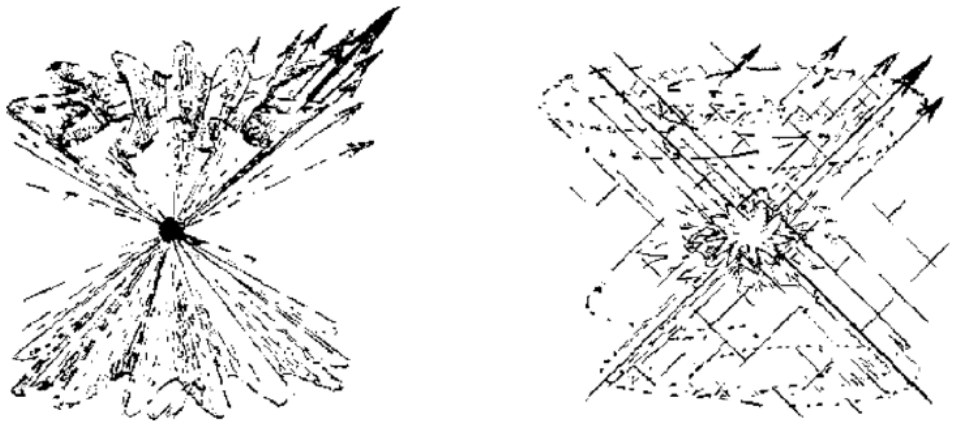
Theorem (LeBrun 1983 following Penrose 1976)

Complex structure of $\mathbb{A}$ *determines M and conformal metric g. Correspondence is stable under deformations of the complex structure of* $P\mathbb{A}$ *that preserve* symplectic potential $\theta = p_\mu \mathrm{d}x^\mu$.



Preserving $\theta \Rightarrow$ gluing is canonical, generated by Hamiltonians.

Motivation from quantum gravity



- **Normal:** fix background space-time manifold and quantize metric components $\rightsquigarrow$ fuzzy lightcone, well-defined events.
- **Alternative:** fix $\mathbb{A}$ as geometric background and quantize X to give fuzzy events but well-defined light-rays.

Emphasis on complex geometry is more quantum $\leftrightarrow$ $\mathbb{C}$ numbers of quantum mechanics.

Amplitudes from ambitwistor strings

Quantize bosonic ambitwistor string:

- $(X, P) : \Sigma \to T^*M$,

$$S_B = \int_\Sigma P_\mu(\bar\partial + \tilde e \partial) X^\mu - e\, P^2/2\,.$$

- Gauge fix $\tilde e = e = 0$, $\rightsquigarrow$ ghosts & BRST Q
- Introduce vertex operators $V_i \leftrightarrow$ field perturbations.

Amplitudes are computed as correlators of vertex ops

$$\mathscr{M}(1, \ldots, n) = \langle V_1 \ldots V_n \rangle$$

For gravity add type II worldsheet susy $S_{\Psi_1} + S_{\Psi_2}$ where

$$S_\Psi = \int_\Sigma \Psi_\mu \bar\partial \Psi^\mu + \chi P \cdot \Psi\,.$$

From deformations of $\mathbb{A}$ to the scattering equations

Gravitons $\leftrightarrow$ vertex operators $V_i =$ def'm of action $\delta S = \int_\Sigma \delta\theta$.

- θ determines complex structure on $P\mathbb{A}$ via $\theta \wedge d\theta^{d-2}$. So:
- Deformations of complex structure $\leftrightarrow [\delta\theta] \in H^1_{\bar\partial}(P\mathbb{A}, L)$.
  For gluing given by a Hamiltonian

$$\delta\theta = \bar\partial h$$

Proposition

For perturbation $\delta g_{\mu\nu} = \mathrm{e}^{ik\cdot x}\epsilon_\mu\epsilon_\nu$ of flat space-time

$$h = \frac{\mathrm{e}^{ik\cdot x}(\epsilon\cdot P)^2}{k\cdot P}\,, \qquad \delta\theta = \bar\partial h = \bar\delta(k\cdot P)\mathrm{e}^{ik\cdot X}(\epsilon\cdot P)^2\,.$$

Ambitwistor repn $\Rightarrow \bar\delta(k\cdot P) \Rightarrow$ scattering equs.

Proposition

CHY formulae for massless tree amplitudes e.g. YM & gravity arise from appropriate choices of worldsheet matter.

The scattering equations

Take n null momenta $k_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, $k_i^2 = 0$, $\sum_i k_i = 0$,

- define $P : \mathbb{CP}^1 \to \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^{n} \frac{k_i}{\sigma - \sigma_i}\,, \qquad \sigma, \sigma_i \in \mathbb{CP}^1\,.$$

- Solve for $\sigma_i \in \mathbb{CP}^1$ with the *n scattering equations*

$$k_i \cdot P(\sigma_i) = \mathrm{Res}_{\sigma_i} P(\sigma) \cdot P(\sigma) = \sum_{j=1}^{n} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,.$$

$\Rightarrow\ P(\sigma) \cdot P(\sigma) = 0\ \forall \sigma$.

- For Mobius invariance $\Rightarrow P \in \mathbb{C}^d \otimes K$, $K = \Omega^{1,0}\mathbb{CP}^1$
- There are $(n-3)!$ solutions.

Arise in large α' strings [Gross-Mende 1988] & twistor-strings [Witten 2004].

Amplitude formulae for massless theories.

Theorem (Cachazo, He, Yuan 2013,2014)

Tree-level massless amplitudes in d-dims are integrals/sums

$$\mathcal{M}(1,\ldots,n) = \delta^d\left(\sum_i k_i\right)\int_{\mathbb{CP}^{1\,n}} \frac{I^l(\epsilon_i^l,k_j)I^r(\epsilon_i^r,k_j)}{\mathrm{Vol}\ \mathrm{SL}(2,\mathbb{C})}\prod_i{}'\bar{\delta}(k_i{\cdot}P(\sigma_i))d\sigma_i$$

where $I^l(\epsilon_i^l,k_i,\sigma_i)$ and $I^r(\epsilon_i^r,k_i,\sigma_i)$ depend on the theory.

- polarizations ϵ_i^l for spin 1, $\epsilon_i^l\otimes\epsilon_i^r$ for spin-2 ($k_i\cdot\epsilon_i=0\ldots$).
- Introduce skew $2n\times 2n$ matrices $M=\begin{pmatrix} A & C \\ -C^t & B\end{pmatrix}$,

$$A_{ij}=\frac{k_i\cdot k_j}{\sigma_i-\sigma_j},,\quad B_{ij}=\frac{\epsilon_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\quad C_{ij}=\frac{k_i\cdot\epsilon_j}{\sigma_i-\sigma_j},\quad \text{for } i\neq j$$

  and $A_{ii}=B_{ii}=0$, $C_{ii}=\epsilon_i\cdot P(\sigma_i)$.
- For YM, $I^l=Pf'(M)$, $I^r=\prod_i\frac{1}{\sigma_i-\sigma_{i-1}}$.
- For GR $I^l=Pf'(M^l)$, $I^r=Pf'(M^r)$, & many more.

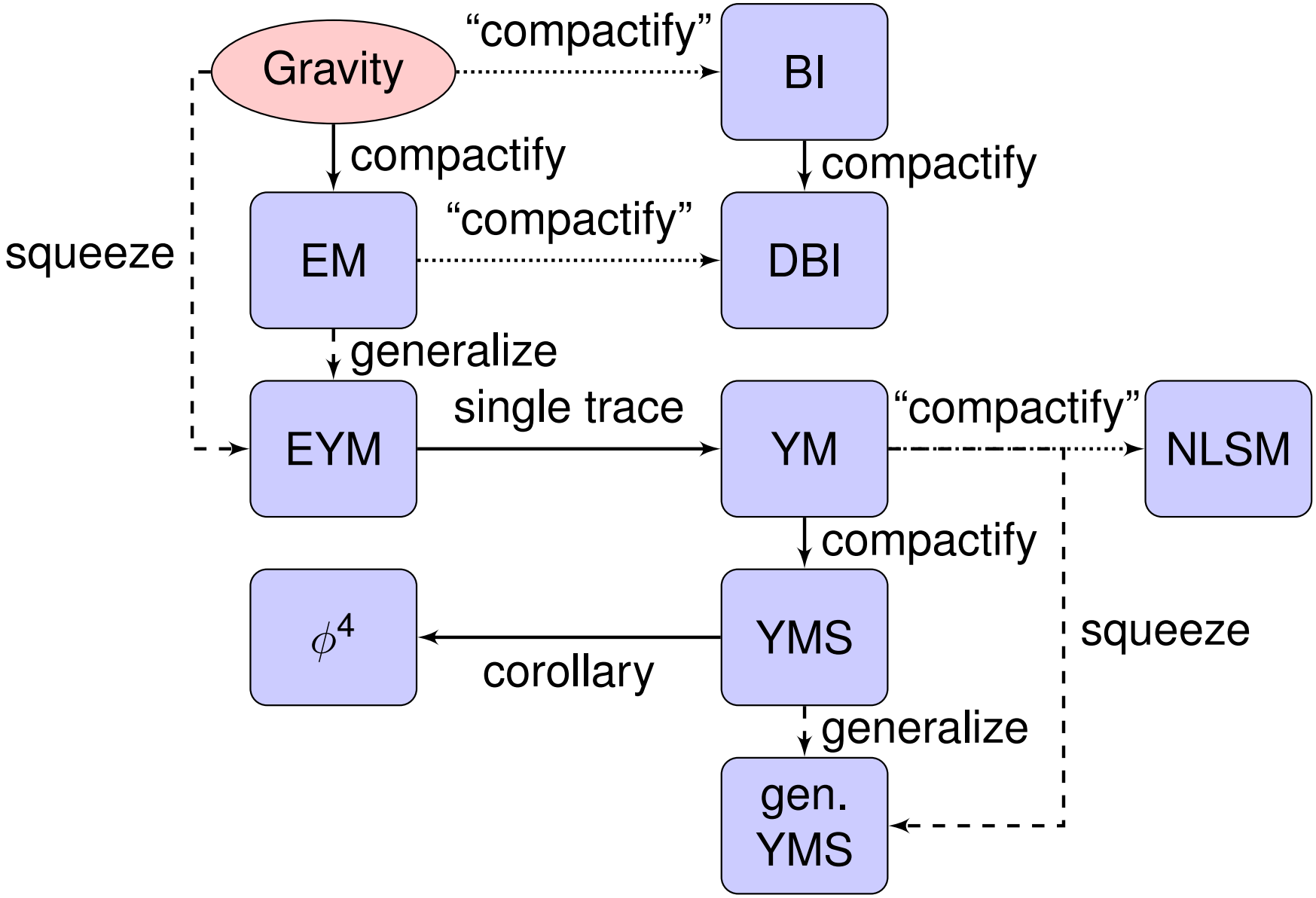


Figure: Theories studied by CHY and operations relating them.

Ambitwistor strings with combinations of matter

$S^l \backslash S^r$	S_Ψ	S_{Ψ_1,Ψ_2}	$S^{(\tilde{m})}_{\rho,\Psi}$	$S^{(\tilde{N})}_{CS,\Psi}$	$S^{(\tilde{N})}_{CS}$
S_Ψ	E				
S_{Ψ_1,Ψ_2}	BI	Galileon			
$S^{(m)}_{\rho,\Psi}$	*EM* $\mathrm{U}(1)^m$	DBI	*EMS* $\mathrm{U}(1)^m \times \mathrm{U}(1)^{\tilde{m}}$		
$S^{(N)}_{CS,\Psi}$	EYM	ext. DBI	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*EYMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	
$S^{(N)}_{CS}$	YM	Nonlinear σ	*EYMS* $\mathrm{SU}(N) \times \mathrm{U}(1)^{\tilde{m}}$	*gen. YMS* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$	*Biadjoint Scalar* $\mathrm{SU}(N) \times \mathrm{SU}(\tilde{N})$

Table: Theories arising from the different choices of matter models.

Models from different geometric realizations of $\mathbb{A}$

We can start with other formulations of null superparticles

- Green-Schwarz version:

$$S = \int P \cdot \bar{\partial} X + P_\mu \gamma^\mu_{\alpha\beta} \theta^\alpha \bar{\partial} \theta^\beta \, .$$

- Pure spinor version (Berkovits) $S = \int P \cdot \bar{\partial} X + p_\alpha \bar{\partial} \theta^\alpha + \ldots$.
- $d = 4$, Twistor-strings of Witten, Berkovits & Skinner

$$\mathbb{A} = \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* | Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}$$

$$S = \int_\Sigma W \cdot \bar{\partial} Z + a Z \cdot W$$

- In 4d have full ambitwistor representation [Geyer, Lipstein, M. 1404.6219]

$$S = \int_\Sigma Z \cdot \bar{\partial} W - W \cdot \bar{\partial} Z + a Z \cdot W$$

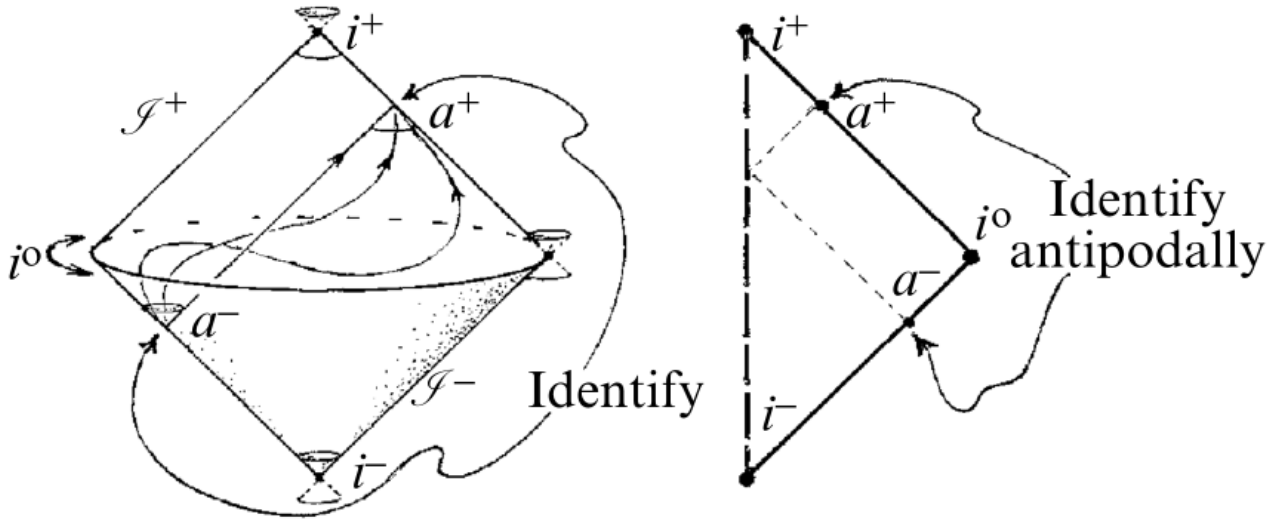
Not twistor string: $(Z, W) \in K^{1/2}$ gives simpler 4d formulae with no moduli. Nonchiral, working at $N = 0$.

Also can use null infinity:

Relation to null infinity, BMS and soft gravitons

Geyer,Lipstein & M. 1406.1462 (cf. Adamo, Casali & Skinner 1405.5122, 1503.02304).

Take space-time asymptotically simple:



Real light rays intersect $\mathscr{I}^+$ and $\mathscr{I}^-$, $\mathbb{A}_{\mathbb{R}} = T^*\mathscr{I}^+ = T^*\mathscr{I}^-$.

- **Flat space-time:** identification is identity and global.
- **Curved space-time:** identification only for real light rays:

$$\mathbb{A} = T^*\mathscr{I}^+_{\mathbb{C}} \cup T^*\mathscr{I}^-_{\mathbb{C}} \qquad \text{glued over } \mathbb{A}_{\mathbb{R}}\,.$$

Infinitesimally glued by Hamiltonian h for light ray scattering

$$\text{Vertex op} = \delta\theta = \bar{\partial} h\,.$$

BMS$^{\pm}$ group acts on $\mathscr{I}^{\pm}$ hence on $T^*\mathscr{I}^{\pm}$ with Hamiltonians h.

- worldsheet generators have same form as Vertex Ops:

$$\oint_{\Sigma} h = \int_{\Sigma} \bar{\partial} h.$$

- **Soft gravitons:** $k \to 0$ then $h \rightsquigarrow$ supertranslation.
- Subleading term generates 'superrotation'.
- Diagonal subgroup $\subset$ BMS$^{+}\times$BMS^{-} are symmetries.

$\rightsquigarrow$ versions of (subleading) soft graviton thm as Ward identity.

Theorem (Lysov/Strominger/Weinberg)

Weinberg soft theorem: as $k_n \to 0$

$$\mathcal{M}(1,\ldots,n) \to \mathcal{M}(1,\ldots,n-1)\sum_{i=1}^{n-1}\frac{(\epsilon_n\cdot k_i)^2}{k_n\cdot k_i}\,,$$

follows from supertranslation equivariance.

The quantum gravity loop integrand

[Adamo, Casali, Skinner 2013, Casali Tourkine 2015 Geyer, M., Monteiro, Tourkine. . .]]

10d type II gravity model is critical so extends to higher genus:

- At genus g, P is a 1-form and acquires dg zero-modes.
- These are the loop momenta for g-loops.
- Standard string technology can be adapted at all g.
- E.g., at 1-loop, $n = 4$, obtain modular invariant sum over spin structures of

$$\mathcal{M}_n^{(1)}(\alpha;\beta) = \delta^{10}\left(\sum_i k_i\right)\int \mathrm{d}^{10}p \wedge \mathrm{d}\tau \wedge \bar{\delta}\left(P^2(\sigma_1;\tau)\right)$$

$$\prod_{j=2}^{4} d\sigma_j\, \bar{\delta}(k_j \cdot P(\sigma_j))\, \frac{\vartheta_\alpha(\tau)^4 \vartheta_\beta(\tau)^4}{\eta(\tau)^{24}}\, \mathrm{Pf}(M_\alpha)\, \mathrm{Pf}(\widetilde{M}_\beta)$$

Now checked in many ways and related to standard integrand.

Is power counting better than that from space-time?

Summary & Outlook

Chiral $\alpha' = 0$ ambitwistor strings use LeBrun's correspondence to give theories underlying CHY formulae old & new.

- Incorporates colour/kinematics Yang-Mills/gravity ideas. Any insight into geometry of kinematic factors?
- Quantization ties scattering of null geodesics into that for gravitational waves.
- Critical models extend to loops .
- Does new representation give new insights into loop integrands?
- Insight into nonperturbative phenomena? Creation of mass? Black holes?

Thank You

Evaluation of amplitude

- Take $\mathrm{e}^{ik_i \cdot X(\sigma_i)}$ factors into action to give

$$S = \frac{1}{2\pi} \int_\Sigma P \cdot \bar{\partial} X + 2\pi \sum_i ik \cdot X(\sigma_i) \,.$$

- Gives field equations $\bar{\partial} X = 0$ and,

$$\bar{\partial} P = 2\pi \sum_i ik \delta^2(\sigma - \sigma_i) \,.$$

- Solutions $X(\sigma) = X = \text{const.}$, $P(\sigma) = \sum_i \frac{k_i}{\sigma - \sigma_i} d\sigma$.

Thus path-integral reduces to

$$\mathcal{M}(1, \ldots, n) = \delta^d(\sum_i k_i) \int_{(\mathbb{CP}^1)^{n-3}} \frac{\prod_i{}' \bar{\delta}(k_i \cdot P) \, (\epsilon_i \cdot P(\sigma_i))^2}{\text{Vol } G}$$

We see $P(\sigma)$ appearing and scattering equations.

Unfortunately: amplitudes for $S \sim \int_M R + R^3$.

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