Limits on the reliable storage of information in a volume of space

Steve Flammia

SF, J. Haah, M. Kastoryano, and I. Kim
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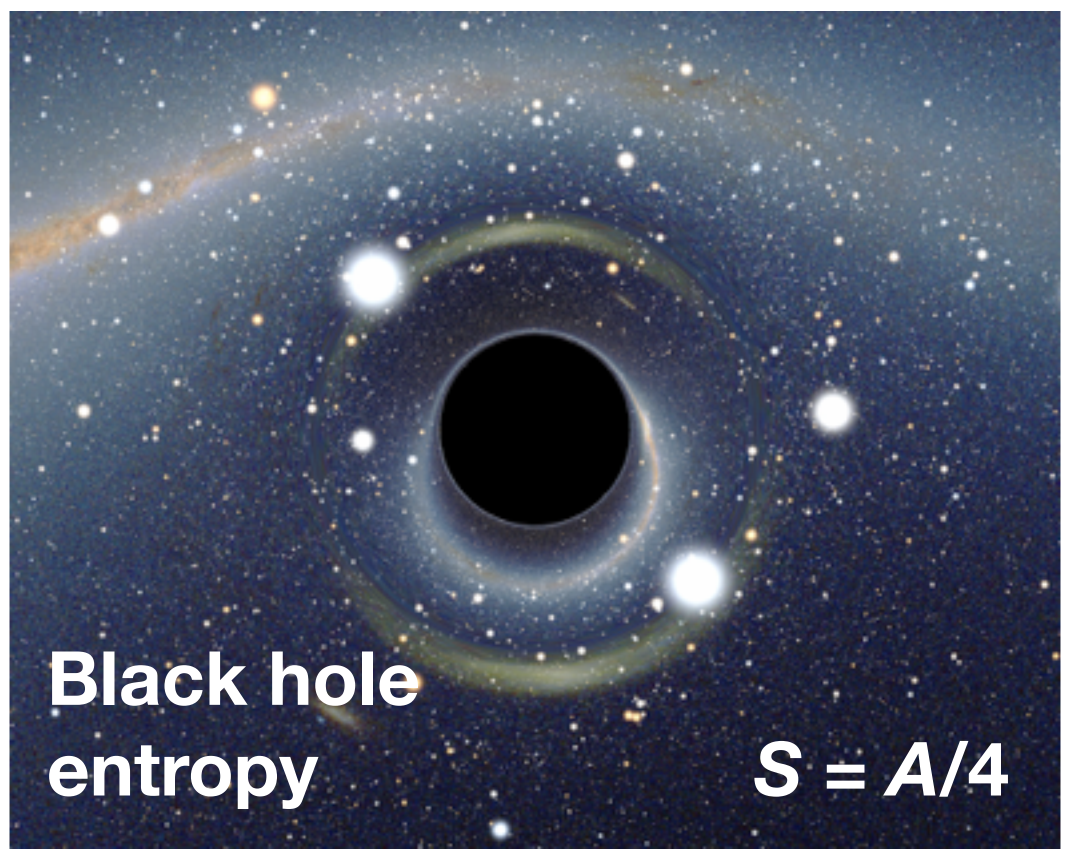
Information, gravity, and geometry

This talk is about **limitations on information storage** in a volume of space, and the relationship between **information** and **geometry** of space.



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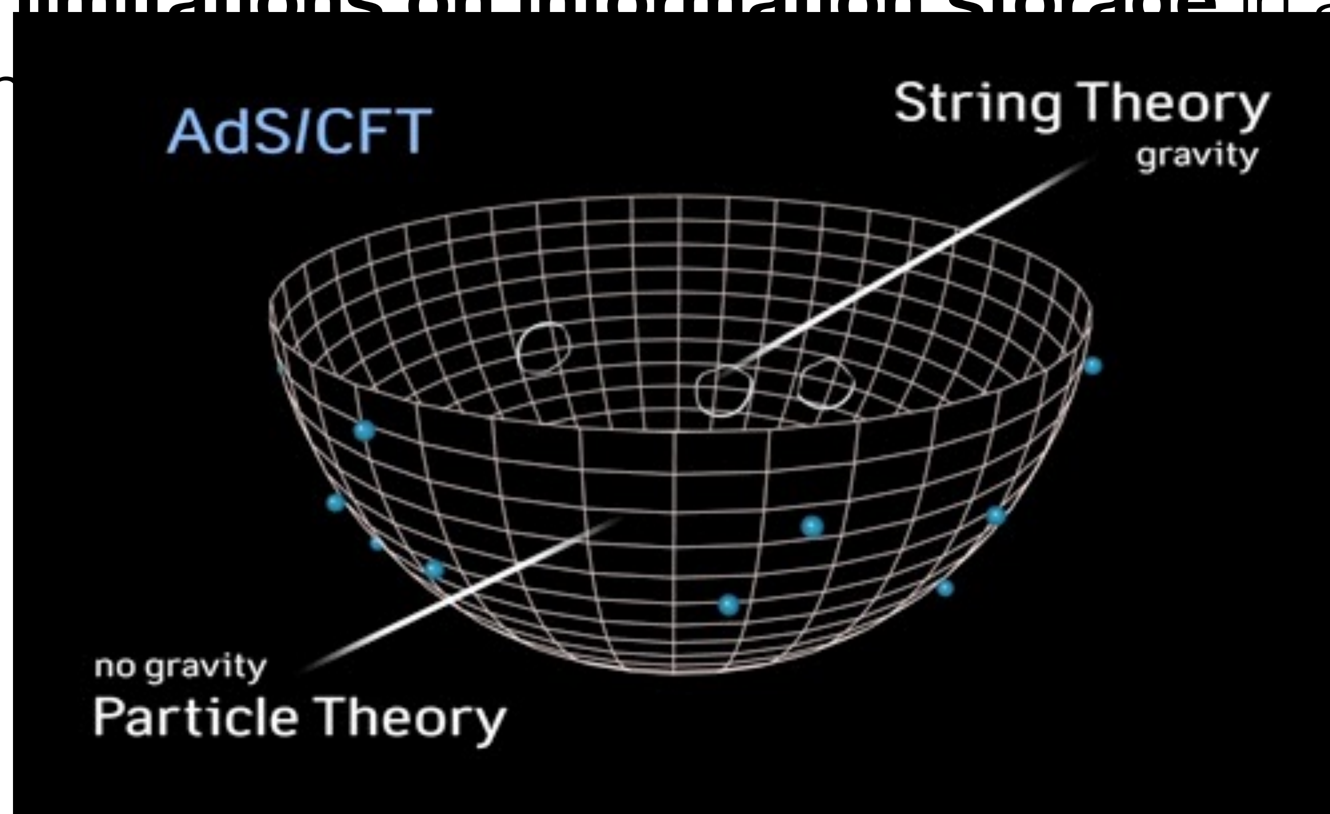

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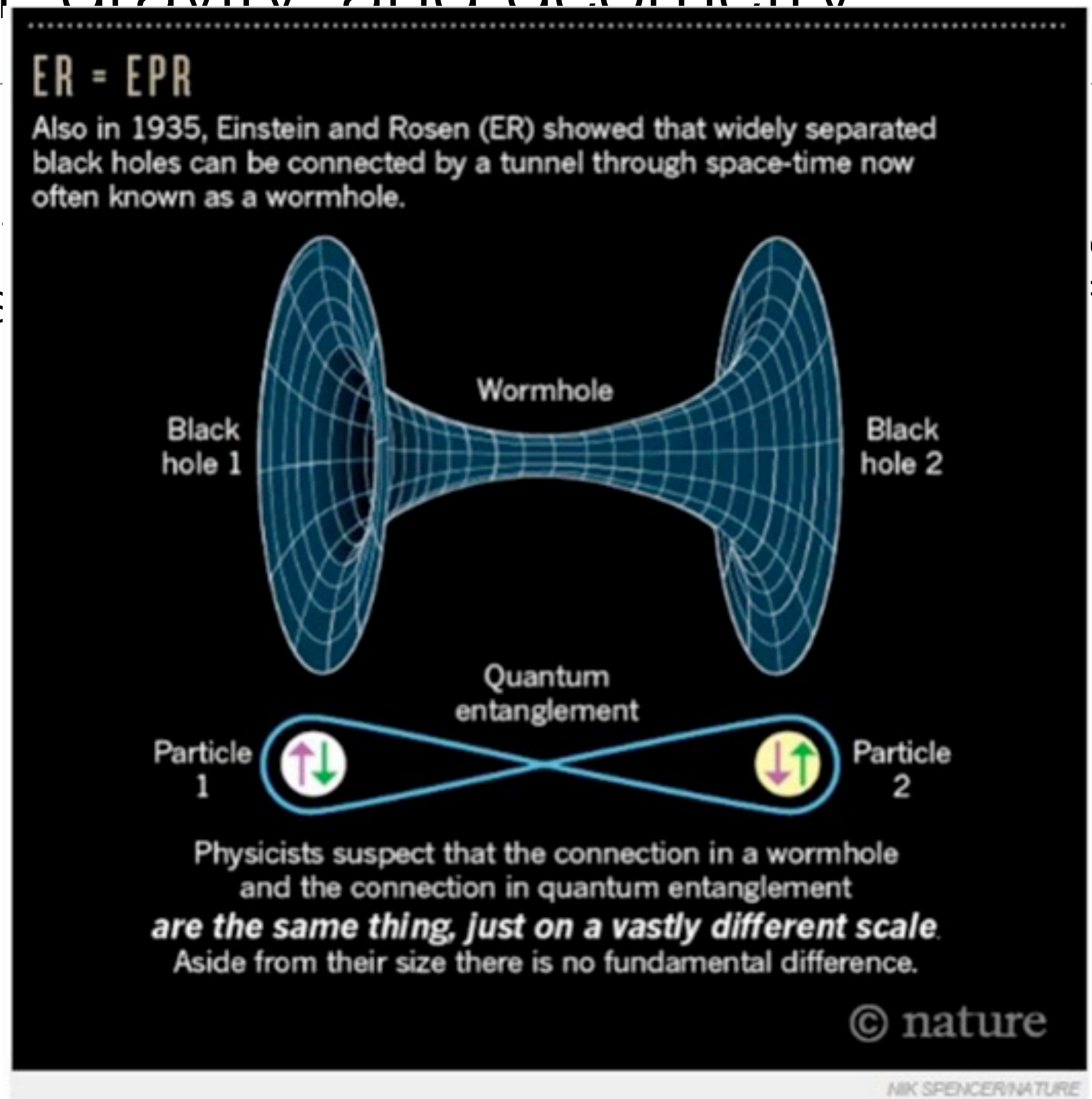

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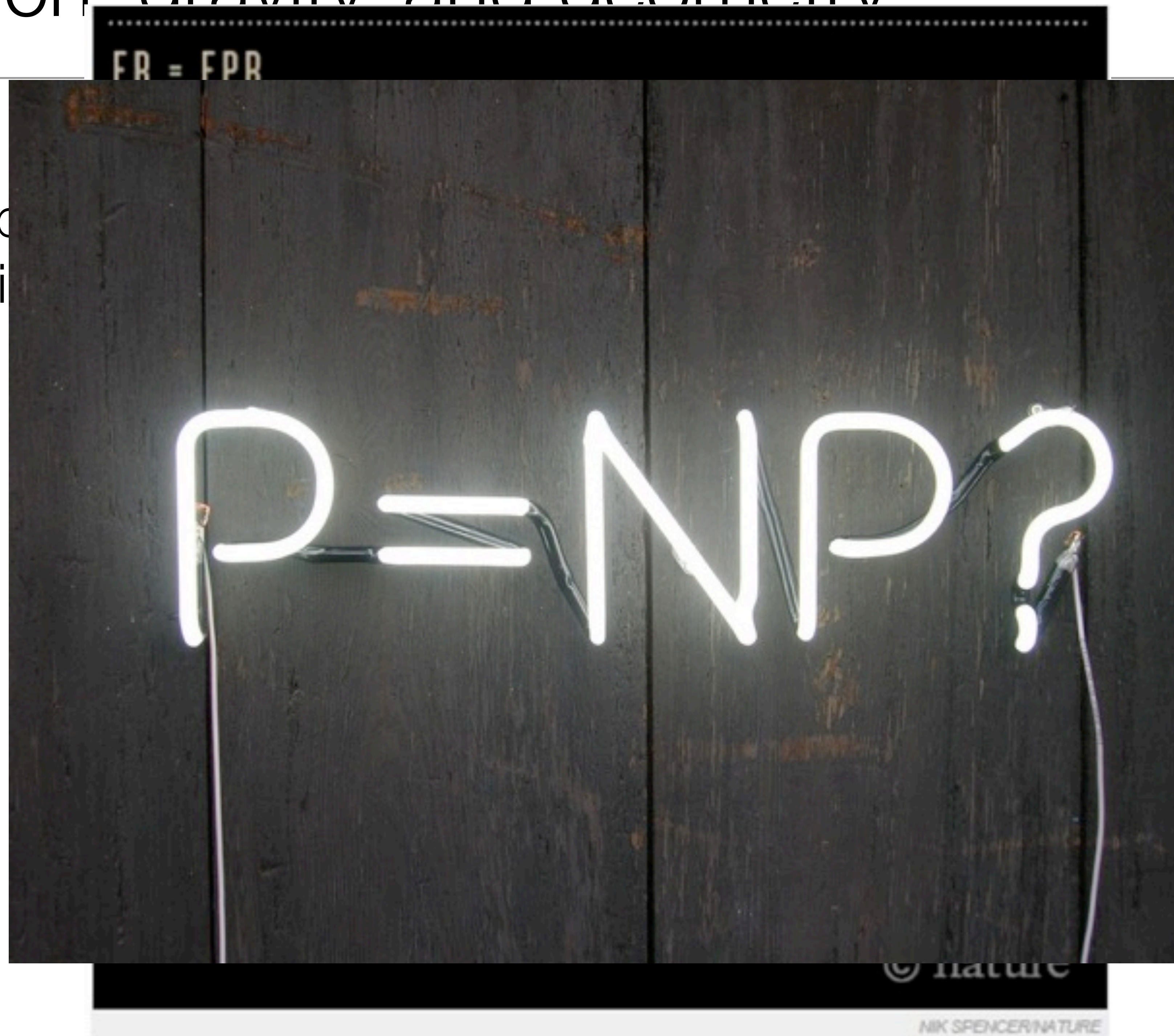

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Quantum error correcting codes

No **gravity** in this talk, only **codes**, **information**, and **geometry.**

An [[n,k,d]] quantum error correcting code is defined by three parameters:

- number of physical qubits n
- number of logical qubits k
- code distance d

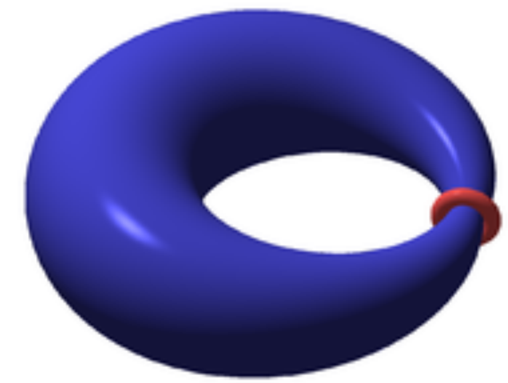
Obviously, $k \leq n$ and $d \leq n$, but not all triples are possible.

However, it is known that *random* codes can achieve $k \geq \Omega(n)$ and $d \geq \Omega(n)$.

What happens when we insist that our code has additional locality structure?

Topological codes

Distance *d* relates to the systole of the geometry, and we get *tradeoff bounds* for $n = L^D$ in a Euclidean lattice:



- $k\,d^{2/(D-1)} \leq c\,n$

  Subspace and commuting projector codes

  Bravyi, Poulin, Terhal

- $k\,d^{1/(D-1)} \leq c\,n$

  Subsystem codes

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- $k\,d^{1/2} \leq c\,n$

  D=2 classical codes

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- $k\,d^{2} \leq c\,n\,(\log k)^2$

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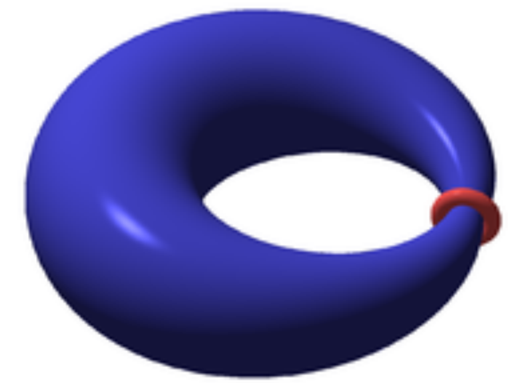

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All of these results assume exact error correction. What happens when we have an ε of room?

Exact vs. approximate QEC

An *exact* quantum code can perfectly correct d-1 erasure errors.

Equivalently, it can perfectly correct any $t = (d\text{-}1)/2$ arbitrary single-qubit errors.

By the no-cloning theorem, no code can correct $n/2$ erasure errors, therefore no code can correct more than $t \leq n/4$ errors (cf. classical codes).

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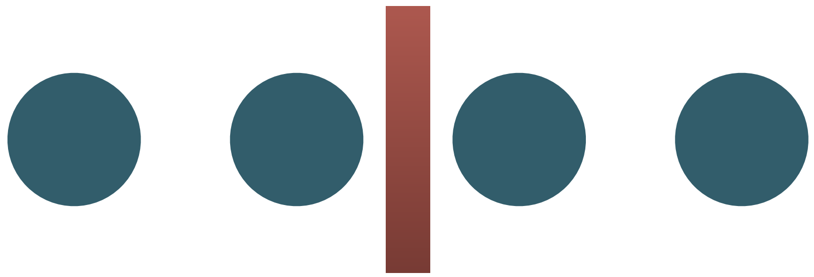

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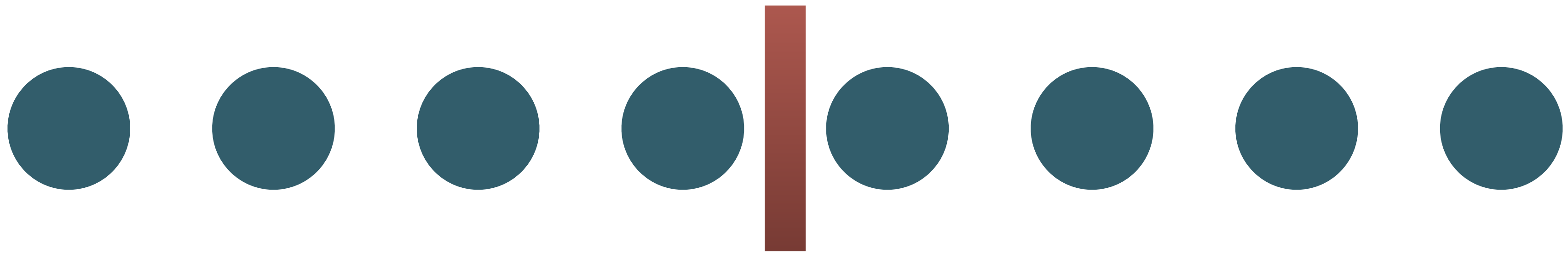

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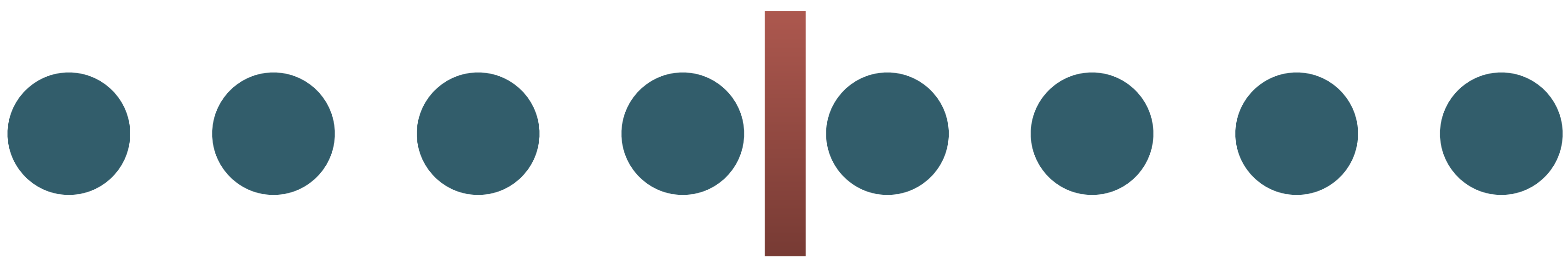


However, *approximate* quantum codes can correct ***t* = *n*/2** errors…
with exponential accuracy!

Crépeau, Gottesman, Smith (2005)

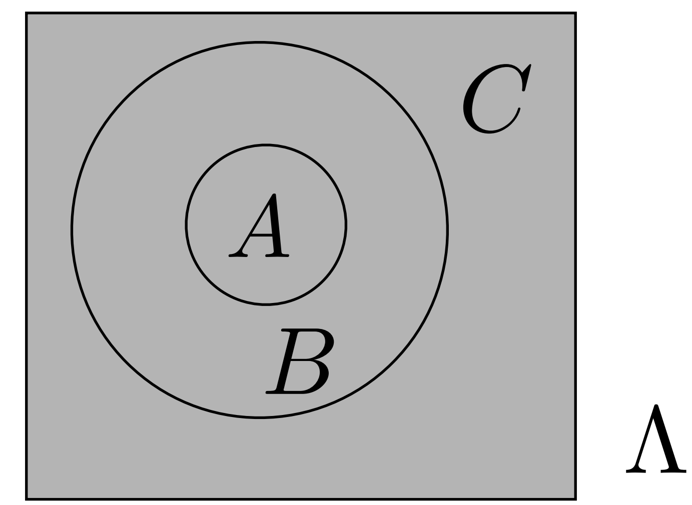
The CGS codes are highly nonlocal. But this example suggests that the approximate case can differ dramatically from the exact case.

Commuting projector codes

$$\{S_j\} \quad [S_j, S_k] = 0 \quad S_j = S_j^2$$

$$\Pi = \prod_j S_j \quad \mathcal{C} = \{|\psi\rangle, \Pi|\psi\rangle = |\psi\rangle\}$$

- $\mathcal{C}$ is the code space
- Λ is the lattice
- Consider erasure errors on the region A



R is a purifying system

- There are several different ways we might define correctability of the code space that would naturally lead to a notion of topological code

Lemma *Let $\mathcal{C}$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between A and C is at least $\ell \geq w$, the interaction range (*e.g. *as in Fig. 3.) Then the following are equivalent:*

(i) *Topological Quantum Order (TQO): for any observable O_A with support on A, any two ground states $|\phi\rangle$ and $|\psi\rangle$ give the same expectation value, $\langle\phi| O^A |\phi\rangle = \langle\psi| O^A |\psi\rangle$.*

(ii) *Decoupling: For any $\rho \in \mathcal{C}$ we have $I_\rho(A : CR) = 0$.*

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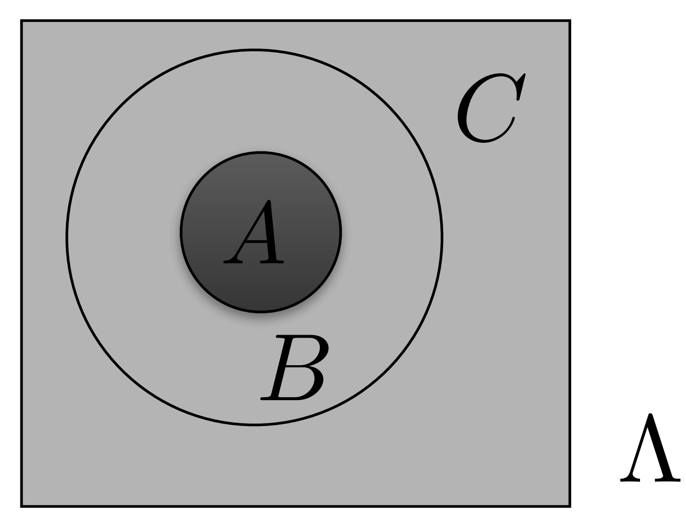

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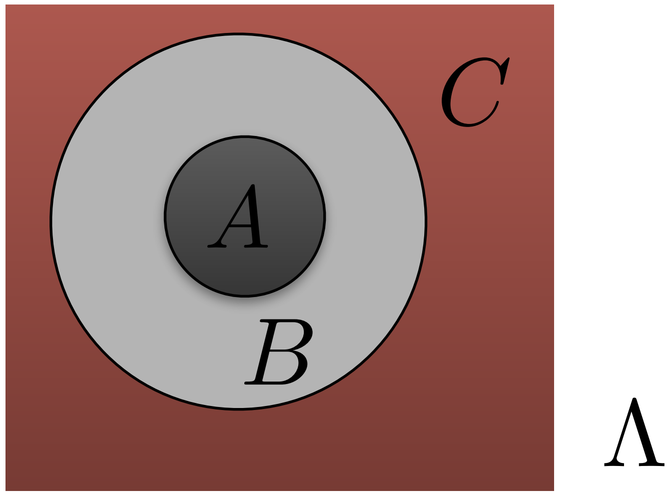
- (ii) Decoupling



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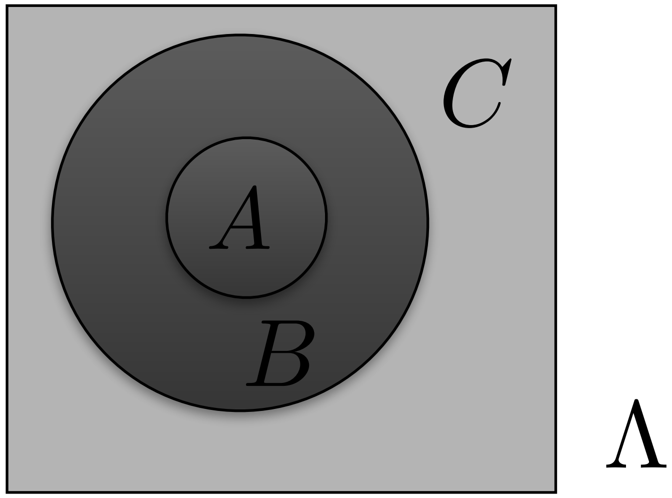

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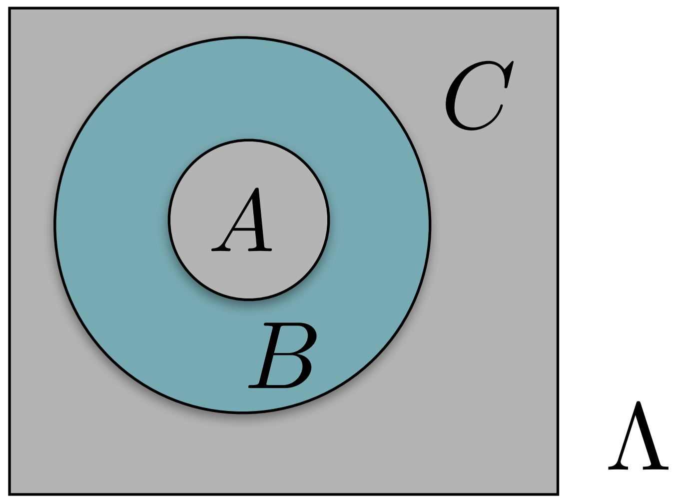

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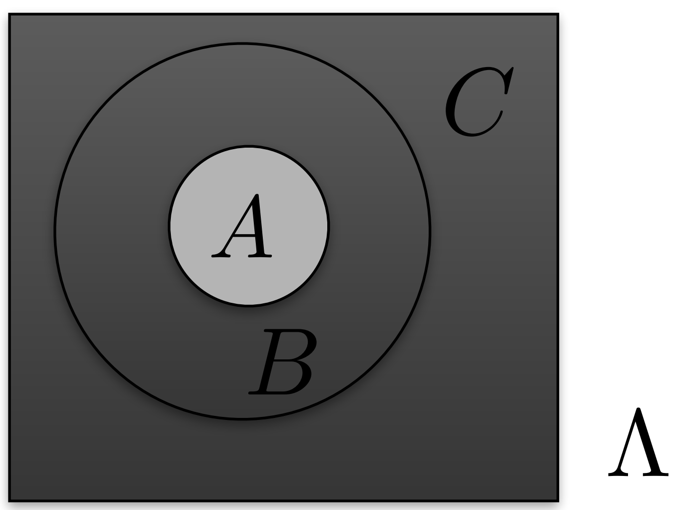

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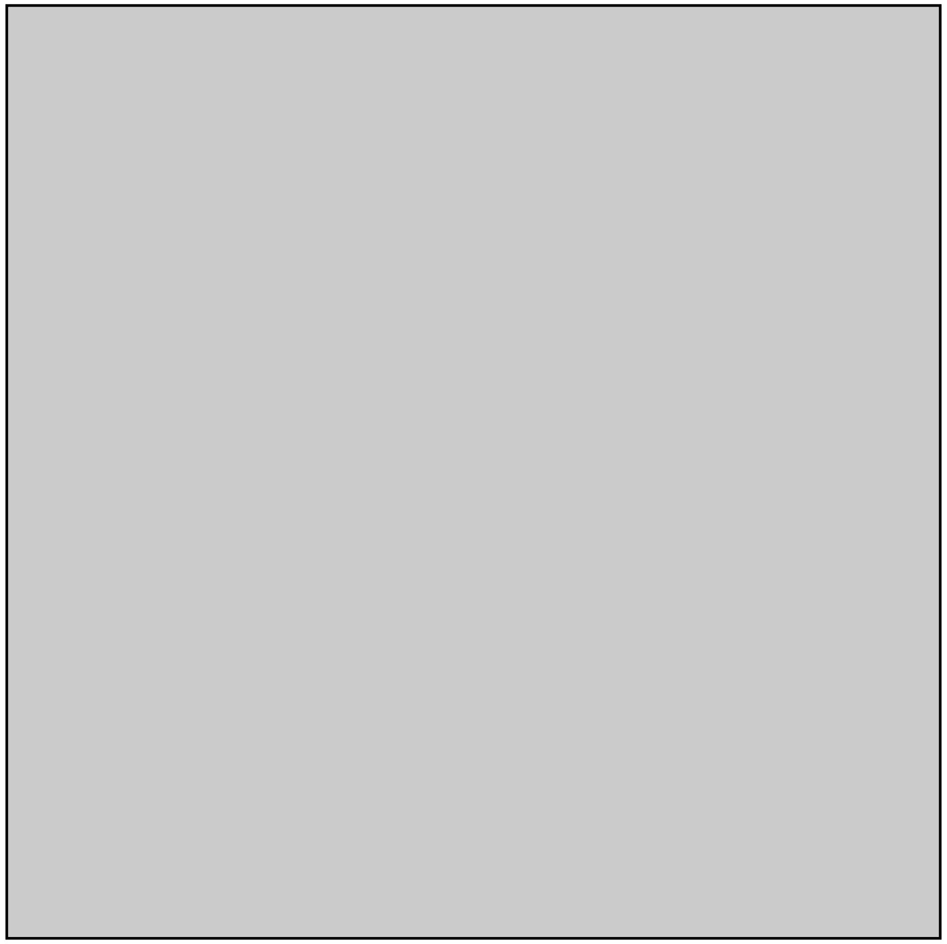
Given a correctable region with a logical operator, we can *clean* it so that the logical operator does not touch the region.



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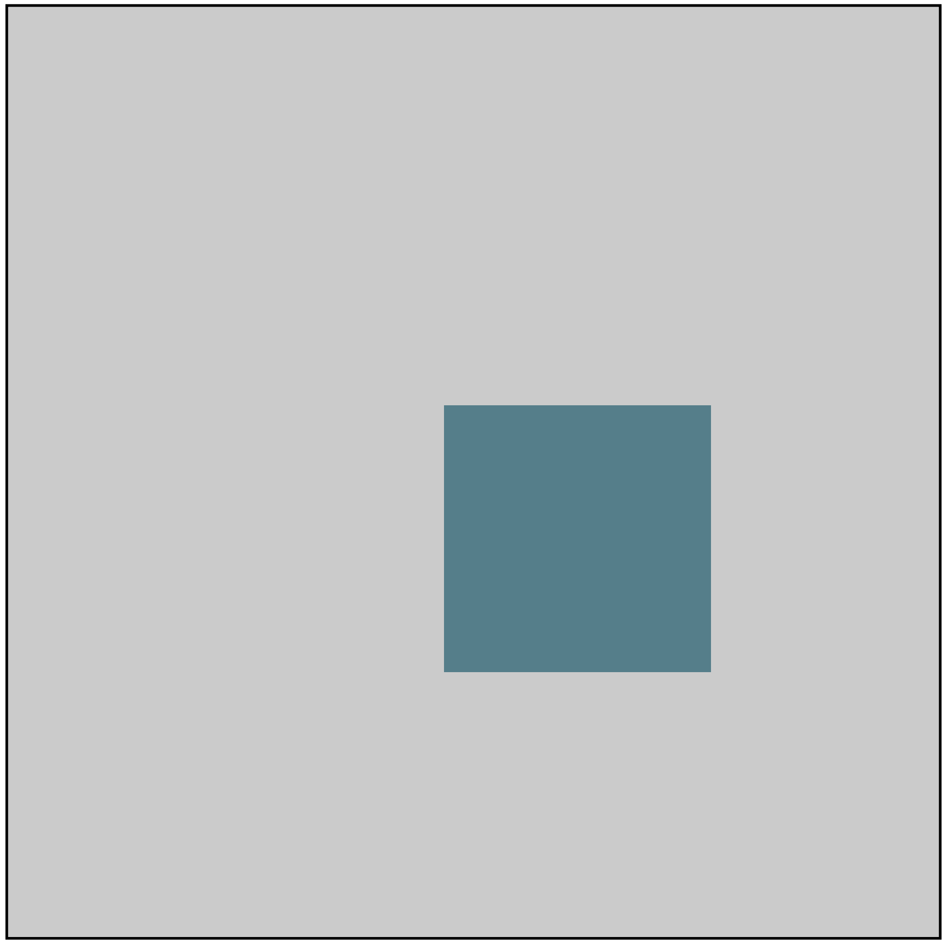

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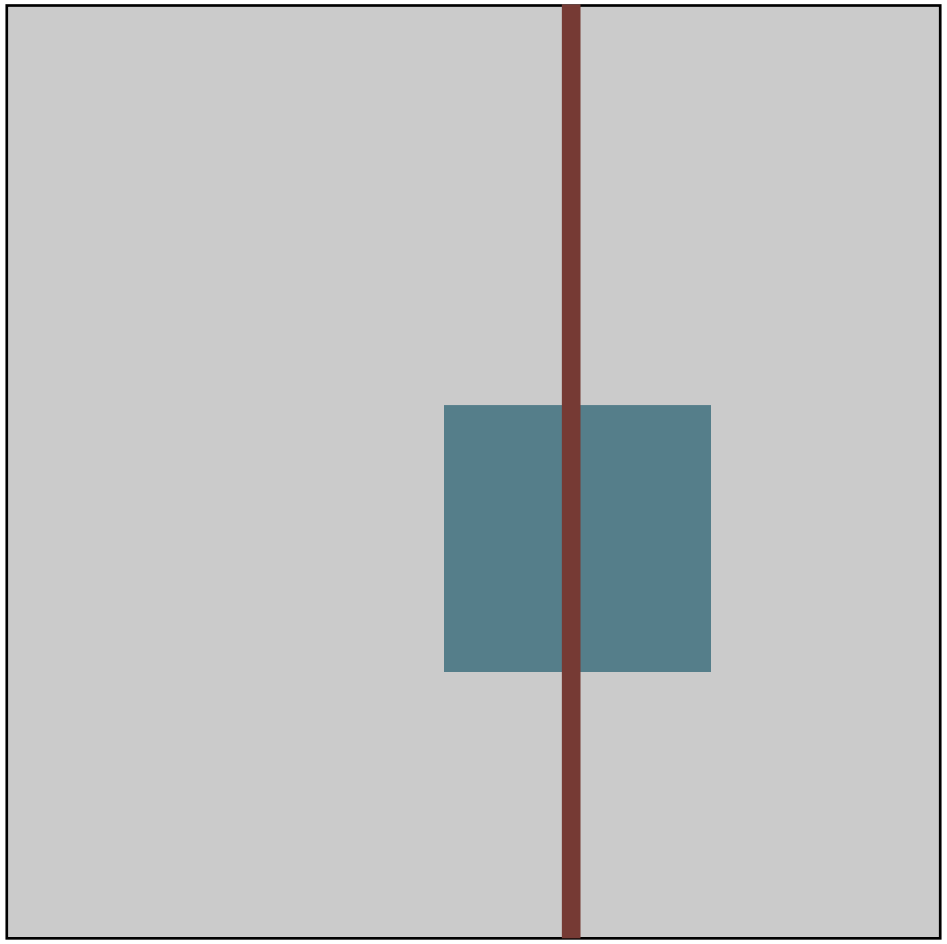

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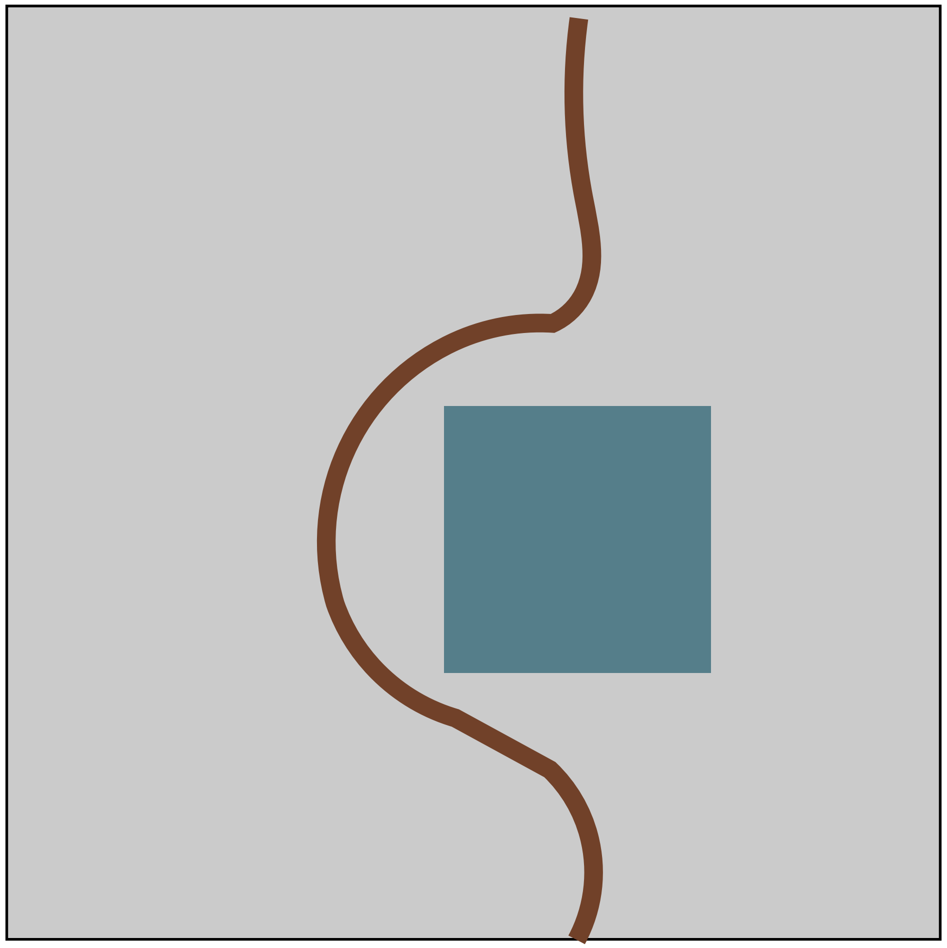

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Which properties can be extended to approximate codes?

➡ Focus on topological codes; tradeoff bounds

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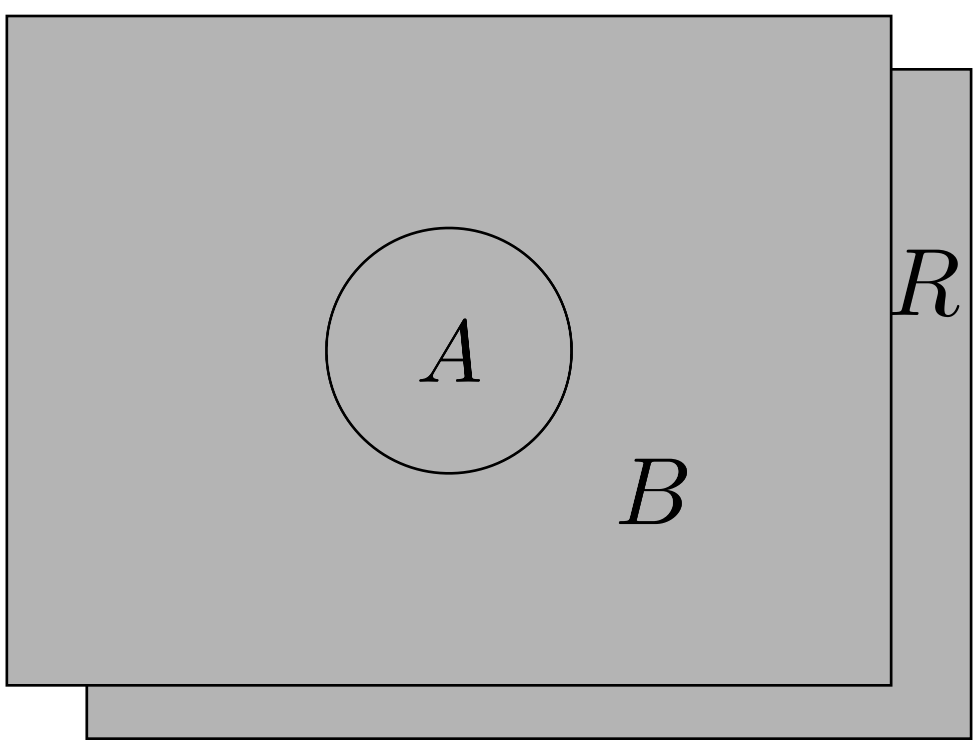
Take as our basic definition

Approximate QEC

Definition (approximate correctability):

There exists a recovery map $\mathcal{R}_B^{AB}$ such that for any code state $\rho^{ABR} \in \mathcal{C}$ the following holds:

$$\mathcal{B}(\rho^{ABR}, \mathcal{R}_B^{AB}(\rho^{BR})) \leq \delta$$



- Bures distance $\mathcal{B}(\rho,\sigma)^2 = 1 - F(\rho,\sigma)$

$$F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$$

- Stabilized distance; R is a copy of the logical space.

Approximate QEC

Definition (local approximate correctability):

There exists a recovery map $\mathcal{R}_B^{AB}$ such that for any code state $\rho^{ABCR} \in \mathcal{C}$ the following holds:

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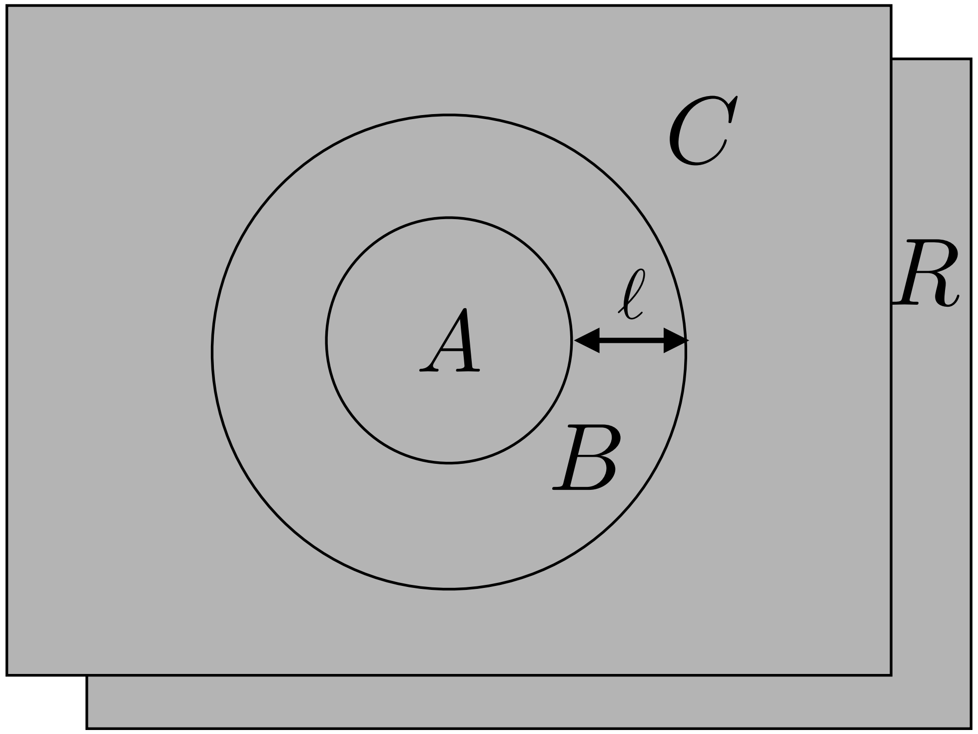


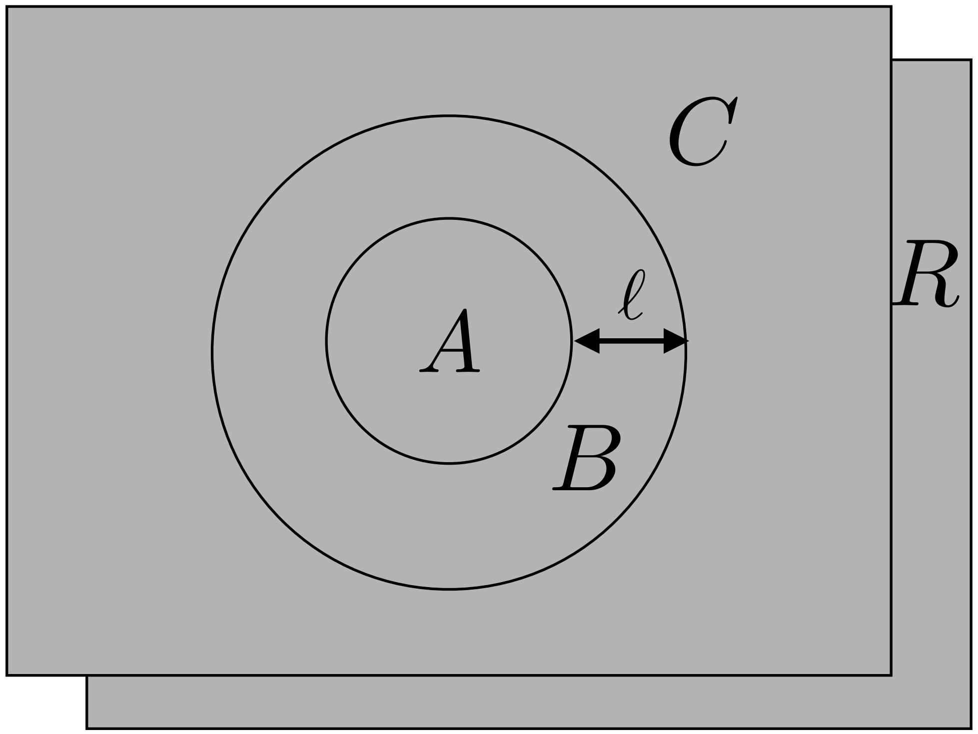
➡ state can be recovered without modifying C

Equivalent formulations

Theorem (information-disturbance tradeoff):

$$\inf_{\omega^A} \sup_{\rho^{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR}) = \inf_{\mathcal{R}_B^{AB}} \sup_{\rho^{ABCR}} \mathcal{B}(\mathcal{R}_B^{AB}(\rho^{BCR}), \rho^{ABCR})$$

$$\delta_\ell(A) := \inf_{\omega^A} \sup_{\rho^{ABCR}} \mathcal{B}(\omega^A \otimes \rho^{CR}, \rho^{ACR})$$



- ρ^{ABCR} is in the code space
- ω^A is some fixed state on A

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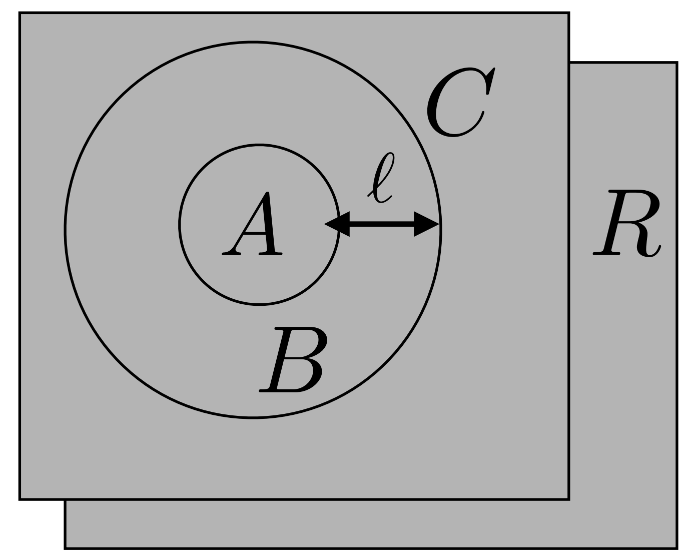


Theorem (decoupling):

$$\frac{1}{9}\delta_\ell(A)^2 \leq \sup_{\rho^{ABCR}} \mathcal{B}(\rho^{ACR}, \rho^A \otimes \rho^{CR}) \leq 2\delta_\ell(A)$$

(Also need to prove $B^2 \leq I(A:R) \leq \delta \log(1/\delta)$)

Lemma *Let $\mathcal{C}$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between A and C is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:*

(i) *Topological Quantum Order (TQO): for any observable O_A with support on A, any two ground states $|\phi\rangle$ and $|\psi\rangle$ give the same expectation value, $\langle\phi| O^A |\phi\rangle = \langle\psi| O^A |\psi\rangle$.*

(ii) *Decoupling: For any $\rho \in \mathcal{C}$ we have $I_\rho(A : CR) = 0$.*

(iii) *Error correction: There exists a recovery map acting on AB such that $\mathcal{R}_B^{AB}(\rho^{BC}) = \rho^{ABC}$ for any $\rho \in \Pi$.*

(iv) *Disentangling unitary: For any $\rho \in \mathcal{C}$ there exists a unitary U^B, such that $U^B \rho U^{B\dagger} = \omega^{AB_1} \otimes \rho^{B_2C}$, for some state ω^{AB_1}.*

(v) *Cleaning: For any unitary U preserving the code space, there exists a unitary V^{BC} such that $U|_{\mathcal{C}} = V^{BC}|_{\mathcal{C}}$*

Which properties can be extended to approximate codes?

(iii) <=> (iv)

(iii) <=> (ii) but with different error order

Error correction $\Rightarrow$ cleanability:

If A is locally correctable: $\mathcal{B}(\mathcal{R}_B^{AB}(\rho^{BCR}), \rho^{ABCR}) \leq \delta$

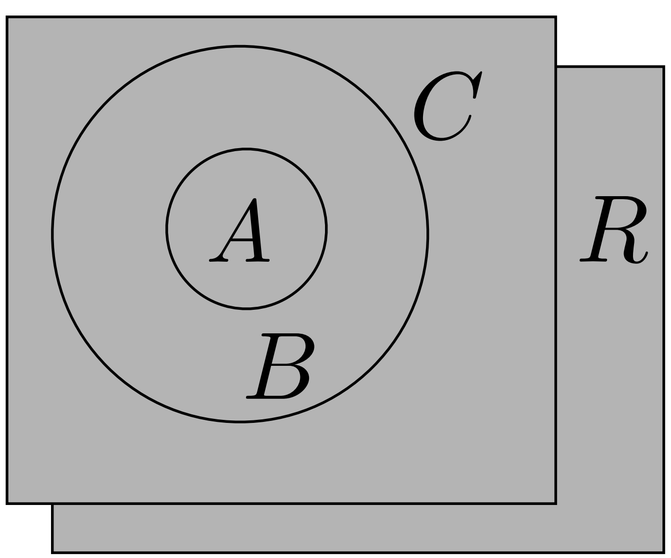
Then for any logical unitary U^{ABC}, the pull-back $V^{BC} = (\mathcal{R}_B^{AB})^*(U^{ABC})$ satisfies

$$||(U^{ABC} - V^{BC})\Pi|| \leq 4\sqrt{\delta}$$

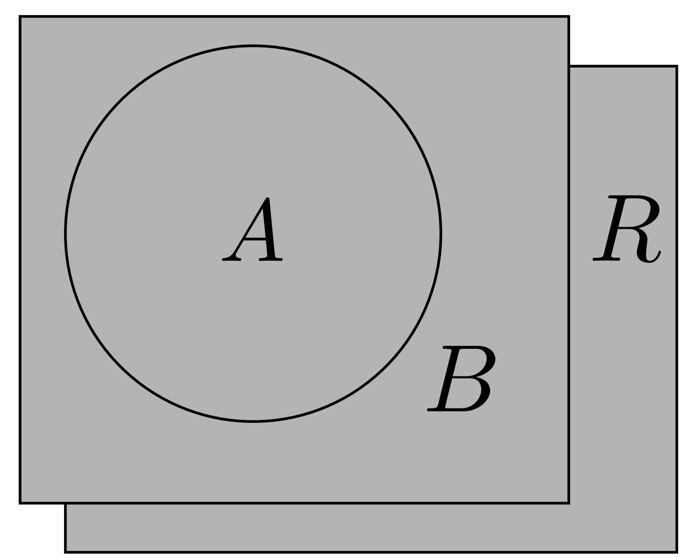
Error correction $\Leftarrow$ cleanability:

If for any U^{AB} there exists a $||V^B|| \leq 1$ on B s.t. $||(U^{ABC} - V^{BC})\Pi|| \leq \delta$

Then there exists ω^A s.t. $||\rho^{AB} - \omega^A \otimes \rho^R||_1 \leq 5\delta$



$\Rightarrow$



$\Leftarrow$

Lemma *Let $\mathcal{C}$ be a commuting projector code, and $ABC = \Lambda$ be decomposition of the lattice such that the distance between A and C is at least $\ell \geq w$, the interaction range (e.g. as in Fig. 3.) Then the following are equivalent:*

(i) *Topological Quantum Order (TQO): for any observable O_A with support on A, any two ground states $|\phi\rangle$ and $|\psi\rangle$ give the same expectation value, $\langle\phi| O^A |\phi\rangle = \langle\psi| O^A |\psi\rangle$.*

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(v) *Cleaning: For any unitary U preserving the code space, there exists a unitary V^{BC} such that $U|_{\mathcal{C}} = V^{BC}|_{\mathcal{C}}$*

Which properties can be extended to approximate codes?

(iii) <=> (iv)

(iii) <=> (ii) but with different error order

(iii) <=> (v) but with different error order and different locality constraints

→ Topological quantum order seems to be different!

BPT bound

Tradeoff bound

$$kd^2 \leq cn$$

Subspace or commuting projector codes
Bravyi, Poulin, Terhal

➡ Toric code saturates the bound in 2D

Proof:

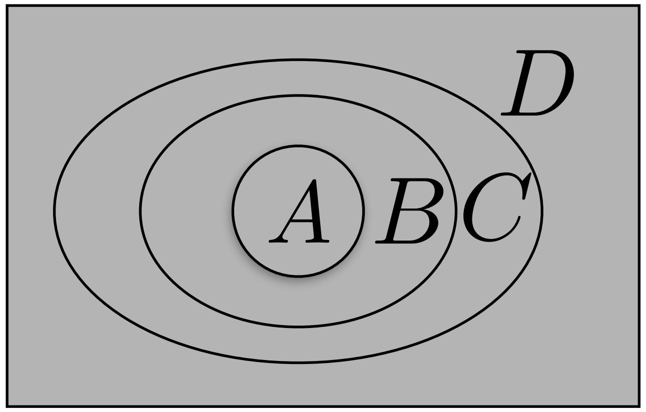
- Expansion lemma
- Union lemma
- Counting degrees of freedom

Approximate BPT

Tradeoff bound

$kd^2 \leq cn$ becomes $\big(1 - c\,\epsilon\log(1/\epsilon)\big)kd^2 \leq c' n \ell^4$

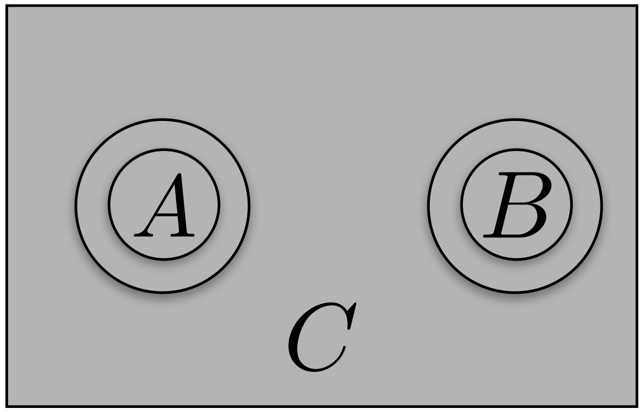
Proof:



$$\epsilon = \frac{\delta}{d/n}$$

- Approximate expansion bound
  - Need (iv) and (iii)

- Approximate union bound
  - Need locality of recovery



$$kd^2 \leq cn$$

BPT bound

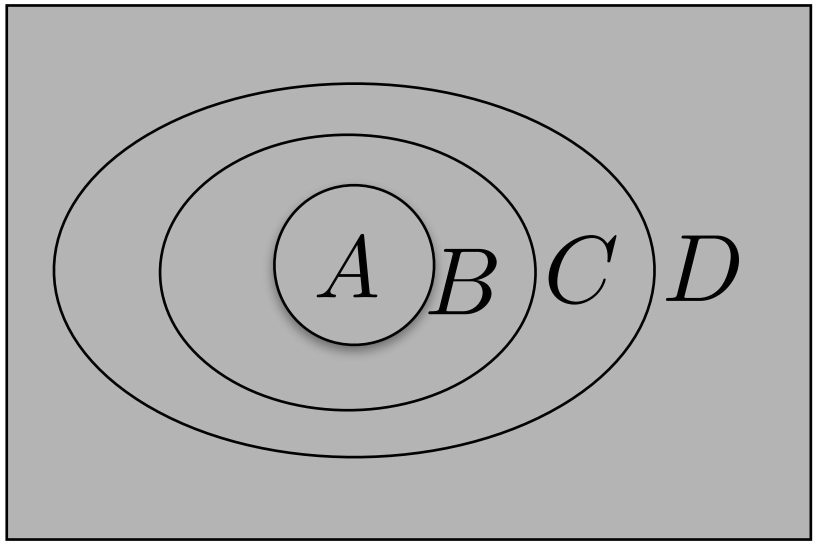
Expansion Lemma:

If A is correctable and B is correctable, then $A \cup B$ is correctable.

Proof:

A correctable $\Rightarrow \quad \rho^{ACD} = \omega^A \otimes \rho^{CD}$ (iv)

B correctable $\Rightarrow \mathcal{R}_{AC}^{ABC}(\rho^{ACD}) = \rho^{ABCD}$ (iii)



Λ

Define a map $\mathcal{F}_C^{ABC}(\rho^{CD}) = \mathcal{R}_{AC}^{ABC}(\omega^A \otimes \rho^{CD})$

Show (iii) $\mathcal{F}_C^{ABC}(\rho^{CD}) = \mathcal{R}_{AC}^{ABC}(\omega^A \otimes \rho^{CD}) = \mathcal{R}_{AC}^{ABC}(\rho^{ACD}) = \rho^{ABCD}$

$\square$

$$kd^2 \leq cn$$

BPT bound

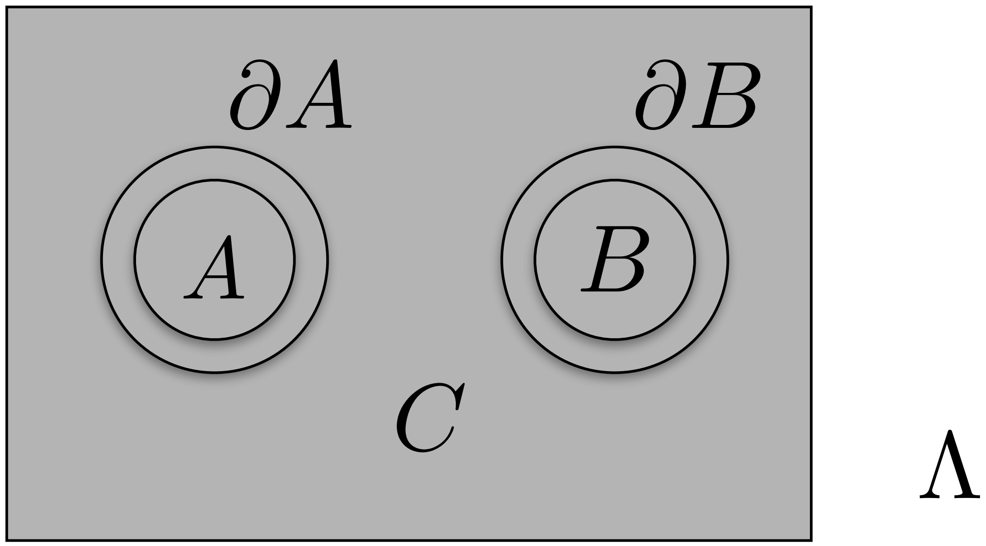
Union Lemma:

If A is correctable and B is correctable, then $A \cup B$ is correctable.

Proof:

A correctable $\Rightarrow$ $\mathcal{R}_{\partial A}^{A\partial A}(\rho^{\Lambda \backslash A}) = \rho^{\Lambda}$ (iii)

B correctable $\Rightarrow$ $\mathcal{R}_{\partial B}^{B\partial B}(\rho^{\Lambda \backslash B}) = \rho^{\Lambda}$ (iii)



Clearly, $\mathcal{R}_{\partial AB}^{AB\partial B}(\rho^{\Lambda \backslash AB}) = \rho^{\Lambda}$

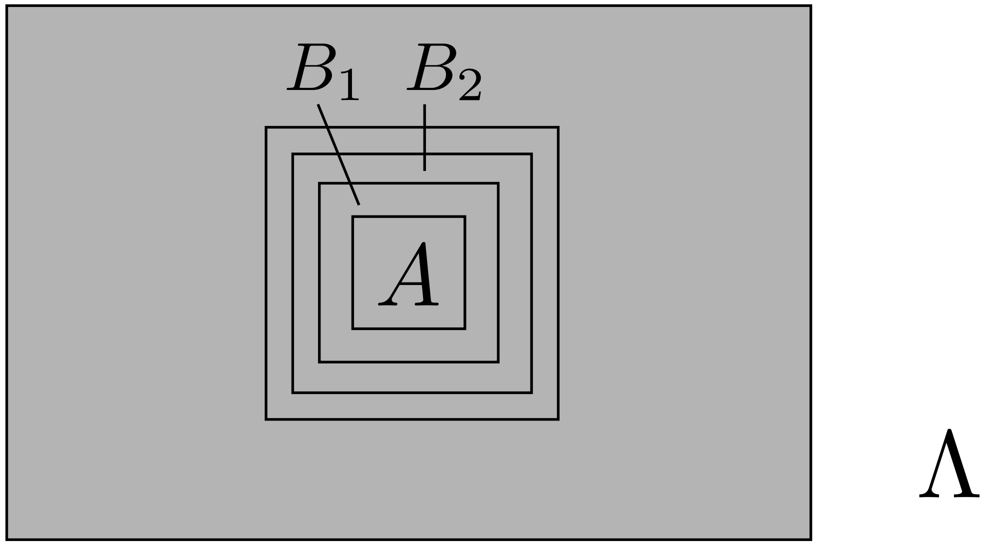
$\square$

$$(1 - c\,\epsilon \log(1/\epsilon))kd^2 \leq c' n \ell^4$$

Proof:

Construct the largest square correctible region by adding ‘onion’ rings.

➡ Largest square region d^2



Decompose the lattice as in Fig 2.

X and Y are correctable

$$I(X:R) = S(X) + S(R) - S(XR) = 0$$
$$S(Y) + S(R) - S(YR) = 0$$

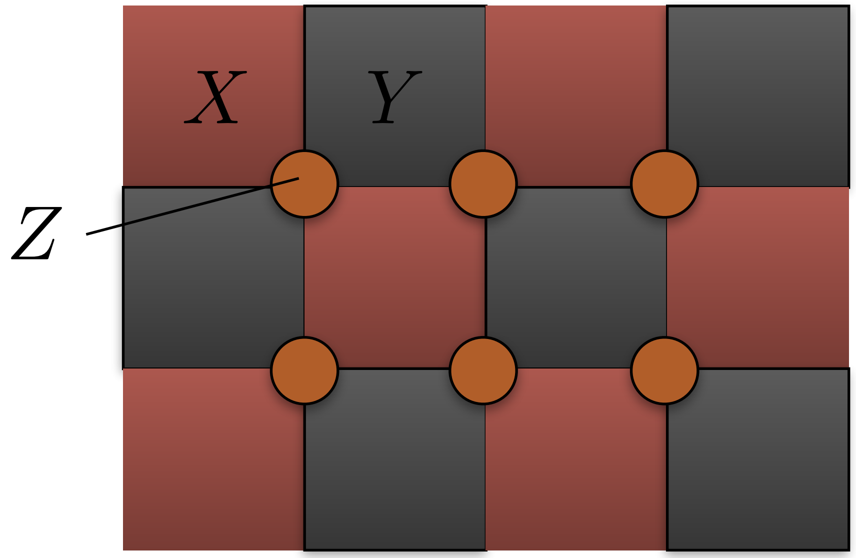


Fig 2

Sum the two and use subadditivity to get

$$S(R) \leq S(Z)$$

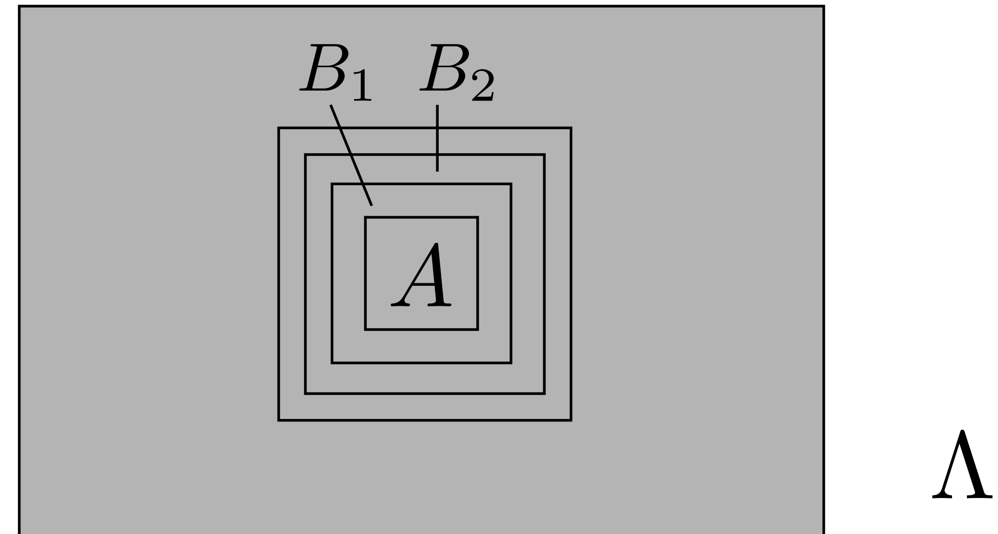
Take identity state on code space

$S(R) = k\log(2)$ and $S(Z) \leq cn/d^2$ $\Rightarrow$ $kd^2 \leq cn$ $\square$

$$\left(1 - c\,\epsilon \log(1/\epsilon)\right) k d^2 \leq c' n \ell^4$$

Proof:

Construct the largest square correctible region by adding 'onion' rings.



➡ Largest square region d^2

Decompose the lattice as in Fi

Need (iii) = (iv)

X and Y are correctable

$$I(X:R) = S(X) + S(R) - S(XR) = 0$$

$$S(Y) + S(R) \dots$$

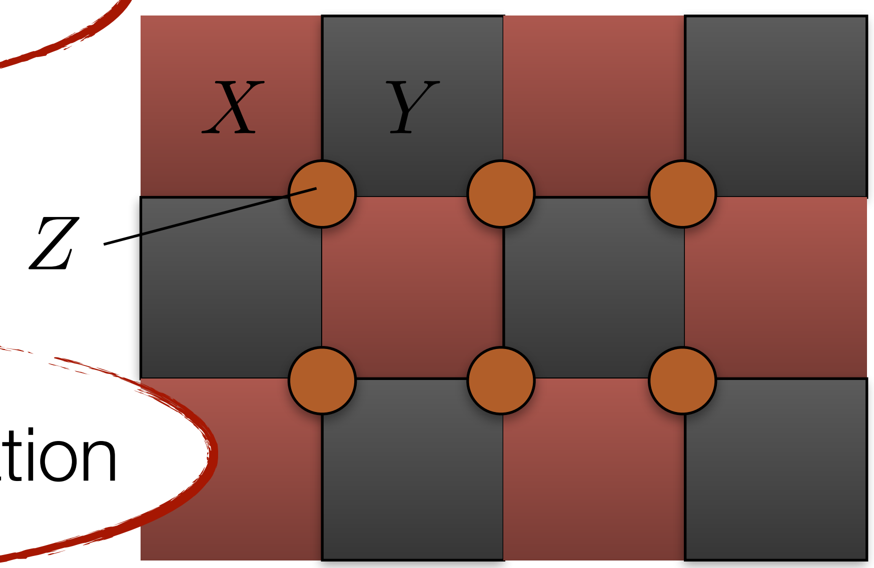


Fig 2

continuity of mutual information

Sum the two and

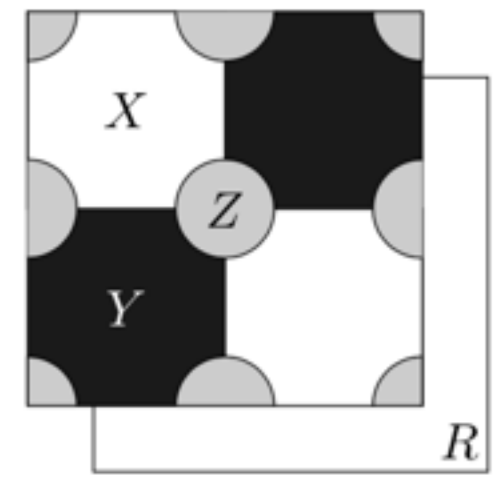
$$S(R) \leq S(Z)$$

Take identity state on code space

$S(R) = k \log(2)$ and $S(Z) \leq cn/d^2$ $\Rightarrow$ $kd^2 \leq cn$ $\square$

Finite anyon types

If the unitaries that commute with the code space are “flexible strings”, then the code space has bounded degeneracy.



Thus we can *derive* one of the key assumptions of the algebraic theory of anyons, namely that there are only a bounded number of simple objects (anyon types).

Definition 14. *A subspace* Π *on a two-dimensional system* **admits flexible (logical) operators** *if for any logical unitary operator* U^{XYZ} *there exist operators* V_1^{YZ} *supported on* YZ *and* V_2^{XZ} *on* XZ *such that* $\|V_i\| \leq 1$, $\|\Pi(U - V_i)\| \leq \epsilon_\ell$, *and* $\|(U - V_i)\Pi\| \leq \epsilon_\ell$, *where* $i = 1, 2$ *and* ϵ_ℓ *is independent of system size and vanishes as* $\ell \to \infty$.

Assuming flexible logical operators, we find that:

$$\dim \mathcal{C} \leq \exp\bigl(O(\ell^2)\bigr)$$

Further applications

Support of logical operators

Theorem . *For any* (δ, ℓ)*-correctable code* $\mathcal{C}$ *with* $\dim \mathcal{C} > 1$ *on a* D*-dimensional lattice of linear size* L*, if* $10L\delta < \ell$*, then the code distance is bounded from above by* $5\ell L^{D-1}$.

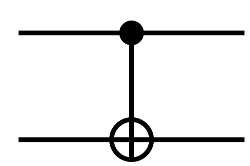
Theorem . *For any* (δ, ℓ)*-correctable code of code distance* d *on a* D*-dimensional lattice with Euclidean geometry of linear size* L*, there exists a region* Y *that contains* $\tilde{d}$ *qubits such that every logical operator* U *can be approximated by an operator* V *on* Y *where*

$$\|(U - V)\Pi\| \leq O\left(\sqrt{n\delta/d}\right)$$

$$\tilde{d}d^{\frac{1}{D-1}} \leq O(n\ell^{\frac{D}{D-1}})$$

Saturating these bounds: From Circuits to Codes

- Begin with a stabilizer code of your choice
- Write a Clifford quantum circuit for **measuring the stabilizers** of this code.
- Turn the circuit elements into **input/output qubits**
- Add gauge generators via **Pauli circuit identities**
- This defines the code

Circuit element	**Gauge generators**
$-\boxed{I}-$	XX, ZZ
$-\boxed{H}-$	ZX, XZ
$-\boxed{P}-$	YX, ZZ
	$\begin{matrix}XX\\XI\end{matrix}, \begin{matrix}II\\XX\end{matrix}, \begin{matrix}ZZ\\II\end{matrix}, \begin{matrix}ZI\\ZZ\end{matrix}$
$\langle 0\| -$	Z
$- \|0\rangle$	Z

Bacon, SF, A. Harrow, J. Shi, 2014

Circuits to codes

- Concatenation of codes, localized on a 3D lattice
- Local subsystem codes exist with

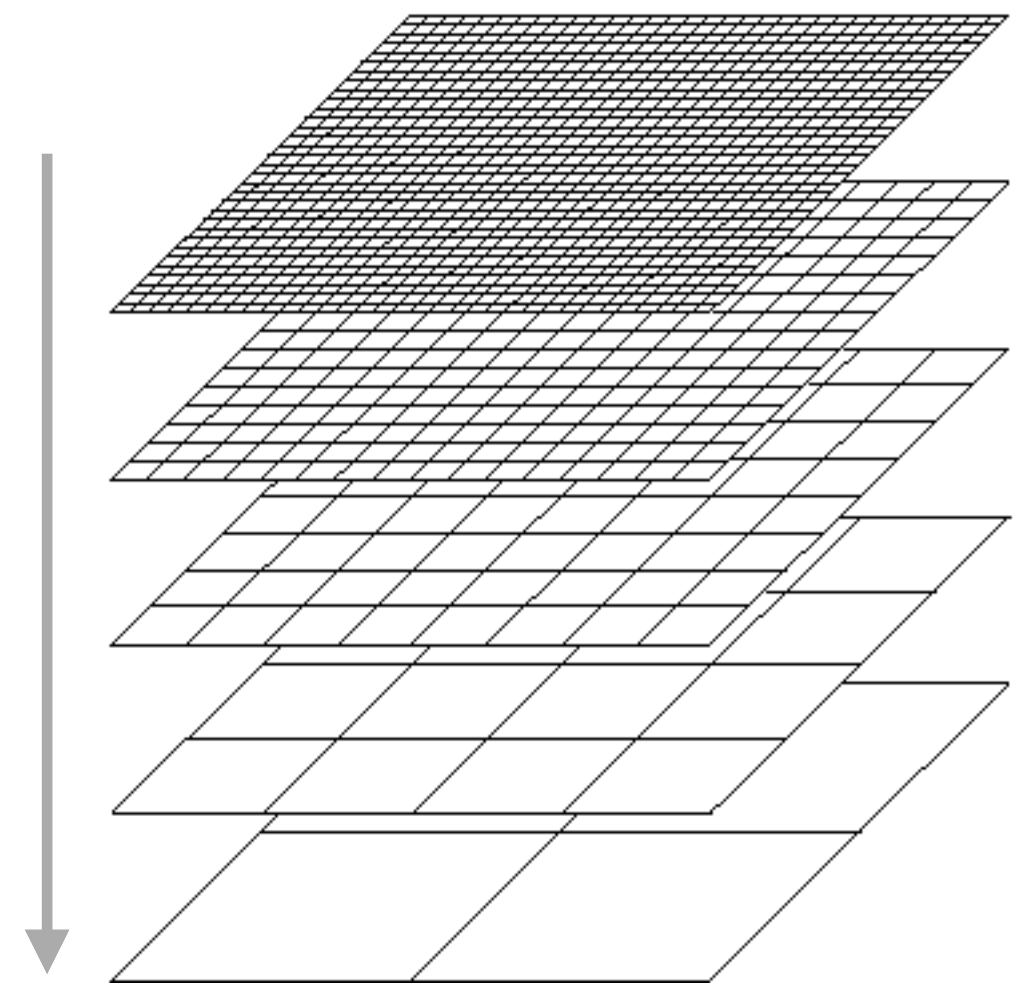
$$d = O(L^{D-1-\varepsilon})$$

and

$$\varepsilon = O(1/\sqrt{\log n})$$

- These codes reliably encode *almost as much information* as there is on the boundary.

Holographic information encoding



Highest level of concatenation

Total volume is $n = L^D$

Bacon, SF, A. Harrow, J. Shi, 2014

Conclusions

Consistent definition of approximate topological quantum codes

Geometry alone constrains information storage, even with an ε .

Fractional quantum Hall states?

Applications to Holography? (MERA codes, Kastoryano & Kim?)

Approximate Eastin-Knill theorem?

Subsystem code version?

SF, J. Haah, M. Kastoryano, I. Kim, Quantum **1**, 4 (2017), arXiv:1610.06169

Bacon, SF, A. Harrow, J. Shi, IEEE Tr. Inf. Th. **63**, 2464 (2017), arXiv:1411.3334