eLOQC[a] I

Linear Optics

- Optical/bosonic modes.
- Linear optics.
- Bosonic qubits.

[a] Efficient Linear Optics Quantum Computation [1]

knill@lanl.gov, http://www.lanl.gov/~knill

One Mode

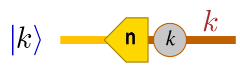
- States: $|0\rangle, |1\rangle, |2\rangle, \ldots$.
  $|n\rangle$: The mode has n photons.
- Operators:
  $\mathbf{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ (annihilation operator)
  $\mathbf{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$ (creation operator)
  $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a} = \sum_n n\,|n\rangle\langle n|$ (number operator)
- Commutation relationship:
  $[\mathbf{a}, \mathbf{a}^\dagger] = \mathbb{I}$.
- Examples:
  - Photons in the fundamental mode of a cavity.
  - Photon wave packet in motion.

To learn about quantum optics see Walls&Milburn 1994 [2]

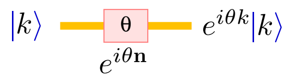
Gates for One Mode

- Mode preparations.

- Photon counter:



- Phase shifter:



…the simplest *linear optics* element.

Two Modes

- States of system with modes A and B:
  $|0\rangle_A|0\rangle_B, |1\rangle_A|0\rangle_B, |0\rangle_A|1\rangle_B, |2\rangle_A|0\rangle_B, \ldots$
  Define: $|a\rangle_A|b\rangle_B = |b\rangle_B|a\rangle_A = |ab\rangle_{AB}$.
- Operators:
  $(\mathbf{a})_B\,|a\rangle_A|b\rangle_B = |a\rangle_A\ (\mathbf{a})_B\,|b\rangle_B = \sqrt{b}\,|a\rangle_A|b-1\rangle_B$.
  $(\mathbf{a}^\dagger)_A\,|a\rangle_A|b\rangle_B = \sqrt{a+1}\,|a+1\rangle_A|b\rangle_B$.
  Define: $\mathbf{a}_S = (\mathbf{a})_S$
  $\mathbf{a}^\dagger{}_S = (\mathbf{a}^\dagger)_S$
- Beam splitter:
  $(\mathbf{B}_{\theta,\phi})_{AB} = e^{-i\left(e^{i\phi}\mathbf{a}^\dagger{}_A\mathbf{a}_B + e^{-i\phi}\mathbf{a}_A\mathbf{a}^\dagger{}_B\right)\theta}$

The Beamsplitter I

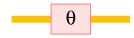
- The beamsplitter as a gate:

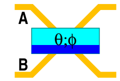
$$|1\rangle_A|0\rangle_B \;\rightarrow\; \cos(\theta)|1\rangle_A|0\rangle_B + e^{-i\phi}\sin(\theta)|0\rangle_A|1\rangle_B$$
$$|0\rangle_A|1\rangle_B \;\rightarrow\; -e^{i\phi}\sin(\theta)|1\rangle_A|0\rangle_B + \cos(\theta)|0\rangle_A|1\rangle_B$$

- Define: $\mathbf{B}_\theta = \mathbf{B}_{\theta,0}$.
- Example: Balanced beamsplitter $\mathbf{B}_{45^\circ}$.
  $\mathbf{B}_{45^\circ}|00\rangle = |00\rangle$.
  $\mathbf{B}_{45^\circ}|10\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$
  $\mathbf{B}_{45^\circ}|01\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$
  - **Problem 1:** $\mathbf{B}_{45^\circ}|11\rangle = ?$

Linear Optics

- $\mathcal{G}$: Operations obtained by composing phase shifters and beam splitters.

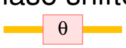


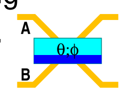


- $\mathcal{U}$: Unitary U such that for each mode k,
  $U\mathbf{a}^\dagger{}_k U^\dagger = \sum_l u_{lk}\mathbf{a}^\dagger{}_l$. Fock space view: No photon interactions.
  Let $\hat{u} = \hat{u}(U)$ be the matrix $(u_{lk})_{lk}$.
- **Theorem 2:** $U \in \mathcal{U}$ satisfies that $\hat{u}(U)$ is unitary.
  Proof: $\delta_{rs} = U\delta_{rs}U^\dagger$
  $$\begin{aligned} &= U[\mathbf{a}_r, \mathbf{a}^\dagger{}_s]U^\dagger \\ &= [U\mathbf{a}_r U^\dagger, U\mathbf{a}^\dagger{}_s U^\dagger] \\ &= \textstyle\sum_l \bar{u}_{lr}u_{ls}[\mathbf{a}_l, \mathbf{a}^\dagger{}_l] \\ &= (\hat{u}^\dagger\hat{u})_{rs} \end{aligned}$$
- **Corollary 3:** $U \in \mathcal{U}$ preserves the number of photons.

Linear Optics

- $\mathcal{G}$: Operations obtained by composing phase shifters and beam splitters.





- $\mathcal{U}$: Unitary U such that for each mode k,
  $U\mathbf{a}^\dagger{}_k U^\dagger = \sum_l u_{lk}\mathbf{a}^\dagger{}_l$. Fock space view: No photon interactions.
  Let $\hat{u} = \hat{u}(U)$ be the matrix $(u_{lk})_{lk}$.
- **Theorem 4:** $\mathcal{U} = \mathcal{G}$. Reck&Zeilinger&Bernstein 1994 [3]
- $\mathcal{U} = \mathcal{G}$ is the set of *passive linear optics* operations.
- Strategy: 1. Construct $\hat{u}$ as needed.
  2. Realize with phase shifters and beam splitters.

Go to: Example realization

Photons as Indistinguishable Particles

- Photons in second quantization for modes A, B, ...
  One photon wavefunctions: $\{\alpha|A\rangle + \beta|B\rangle + \ldots\}$.
  Two photon wavefunctions: Symmetrize state space for distinguishable particles (e.g. with names p and q).
  Ex.: $|A\rangle_p|B\rangle_q \rightarrow \frac{1}{\sqrt{2}}(|A\rangle_p|B\rangle_q + |B\rangle_p|A\rangle_q)$

  ⋮

- U unitary: $\left\{\begin{array}{l}\text{No interactions}\\ \text{indistinguishability}\\ \text{particle \# preserving}\end{array}\right\} \rightarrow U$ is $\hat{u}_k^{\otimes k}$ on k photons.
- U is a passive linear optics operation iff for some $\hat{u}$, U acts with $\hat{u}$ on each photon.
  - $\hat{u}$ is $\hat{u}(U)$ as defined before.

The Beamsplitter II

- $\hat{u}$ for a balanced beamsplitter:

$$\left.\begin{aligned} \mathbf{B}_{45^\circ}|10\rangle &= \tfrac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \\ \mathbf{B}_{45^\circ}|01\rangle &= \tfrac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{aligned}\right\} \quad \hat{u}(\mathbf{B}_{45^\circ}) = \frac{1}{\sqrt{2}} \begin{array}{cc} \begin{array}{cc} \mathsf{A} & \mathsf{B} \end{array} & \\ \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} & \begin{array}{c} \mathsf{A} \\ \mathsf{B} \end{array} \end{array}$$

- Compute $\mathbf{B}_{45^\circ}|11\rangle_{\mathsf{AB}}$: See Prob. 1.

  Using mode algebra (first quantization):

$$\begin{aligned} \mathbf{B}_{45^\circ}|11\rangle_{\mathsf{AB}} &= \mathbf{B}_{45^\circ}\mathbf{a}^\dagger{}_1\mathbf{a}^\dagger{}_2|\mathbf{0}\rangle \\ &= \left(\mathbf{B}_{45^\circ}\mathbf{a}^\dagger{}_1\mathbf{B}^\dagger_{45^\circ}\right)\left(\mathbf{B}_{45^\circ}\mathbf{a}^\dagger{}_2\mathbf{B}^\dagger_{45^\circ}\right)\mathbf{B}_{45^\circ}|\mathbf{0}\rangle \\ &= \tfrac{1}{\sqrt{2}}(\mathbf{a}^\dagger{}_1 + \mathbf{a}^\dagger{}_2)\tfrac{1}{\sqrt{2}}(\mathbf{a}^\dagger{}_2 - \mathbf{a}^\dagger{}_1)|\mathbf{0}\rangle \\ &= \tfrac{1}{2}(\mathbf{a}^{\dagger 2}_1 + \mathbf{a}^{\dagger 2}_2)|\mathbf{0}\rangle \\ &= \tfrac{1}{\sqrt{2}}(|20\rangle_{\mathsf{AB}} + |02\rangle_{\mathsf{AB}}) \end{aligned}$$

The Beamsplitter II

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- Compute $\mathbf{B}_{45^\circ}|11\rangle_{\mathsf{AB}}$: See Prob. 1.

  Using particles (second quantization):

$$\begin{aligned} \mathbf{B}_{45^\circ}|11\rangle_{\mathsf{AB}} &\simeq (\hat{u}\otimes\hat{u})\ \tfrac{1}{\sqrt{2}}\left(|\mathsf{A}\rangle_{\mathsf{p}}|\mathsf{B}\rangle_{\mathsf{q}} + |\mathsf{B}\rangle_{\mathsf{p}}|\mathsf{A}\rangle_{\mathsf{q}}\right) \\ &= \tfrac{1}{2\sqrt{2}}\Big((|\mathsf{A}\rangle_{\mathsf{p}} + |\mathsf{B}\rangle_{\mathsf{p}})(|\mathsf{B}\rangle_{\mathsf{q}} - |\mathsf{A}\rangle_{\mathsf{q}}) \\ &\qquad + (|\mathsf{B}\rangle_{\mathsf{p}} - |\mathsf{A}\rangle_{\mathsf{p}})(|\mathsf{A}\rangle_{\mathsf{q}} + |\mathsf{B}\rangle_{\mathsf{q}})\Big) \\ &= \tfrac{1}{\sqrt{2}}\left(|\mathsf{A}\rangle_{\mathsf{p}}|\mathsf{A}\rangle_{\mathsf{q}} + |\mathsf{B}\rangle_{\mathsf{p}}|\mathsf{B}\rangle_{\mathsf{q}}\right) \\ &\simeq \tfrac{1}{\sqrt{2}}(|20\rangle_{\mathsf{AB}} + |02\rangle_{\mathsf{AB}}) \end{aligned}$$

Physical Modes

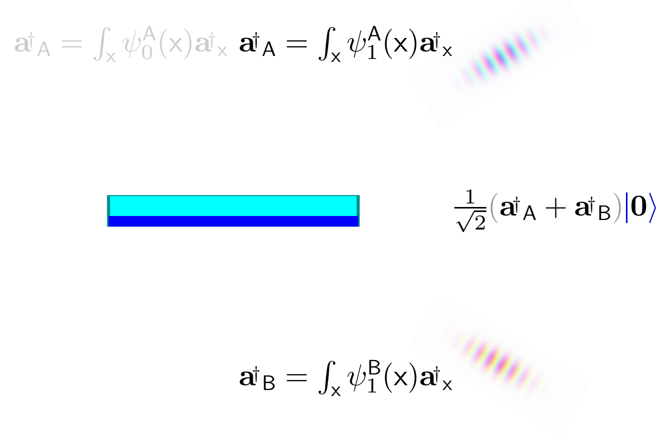
- One pure-state photon in space:
  $|\psi\rangle = \int_{\mathsf{x}} \psi_t(\mathsf{x})\mathbf{a}^\dagger{}_{\mathsf{x}}|\mathbf{0}\rangle$, with $\int_{\mathsf{x}} |\psi_t(\mathsf{x})|^2 = 1$.

  - Free space: $\psi_t(\mathsf{x})$ satisfies the wave equation.
  - Confinements: Cavities, fiber optics, mirrors.
- Identify an abstract mode A by its creation operator:
  $\mathbf{a}^\dagger{}_{\mathsf{A}} = \int_{\mathsf{x}} \psi^{\mathsf{A}}_t(\mathsf{x})\mathbf{a}^\dagger{}_{\mathsf{x}}$.
- For computing with modes A, B:
  $\psi^{\mathsf{A}}(\mathsf{x})$ and $\psi^{\mathsf{B}}(\mathsf{x})$ must be orthogonal.
  For example: Well separated in normal or in frequency space.

The basis-independent approach: I. Deutsch 1991 [4]

Physical Beamsplitter

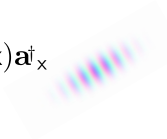
- Balanced beam splitter, before and after:



Physical Beamsplitter

- Balanced beam splitter, before and after:

$a^\dagger_A = \int_x \psi_0^A(x) a^\dagger_x$ $\mathbf{a}^\dagger_A = \int_x \psi_1^A(x)\mathbf{a}^\dagger_x$



$\frac{1}{\sqrt{2}}(\mathbf{a}^\dagger_B - \mathbf{a}^\dagger_A)|\mathbf{0}\rangle$

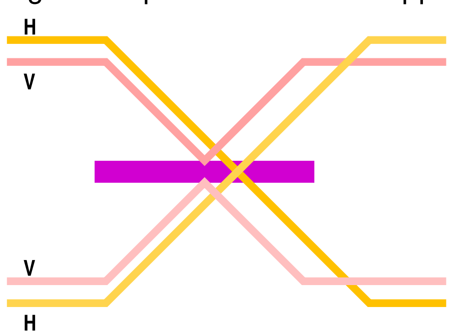
$a^\dagger_B = \int_x \psi_0^B(x) a^\dagger_x$ $\mathbf{a}^\dagger_B = \int_x \psi_1^B(x)\mathbf{a}^\dagger_x$

- Passive linear optics implies:

$\mathbf{a}^\dagger_B\mathbf{a}^\dagger_A|\mathbf{0}\rangle \quad \Rightarrow \quad \frac{1}{2}(\mathbf{a}^{\dagger 2}_B - \mathbf{a}^{\dagger 2}_A)$

Polarization

- Each spatial mode supports two polarization modes.
  "Vertical": $\int_x \psi_t^A(x)\mathbf{a}^\dagger_{x,V}$.
  "Horizontal": $\int_x \psi_t^A(x)\mathbf{a}^\dagger_{x,H}$.
- Polarizing beamsplitter: A mode swapping device.

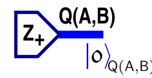


Bosonic Qubits: State space

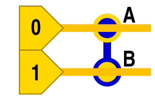
- Statespace of qubit Q: $\{\alpha|0\rangle_Q + \beta|1\rangle_Q\}$.
- Logical mapping to modes A, B:

$$|0\rangle_Q = |0\rangle_{Q(A,B)} \rightarrow |01\rangle_{AB}$$
$$|0\rangle_Q = |1\rangle_{Q(A,B)} \rightarrow |10\rangle_{AB}$$

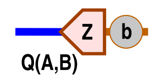
  Q(A, B) is defined by one photon in two modes.
- Preparation networks:



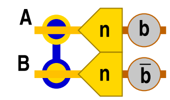
$\Rightarrow$



- Measurement networks:

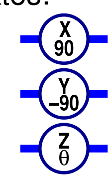


$\Rightarrow$

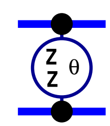


Rotations for Universal Computation

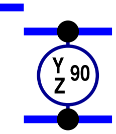
- One qubit rotations and gates:
  - $X_{90^\circ} = e^{-i\sigma_x\pi/4}$
  - $Y_{-90^\circ} = e^{i\sigma_y\pi/4}$
  - $Z_\theta = e^{-i\sigma_z\theta/2}$



- Two qubit rotations and gates:
  - $\left((Z)_1(Z)_2\right)_\theta = e^{-i(\sigma_z)_1(\sigma_z)_2\theta/2}$



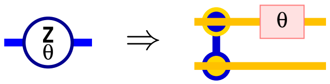
  - $\left((Y)_1(Z)_2\right)_{90^\circ} = e^{-i(\sigma_y)_1(\sigma_z)_2\pi/2}$



Bosonic Qubits: Rotations

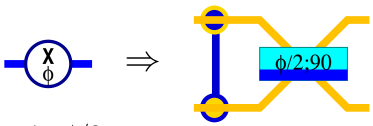
- $Z = \sigma_z$ rotation by θ, $e^{-i\sigma_z\theta/2}$:

$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$:



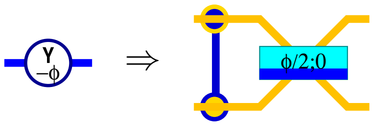
- $X = \sigma_x$ rotation by ϕ, $e^{-i\sigma_x\phi/2}$:

$\begin{pmatrix} \cos(\phi) & -i\sin(\phi) \\ -i\sin(\phi) & \cos(\phi) \end{pmatrix}$:



- $Y = \sigma_y$ rotation by $-\phi$, $e^{i\sigma_y\phi/2}$:

$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$:



Bosonic Qubits: Interactions?

- Sufficient to realize the conditional sign flip:
  CS : $|b_1b_2\rangle \rightarrow (-1)^{b_1b_2}|b_1b_2\rangle$.
- Passive and active linear optics:
  Evolutions of the form $\exp\left(-i(\text{quadratic in } \mathbf{a}\text{'s and } \mathbf{a}^\dagger\text{'s})t\right)$.
  - CS is not linear.
- Intuition: No interactions $\Rightarrow$ like classical waves.
- Facts:
  - One photon, N modes $\simeq$ N-level system
    $\simeq$ N classical coupled oscillators.
  - Coherent states: Preserved by passive linear optics.
  - Gaussian states: Preserved by linear optics.

Linear Optics No-Go?

- Not universal:
  Group of linear optics operations on n modes:
  $\simeq$ A group of affine linear operations on $\mathbb{R}^{2n}$.
- **Theorem 5:** The complete Bell measurement is impossible with linear optics, helper modes in known states, photon counters and feedback. Lütkenhaus&Calsamiglia&Suominen 1999 [5]
- **Conjecture 6:** Number state preparations, linear optics and measurement of exactly one mode (no feedback) are efficiently classically simulatable.
- **Conjecture 7:** Coherent state preparations, passive linear optics and photodectors with feedback are efficiently classically simulatable. Ivan Deutsch 2001

More No-Go's?

- **Theorem 8:** Linear optics and particle detectors for fermions are efficiently classically simulatable.
  Valiant 2001 [6], Terhal&DiVincenzo 2001 [7], Knill 2001 [8]
- **Theorem 9:** Linear optics and measurement of $\hat{x} \propto \mathbf{a}^\dagger + \mathbf{a}$ is efficiently classically simulatable. Bartlett&*al.* 2001 [9]
- Explanation:
  The projectors associated with measurement outcomes are in the closure of $e^{\mathbb{C}\mathfrak{g}}$, where $\mathfrak{g}$ is the linear optics Lie algebra in each case.
  But for bosons, $\mathbb{I} - |\mathbf{0}\rangle\langle\mathbf{0}|$ and $|k\rangle\langle k|$ for $k \neq 0$ (associated with photon counter outcomes) is not of this form.

Contents

References

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[4] I. H. Deutsch. A basis-independent approach to quantum optics. *Am. J. Phys.*, 59:834–839, 1991.

[5] N. Lütkenhaus, J. Calsamiglia, and K.-A. Suominen. Bell measurements for teleportation. *Phys. Rev. A*, 59:3295–3300, 1999.

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[7] B. M. Terhal and D. P. DiVincenzo. Classical simulation of quantum circuits using non-interacting fermions. quant-ph/0108010, 2001.

[8] E. Knill. Fermionic linear optics and matchgates. Technical Report LAUR-01-4472, Los Alamos National Laboratory, 2001. quant-ph/0108033.

[9] S. D. Bartlett, B. C. Sanders, S. L. Braunstein, and K. Nemoto. Efficient classical simulation of continuous variable quantum information processes. quant-ph/0109047, 2001.