eLOQC[a] V

Efficient Computation: Operations and Thresholds

- Z-measurement-robust teleportation.
- Z-measurement-robust encoded operations.
- Z-measurement-error accuracy threshold.

[a] Efficient Linear Optics Quantum Computation [1]

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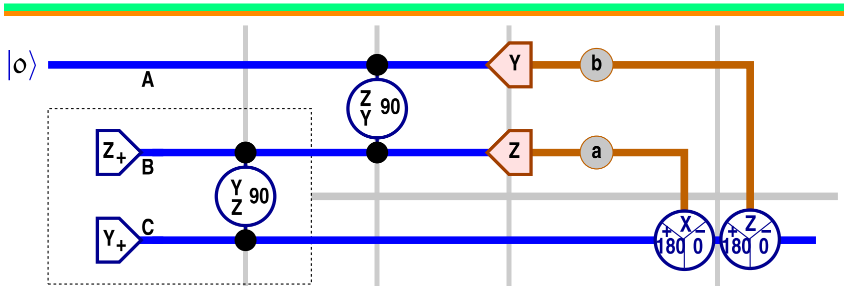
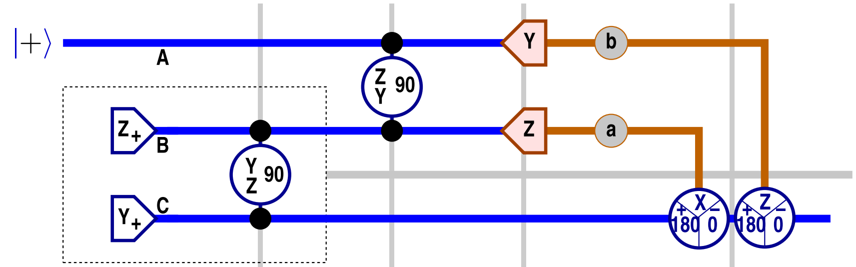
Z-Measurement-Error Preserving Teleportation



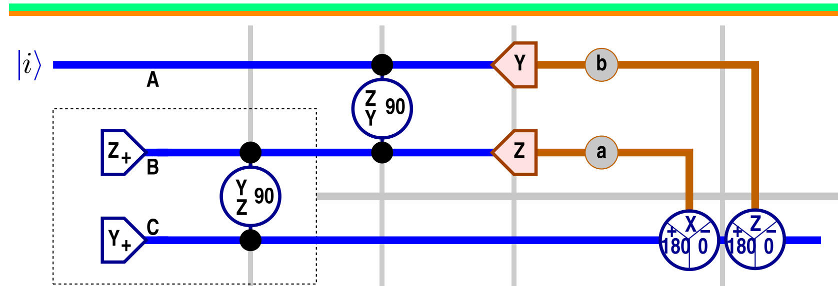
Stabilizer:	ZII	ZII	ZII		
$(\lambda = 1)$	IZI	IXZ	$-ZZZ$		
	IIY	$-IYX$	$-IYX$		
			$\downarrow$		
			ZII	$*$	$*$
			$-IZZ$	$-aZ$	Z
			$-IYX$	$*$	$*$

Z-Measurement-Error Preserving Teleportation



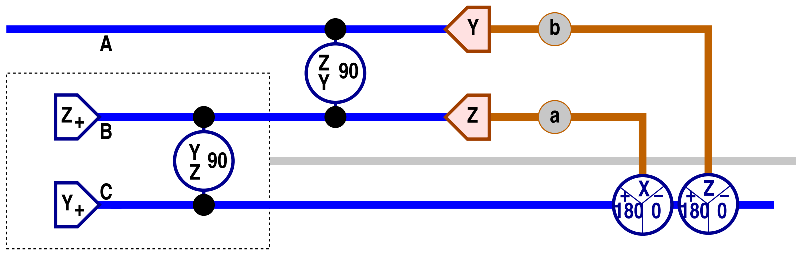
Stabilizer:	XII	XII	YYI		
$(\lambda = 1)$	IZI	IXZ	$-ZZZ$		
	IIY	$-IYX$	$-IYX$		
			$\downarrow$		
			$-YIX$	$-bX$	X
			$-ZZZ$	$*$	$*$
			$-IYX$	$*$	$*$

Z-Measurement-Error Preserving Teleportation



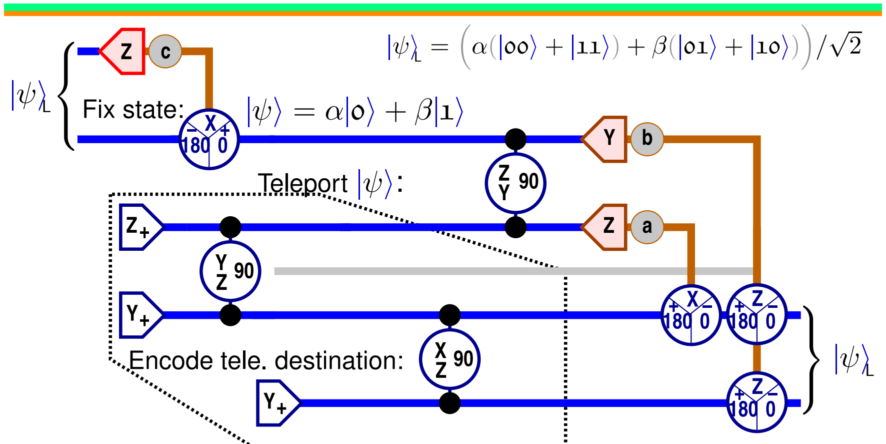
Stabilizer:	YII	YII	$-XYI$		
$(\lambda = 1)$	IZI	IXZ	$-ZZZ$		
	IIY	$-IYX$	$-IYX$		
			$\downarrow$		
			$-XYI$	$*$	$*$
			$-ZZZ$	$*$	$*$
			YZY	baY	Y

Z-Measurement-Error Preserving Teleportation



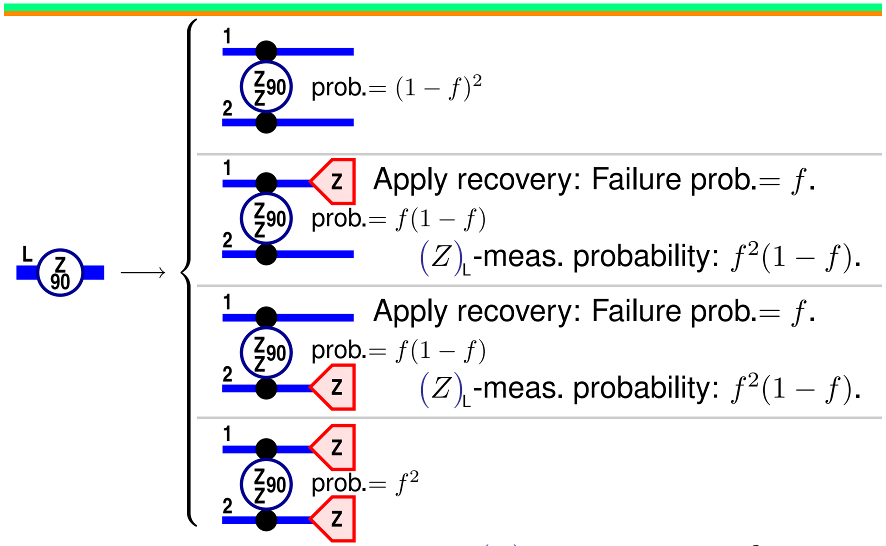
- Use sequential teleportation to implement $(ZYI)_{90^\circ}$:
  1. Teleport B. If failed (prob. f): try again. Else (prob. $1-f$):
  2. Teleport A. Failure (prob. f): Measure $(Z)_\mathrm{A}$.
- Success probability:
  $\big((1-f)+f(1-f)+f^2(1-f)+\ldots\big)(1-f)=1-f$.
- Failure behavior: Z-measurement of A.

Recovery from Z-measurement Error



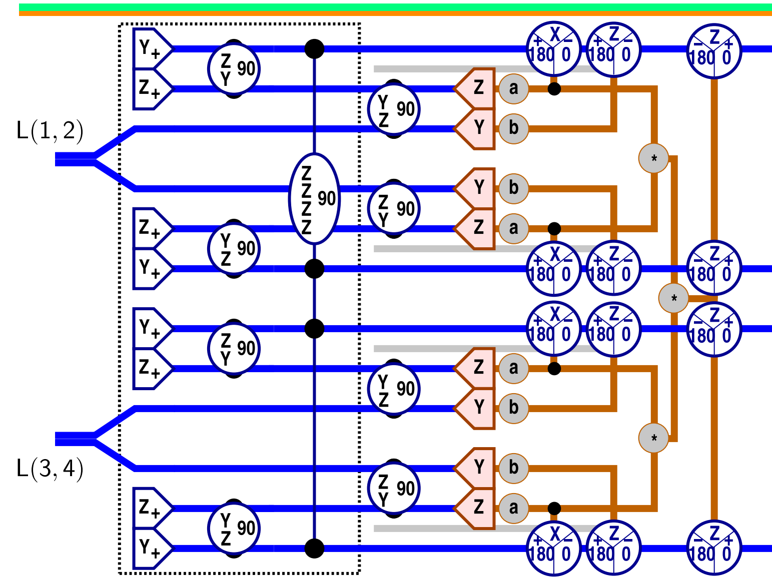
- Analysis:
  - Failure: Probability f, by $(Z)_\mathrm{L}$ measurement.

Failure Analysis of Encoded $(Z)_{90^\circ}$



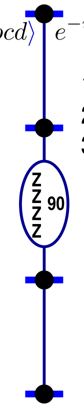
Overall $(Z)_\mathrm{L}$-meas. prob.$= f^2(3-2f)$.

Encoded $(ZZ)_{90^\circ}$

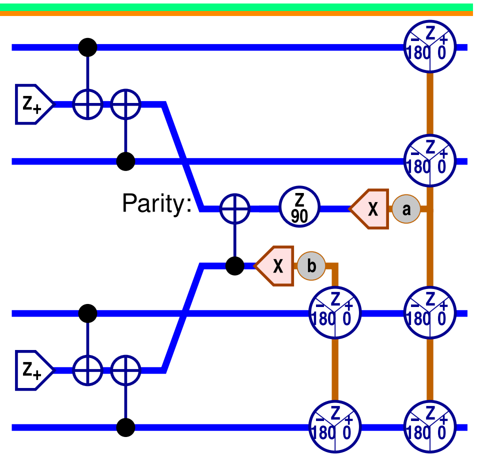


Encoded $(ZZ)_{90^\circ}$: State Preparation

$|abcd\rangle$ $e^{-i((-1)^{a\oplus b\oplus c\oplus d})\pi/4}|abcd\rangle$

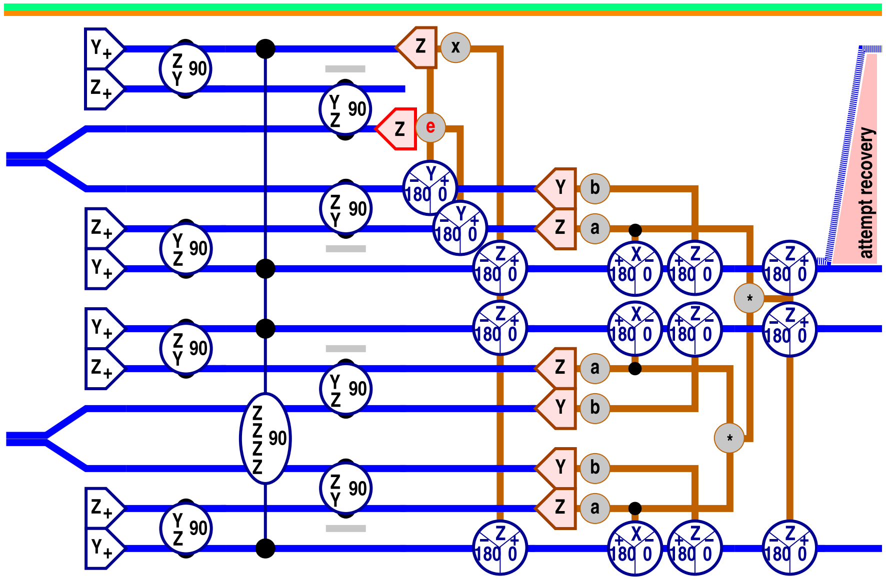


1. Parity tree
2. $(Z)_{90^\circ}$
3. Eliminate parities using $(I \pm X)$. $(I - X)$ kicks in signs!



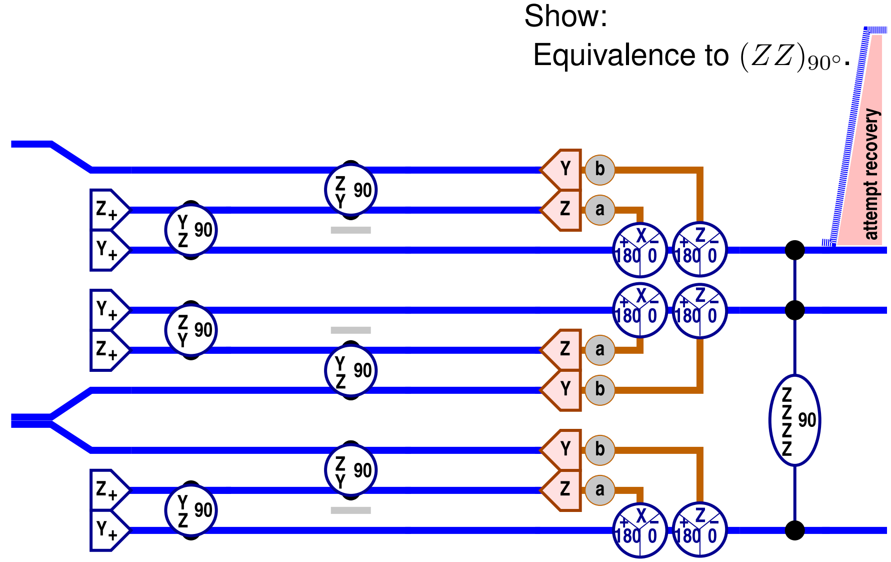
- Generalizes to $(ZZ\ldots Z)_{90^\circ}$:
  - With teleportation state preps: Logarithmic dependency depth $\rightarrow$ polynomial resources.

$(ZZ)_{90^\circ}$: Typical Failure Response



$(ZZ)_{90^\circ}$: Typical Failure Response

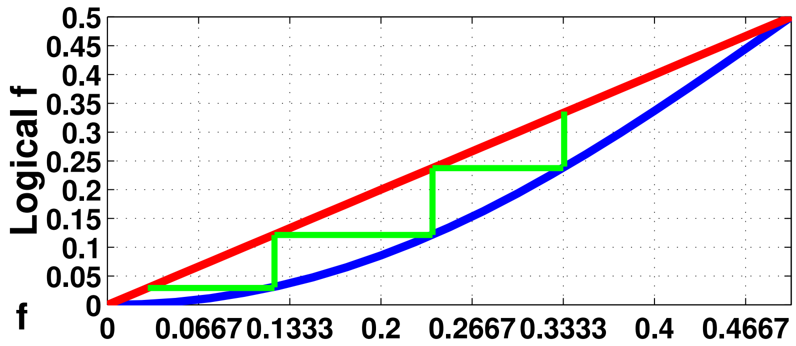
Show:
Equivalence to $(ZZ)_{90^\circ}$.



$(ZZ)_{90^\circ}$: Failure Analysis

- Initiate teleportations simultaneously.
- Independently on each encoded qubit:
  - prob.$= f^2$: Encoded Z-measurement error.
  - prob.$= (1-f)^2$: Success.
  - prob.$= 2(1-f)f$: One Z-measurement error.
    - prob.$= (1-f)$: Re-encoding succeeds.
    - prob.$= f$: Encoded Z-measurement error.
- Encoded Z-measurement errors are independent, effectively after the encoded $(ZZ)_{90^\circ}$.
- Failure probability/encoded qubit: $f^2 + 2f^2(1-f) = f^2(3-2f)$.

Threshold Analysis



- Logical f: $f_L = f^2(3-2f)$.
- Concatenation of $\mathcal{X}_2$ works for $f < .5$.
- Z-measurement-error accuracy threshold is at least $.5$.

Using Other Classical Linear Codes I

- From X-only stabilizers to binary codes: MacWilliams&Sloan (1977) [2]

$$S = \begin{pmatrix} X\,X\,I\,\,I\ldots \\ I\,X\,X\,X\ldots \\ \vdots \end{pmatrix} \quad\longleftrightarrow\quad P = \begin{pmatrix} 1100\ldots \\ 0111\ldots \\ \vdots \end{pmatrix}$$

  - Example: $S_3 = \begin{pmatrix} X\,X\,I \\ I\,X\,X \end{pmatrix} \quad\longleftrightarrow\quad P_3 = \begin{pmatrix} 110 \\ 011 \end{pmatrix}$

- P is the "parity check matrix" for a code C: $C = \{x : Px^T = \mathbf{0}\}$
  - Example: $\begin{pmatrix} 110 \\ 011 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 110 \\ 011 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

    $C = \{000, 111\}$. (In $\mathbb{Z}_2$, i.e. modulo 2.)

- $\big($ Dimension of C over $\mathbb{Z}_2$ $\big) = \big($ # of qubits encoded by $C(S)$ $\big)$.

Using Other Classical Linear Codes II

- Minimum distance d of C: $\min\{|x| : x \in C\}$. ($|x|$: # of 1's in x.)
- **Theorem 1:** If P is the parity check matrix of a minimum distance d code then any combination of up to $d-1$ Z-measurement errors is correctable.

  Proof: Let x's 1's indicate the Z-measurement errors.
  - If $|x| < d$ and $y \leq x$ (coordinatewise), then $Py^T \neq \mathbf{0}$.
    Def: $Z(y)$ is obtained from y by replacing $0 \to I$, $1 \to Z$.
  - $Py^T \neq \mathbf{0}$ if and only if $Z(y)$ is detectable by S.
  - See: The characterization of correctability.

  - Example: S_3 corrects detected $(I \pm Z)I(I \pm Z)/4 \quad\leftrightarrow\quad x = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ldots$
- Conclusion:
  Use the parity check matrix P of a "good" binary code.

$\mathcal{X}_n$: The Repetition Code

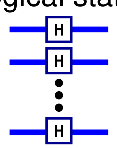
- Codewords, length n: $C_n = \{00\ldots0^T, 11\ldots1^T\}$.
- Parity check matrix $\to$ stabilizer:

$$P_n = \begin{pmatrix} 1100\ldots \\ 1010\ldots \\ 1001\ldots \\ \vdots \end{pmatrix} \quad\longrightarrow\quad S_n = \begin{pmatrix} X\,X\,I\,\,I\ldots \\ X\,I\,X\,I\ldots \\ X\,I\,\,I\,X\ldots \\ \vdots \end{pmatrix}$$

- Logical states: $|\mathfrak{o}\rangle_{\mathsf{L}} = \frac{1}{\sqrt{2^{n-1}}}\left(\sum_{\mathfrak{e}:|\mathfrak{e}|\text{even}} |\mathfrak{e}\rangle\right)$ $\quad$ $|\mathfrak{e}|$ is the # of $\mathfrak{1}$'s in $\mathfrak{e}$.

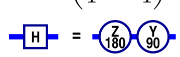
  $|\mathfrak{1}\rangle_{\mathsf{L}} = \frac{1}{\sqrt{2^{n-1}}}\left(\sum_{\mathfrak{o}:|\mathfrak{o}|\text{odd}} |\mathfrak{o}\rangle\right)$
- From "cat" states to $\mathcal{X}_n$'s logical states:

$$\left.\begin{matrix} \frac{1}{\sqrt{2}}\big(|\mathfrak{oo}\ldots\mathfrak{o}\rangle + |\mathfrak{11}\ldots\mathfrak{1}\rangle\big) \\ \frac{1}{\sqrt{2}}\big(|\mathfrak{oo}\ldots\mathfrak{o}\rangle - |\mathfrak{11}\ldots\mathfrak{1}\rangle\big) \end{matrix}\right\}$$



$$\left\{\begin{matrix} |\mathfrak{o}\rangle_{\mathsf{L}} \\ |\mathfrak{1}\rangle_{\mathsf{L}} \end{matrix}\right.$$

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

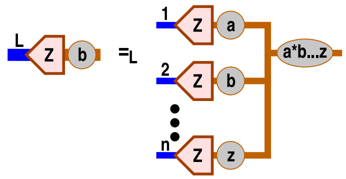


$\mathcal{X}_n$: Encoded Operations

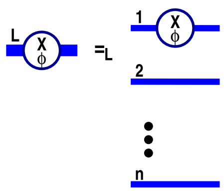
- Encoded observables:

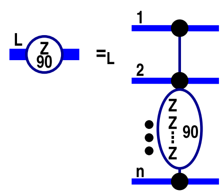
$$(X)_\mathsf{L} = XI\ldots I \qquad (Y)_\mathsf{L} = YZ\ldots Z$$
$$(Z)_\mathsf{L} = ZZ\ldots Z$$

- Encoded Z measurement:



- Some encoded operations:

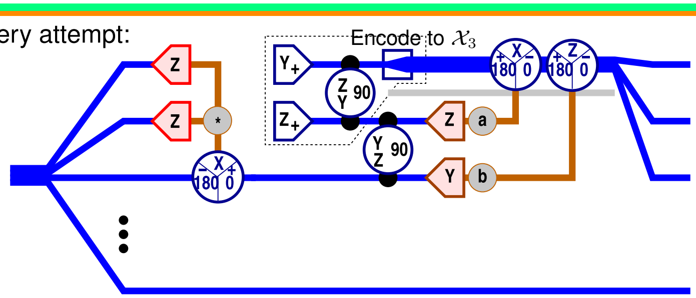




- Encoded $(ZZ)_{90^\circ}$ by teleportation.

The Z-Measurement Threshold I

Recovery attempt:



- Recovery:
  - Requires only one surviving qubit.
  - Succeeds with probability $1-f$.
  - Failure adds one Z-measurement error.
- Overall recovery success probability for $\mathcal{X}_n$: $\to 1$ $(n \to \infty)$.
- Accuracy threshold for the Z-measurement error model is 1.

The Z-Measurement Threshold II

- Encoded Z-measurement error probability with $\mathcal{X}_n$:

$$\sum_{k=1}^{n} \quad \binom{n}{k} \quad f^k(1-f)^{n-k} \quad f^{n-k}$$

↑ k: # of failed qubit teleportations.

↑ # of k-subsets.

↑ k fail, $n-k$ succeed.

↑ All attempted recoveries from good qubits fail.

$$= (f+f(1-f))^n - f^n(1-f)^n$$
$$= f^n((2-f)^n - (1-f)^n) = f_{n,L}.$$

- Thresholds for concatenation with $\mathcal{X}_n$:

n	2	3	4	5	9	13	17
f	0.500	0.630	0.720	0.780	0.883	0.920	0.939

- Can use $\mathcal{X}_3$ with simplest postselected LOQC gates.
  - Large $n \to$ difficult, low-efficiency state preps.

  …tradeoffs need to be optimized.

Contents

References

[1] E. Knill, R. Laflamme, and G. Milburn. A scheme for efficient linear optics quantum computation. *Nature*, 409:46–52, 2001.
[2] F. J. MacWilliams and N. J. Sloane. *The Theory of Error-Correcting Codes*. North-Holland Publishing Company, 1977.