Bit Threads & Holographic Entanglement

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Mostly based on arXiv:1604.00354 [hep-th] with Michael Freedman

Contents

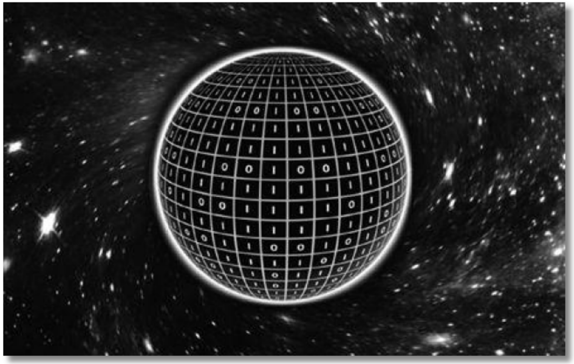
1 How should one think about the minimal surface?

In semiclassical gravity, surface areas are related to entropies

Bekenstein-Hawking '74:

For black hole,

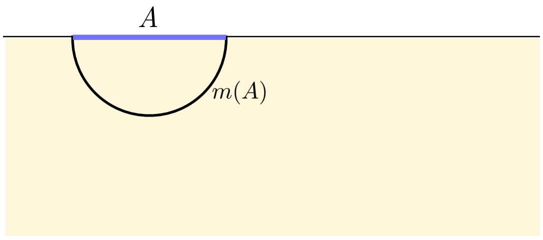
$$S = \frac{1}{4G_\mathrm{N}}\,\mathrm{area}(\text{horizon})$$



Why?

Possible answer:

Microstate bits "live" on horizon, 1 bit/4 Planck areas



Ryu-Takayanagi '06: For region in holographic field theory (classical Einstein gravity, static state)

$$S(A) = \frac{1}{4G_\mathrm{N}}\,\mathrm{area}(m(A))$$

$m(A) =$ bulk minimal surface homologous to A

Bulk geometry packages entanglement entropies in a simple & beautiful way

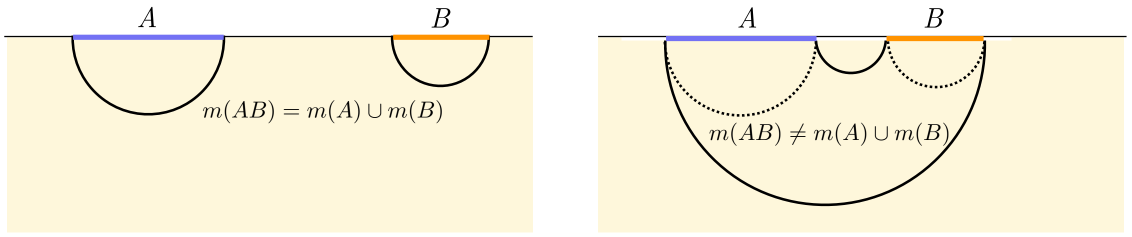
Do microstate bits of A "live" on $m(A)$?

Unlike horizon, $m(A)$ is not a special place; by choosing A, we can put $m(A)$ almost anywhere

Puzzles:

- Under continuous changes in boundary region, minimal surface can jump

  Example: Union of separated regions A, B



- Information-theoretic quantities are given by differences of areas of surfaces passing through different parts of bulk:

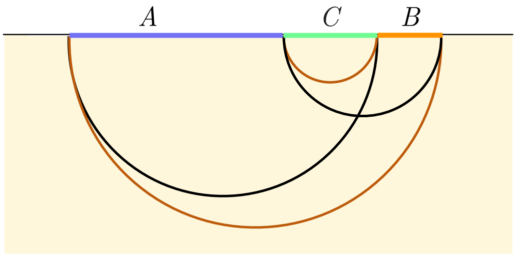
Conditional entropy:	$H(A\|B) = S(AB) - S(B)$
Mutual information:	$I(A:B) = S(A) + S(B) - S(AB)$
Conditional mutual information:	$I(A:B\|C) = S(AB) + S(BC) - S(ABC) - S(C)$

What do differences between areas of surfaces, passing through different parts of bulk, have to do with these measures of information?

- RT obeys strong subadditivity [Headrick-Takayanagi '07]

$$I(A:BC) \geq I(A:C)$$

  What does proof (by cutting & gluing minimal surfaces) have to do with information-theoretic meaning of SSA (monotonicity of correlations)?



To try to answer these questions, I will present a new formulation of RT

- Does not refer to minimal surfaces (demoted to a calculational device)
- Suggests a new way to think about the holographic principle & about the connection between spacetime geometry and information

2 Reformulation of RT

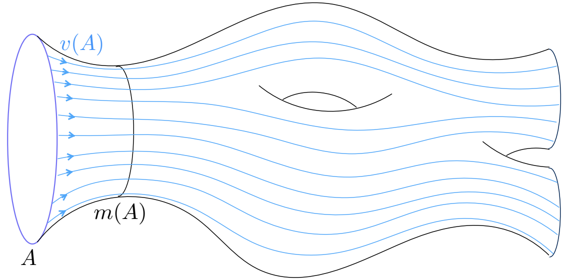
Consider a Riemannian manifold with boundary

Flow: vector field v obeying $\nabla \cdot v = 0$, $|v| \leq 1$
Think of flow as a set of oriented threads (flow lines) beginning & ending on boundary,
transverse density $= |v| \leq 1$

Let A be a subset of boundary

Max flow-min cut theorem (originally on graphs; Riemannian version: [Federer '74, Strang '83, Nozawa '90]):

$$\max_v \int_A v = \min_{m \sim A} \text{area}(m)$$



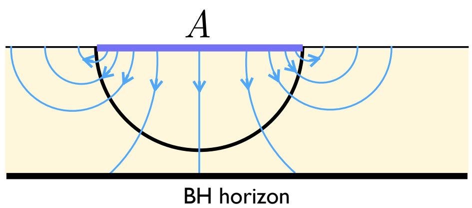
(Headrick-Hubeny '17 contains exposition of proof:
Finding max flow is convex program, related by Lagrangian duality & convex relaxation to finding minimal surface

Also prove a *min flow-max cut* theorem for Lorentzian spacetimes, relating maximal-volume slices to minimal-flux flows, where a *flow* is a future-directed timelike vector field with norm ≥ 1)

Note that max flow is highly non-unique (except on $m(A)$, where $v =$ unit normal)

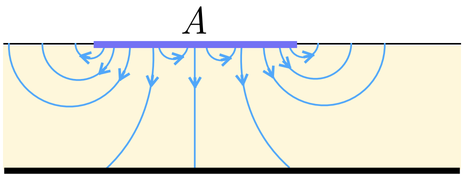
RT version 2.0:

$$\begin{aligned} S(A) &= \max_v \int_A v \qquad\qquad (4G_N = 1) \\ &= \text{max \# of threads beginning on } A \end{aligned}$$



Threads can end on A^c or horizon
Each thread has cross section of 4 Planck areas & is identified with 1 (independent) bit of A

Automatically incorporates homology & global minimization conditions of RT



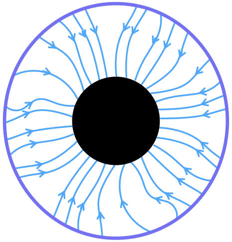
Threads are "floppy": lots of freedom to move them around in bulk & move where they attach to A

Also lots of room near boundary to add extra threads that begin & end on A (don't contribute to $S(A)$)

Role of minimal surface: bottleneck, where threads are maximally packed, hence counted by area

Holographic principle: entropy $\propto$ area because bits are carried by one-dimensional objects

Bekenstein-Hawking:

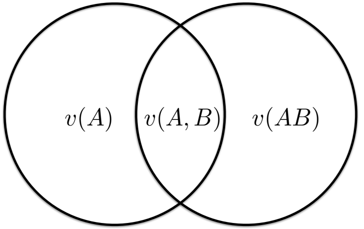


3 Threads & information

Now we address conceptual puzzles with RT raised before

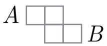
First: even when $m(A)$ jumps, $v(A)$ changes continuously with A

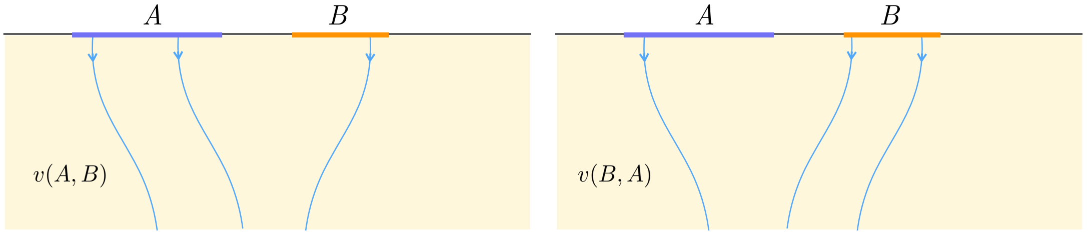
Next, consider two regions A, B



We can maximize flux through A *or* B, not in general both
But we *can* always maximize through A *and* AB (nesting property)
Call such a flow $v(A, B)$

Example 1: $S(A) = S(B) = 2,\ S(AB) = 3 \Rightarrow I(A : B) = 1,\ H(A|B) = 1$



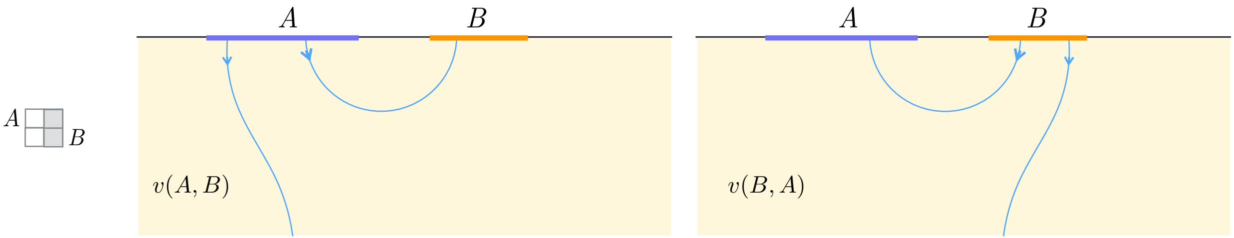


Lesson 1:

- Threads that are stuck on A represent bits unique to A
- Threads that can be moved between A & B represent correlated pairs of bits

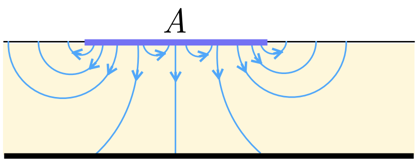
Example 2: $S(A) = S(B) = 2,\ S(AB) = 1 \Rightarrow I(A : B) = 3,\ H(A|B) = -1 \Rightarrow$ entanglement

One thread leaving A *must* go to B, and vice versa



Lesson 2:

- Threads that connect A & B (switching orientation) represent entangled pairs of qubits



Apply lessons to single region:

- freedom to move beginning points around reflects correlations within A
- freedom to add threads that begin & end on A reflects entanglement within A

Equations:

Conditional entropy:

$$
\begin{aligned}
H(A|B) &= S(AB) - S(B) \\
&= \int_{AB} v(AB) - \int_B v(B) \\
&= \int_{AB} v(B,A) - \int_B v(B,A) \\
&= \int_A v(B,A) \\
&= \text{min flux on } A \text{ (maximizing on } AB\text{)}
\end{aligned}
$$

Mutual information:

$$
\begin{aligned}
I(A:B) &= S(A) - H(A|B) \\
&= \int_A v(A,B) - \int_A v(B,A) \\
&= \text{max} - \text{min flux on } A \text{ (maximizing on } AB\text{)} \\
&= \text{flux movable between } A \text{ and } B \text{ (maximizing on } AB\text{)}
\end{aligned}
$$

Subadditivity is clear

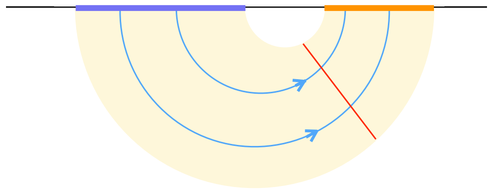
Max flow can be defined even when flux is infinite: flow that cannot be augmented

Regulator-free computation of mutual information:

$$
I(A:B) = \int_A (v(A,B) - v(B,A))
$$

Define

$$v(A:B) = \frac{1}{2}\left(v(A,B) - v(B,A)\right)$$



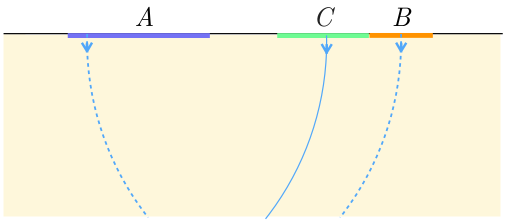
Flow from A to B through homology region $r(AB)$ w/flux $\frac{1}{2}I(A:B)$

Implies

$$\frac{1}{2}I(A:B) \leq \text{cross section of } r(AB)$$

(Takayanagi-Umemoto '17, Nguyen et al. '17 use this to interpret cross section as entanglement of purification)

Conditional mutual information:



$$\begin{aligned} I(A:B|C) &= H(A|C) - H(A|BC) \\ &= \int_A v(C,A,B) - \int_A v(C,B,A) \\ &= \text{max} - \text{min flux on } A \text{ (maximizing on } C \text{ \& } ABC) \\ &= \text{flux movable between } A \text{ \& } B \text{ (maximizing on } C \text{ \& } ABC) \\ &= \text{(flux movable between } A \text{ \& } BC) - \text{(movable between } A \text{ \& } C) \\ &= I(A:BC) - I(A:C) \end{aligned}$$

Strong subadditivity $I(A:B|C) \geq 0$ is clear

In each case, clear connection to information-theoretic meaning of quantity/property

4 Monogamy of mutual information

Work in progress with Shawn Cui, Patrick Hayden, Temple He, Bogdan Stoica, Michael Walter

Given a 3-party state ρ_{ABC}, define *tripartite information*:

$$\begin{aligned} -I_3(A:B:C) &:= S(AB) + S(BC) + S(AC) - S(A) - S(B) - S(C) - S(ABC) \\ &= I(A:BC) - I(A:B) - I(A:C) \end{aligned}$$

Cases:

ρ_{ABC}	$-I_3(A:B:C)$
pure	0
$\rho_A \otimes \rho_{BC}$	0
marginal of higher-party GHZ, e.g. $\lvert\psi\rangle_{ABCD} = (\lvert 0000\rangle + \lvert 1111\rangle)/\sqrt{2}$	< 0
marginal of 4-party perfect tensor (e.g. 4-qutrit code)	> 0

Hayden-Headrick-Maloney '11 used cutting-and-pasting of minimal surfaces to show that RT implies

$$-I_3(A:B:C) \geq 0$$

"Monogamy of mutual information" (MMI)

Suggests that perfect-tensor entanglement dominates over GHZ-type entanglement in holographic states

(also true in random stabilizer tensor networks [Nezami-Walter '16])

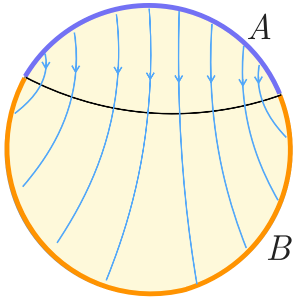
Can prove MMI using flows

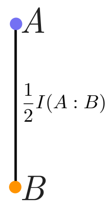
Use *multi-commodity flows* to obtain a geometric decomposition of the bulk into parts representing different kinds of entanglement

First consider simplest case where AB is pure (no C):

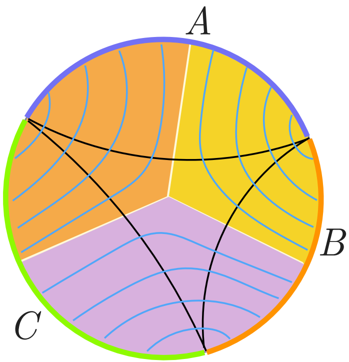
Bulk is "bridge" connecting A and B,

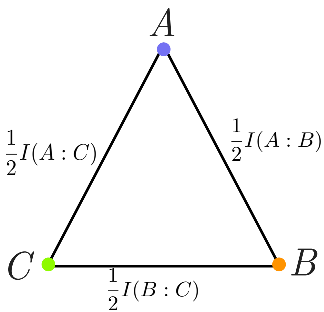
capacity $= S(A) = S(B) = \frac{1}{2}I(A:B)$





Now suppose ABC is pure



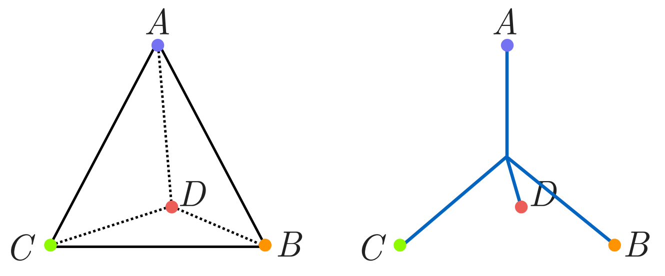


Bulk can be decomposed into

- $A - B$ bridge with capacity $\frac{1}{2}I(A:B)$
- $B - C$ bridge with capacity $\frac{1}{2}I(B:C)$
- $A - C$ bridge with capacity $\frac{1}{2}I(A:C)$

Now the general case: ABC is mixed; let $D =$ rest of boundary, so $ABCD$ is pure
Decomposition of bulk into 7 pieces:

- $A - B$ bridge with capacity $\frac{1}{2}I(A:B)$
- $A - C$ bridge with capacity $\frac{1}{2}I(A:C)$
- $A - D$ bridge with capacity $\frac{1}{2}I(A:D)$
- $B - C$ bridge with capacity $\frac{1}{2}I(B:C)$
- $B - D$ bridge with capacity $\frac{1}{2}I(B:D)$
- $C - D$ bridge with capacity $\frac{1}{2}I(C:D)$
- 4-way bridge with capacity on each leg $-\frac{1}{2}I_3(A:B:C)$



2-way bridges account for all pairwise MIs; 4-way bridge accounts for all of $-I_3(A:B:C)$

These are the extremal rays of 4-party holographic entropy cone

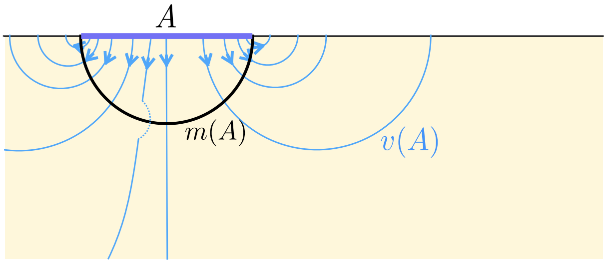
Question:
Does the *state* enjoy a similar (approximate) decomposition, into pairwise Bell pairs and 4-party perfect tensors?

5 No time left

5.1 Quantum corrections

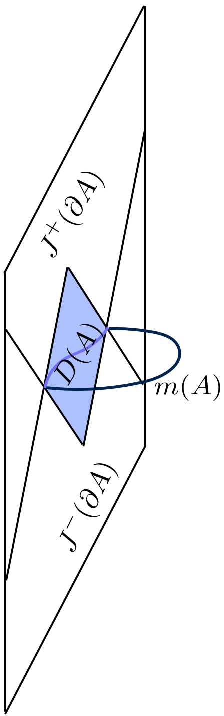
Faulkner-Lewkowycz-Maldacena '13:
Quantum (order G_N^0) correction to RT is from entanglement of bulk fields

May be reproduced by allowing threads to jump from one point to another
(or tunnel through microscopic wormholes, à la ER = EPR [Maldacena-Susskind '13])



5.2 Covariant bit threads

Work in progress with Veronika Hubeny



Hubeny-Rangamani-Takayanagi ['07] covariant entanglement entropy formula:

$$S(A) = \mathrm{area}(m(A))$$

$m(A) =$ minimal extremal surface homologous to A

Generalization of max flow-min cut theorem to Lorentzian setting:

A *flow* is still a divergenceless vector field v

Also have "clock function" ϕ, with boundary condition

$$\phi = 0 \quad \text{on } J^-(\partial A)\,, \qquad \phi = 1 \quad \text{on } J^+(\partial A)$$

Density of threads in rest frame of observer with 4-velocity u is $|v|_u := \sqrt{v^2 + (u \cdot v)^2}$

Density constraint: $|v|_u \le u^\mu \partial_\mu \phi$ for any timelike unit vector u

Observer w/proper time τ sees density $\le d\phi/d\tau$

Theorem (assuming NEC, using results of Wall '12 & Headrick-Hubeny-Lawrence-Rangamani '14):

$$\max_v \int_{D(A)} v = \mathrm{area}(m(A)) \qquad\qquad D(A) = \text{boundary causal domain of } A$$

Finding extremal surface area becomes convex optimization problem

HRT version 2.0:

$$S(A) = \max_v \int_{D(A)} v$$

To maximize flux, threads seek out $m(A)$, automatically confining themselves to entanglement wedge

Threads can lie on common Cauchy slice (equivalent to Wall's ['12] maximin by standard max flow-min cut) or spread out in time