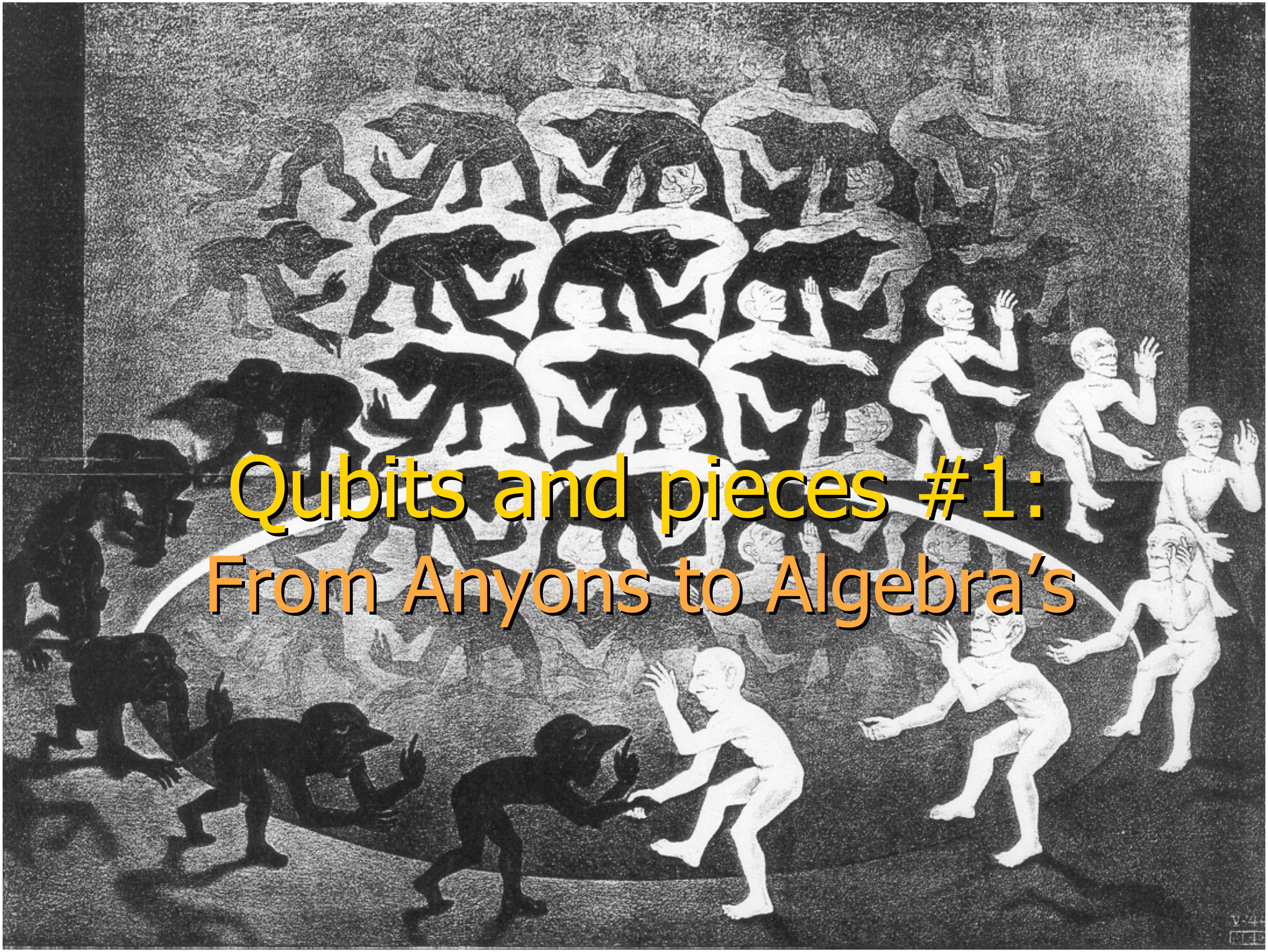
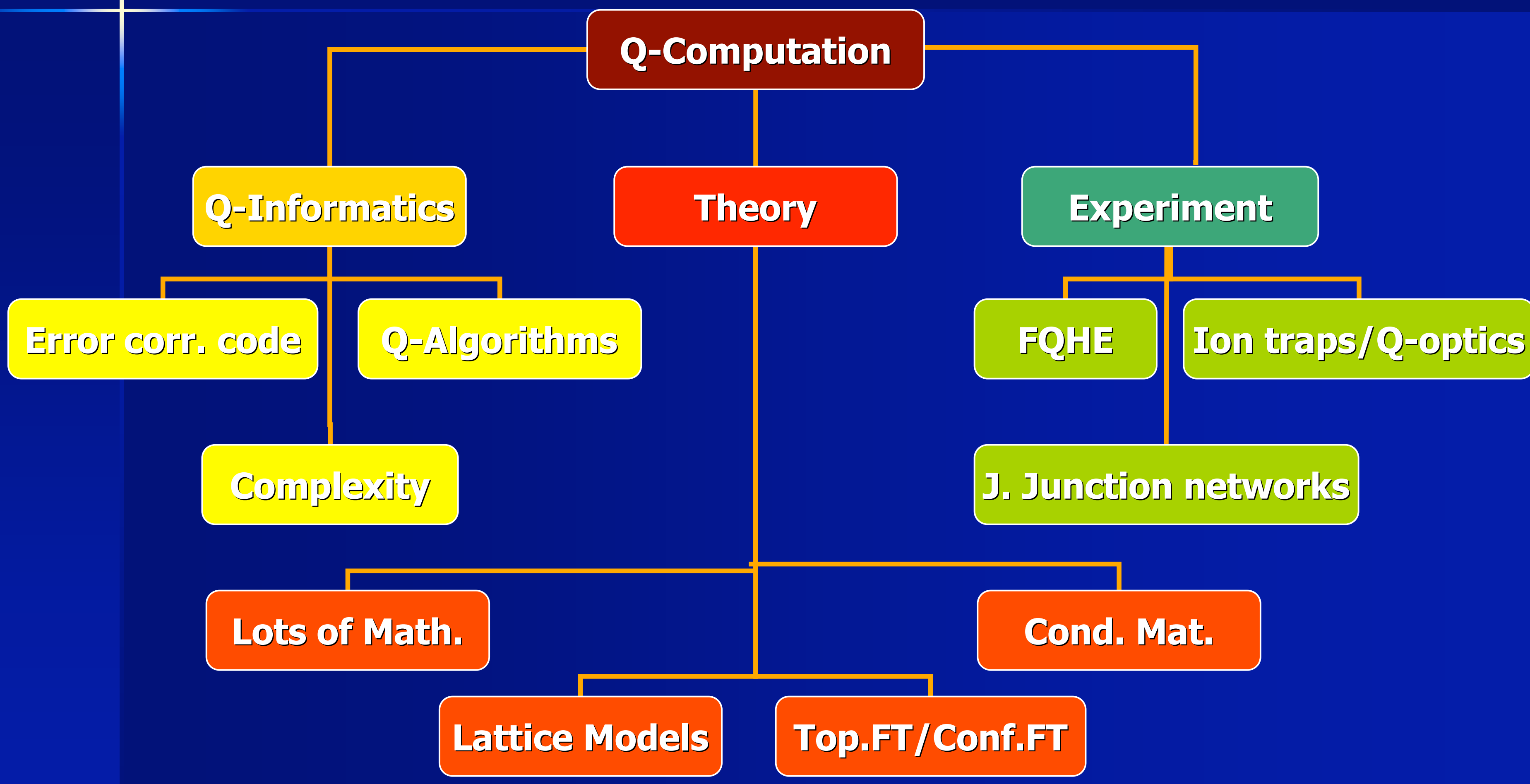
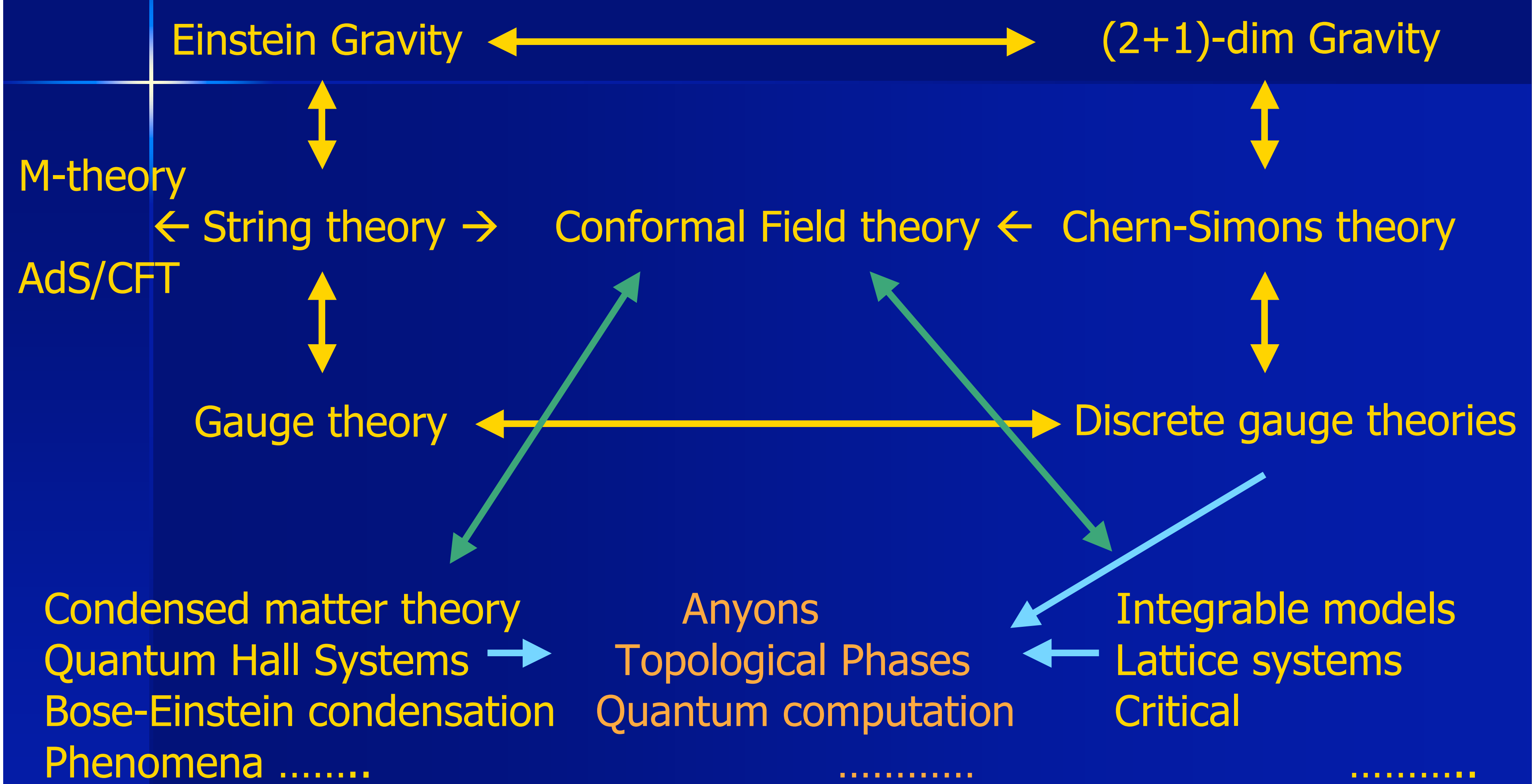
Qubits and pieces #1:
From Anyons to Algebra's



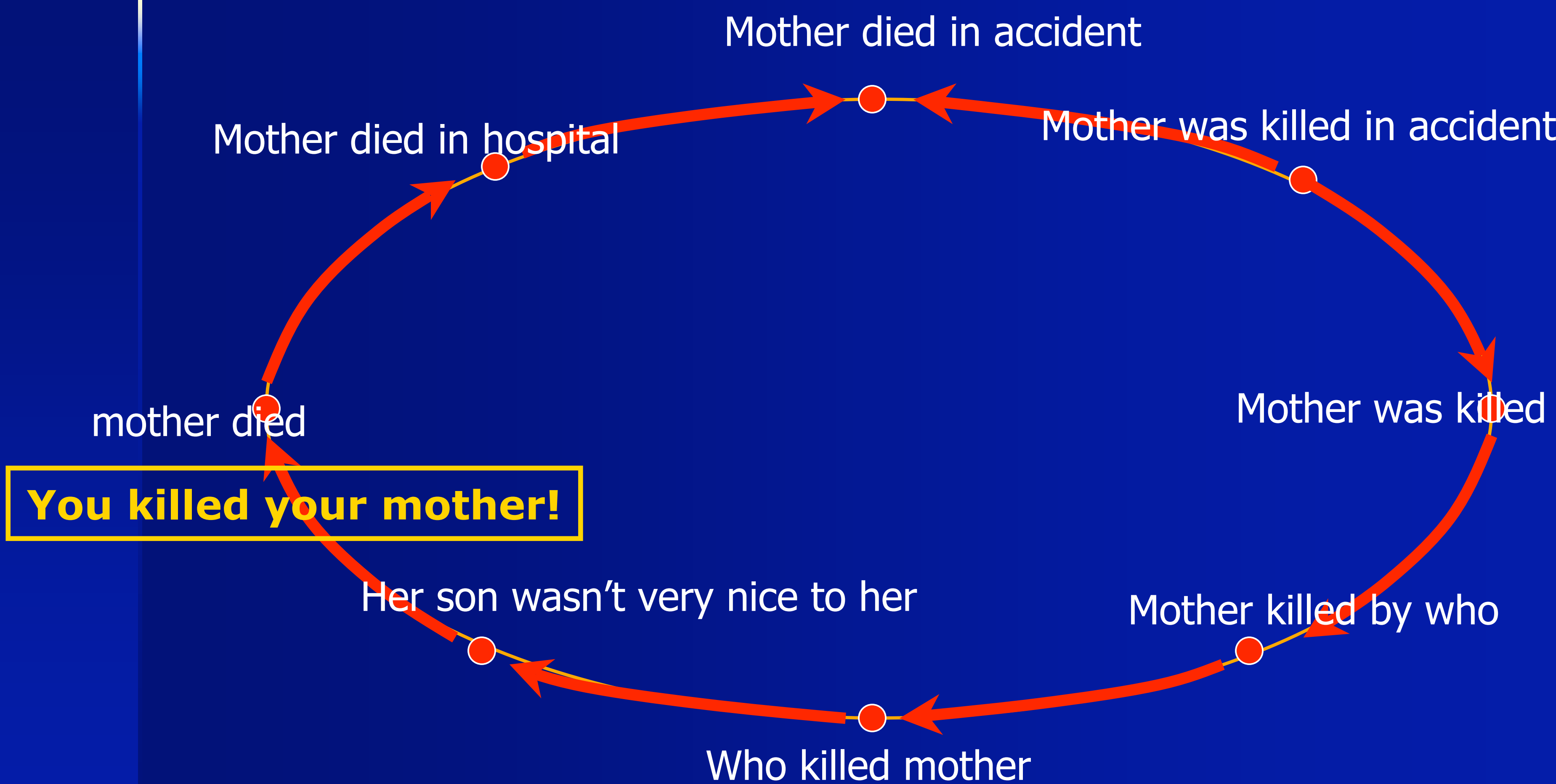
The Qubits and pieces



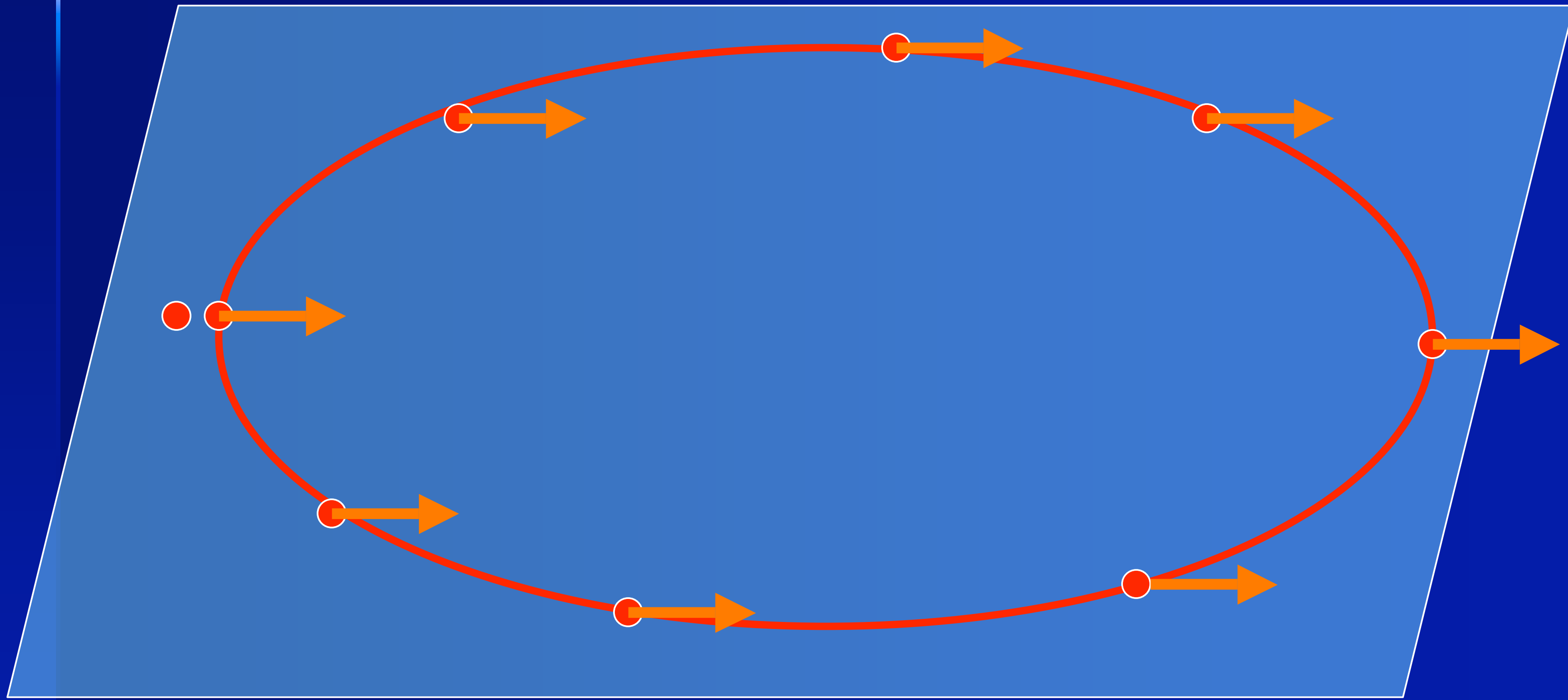
A survey of the theoretical battle field



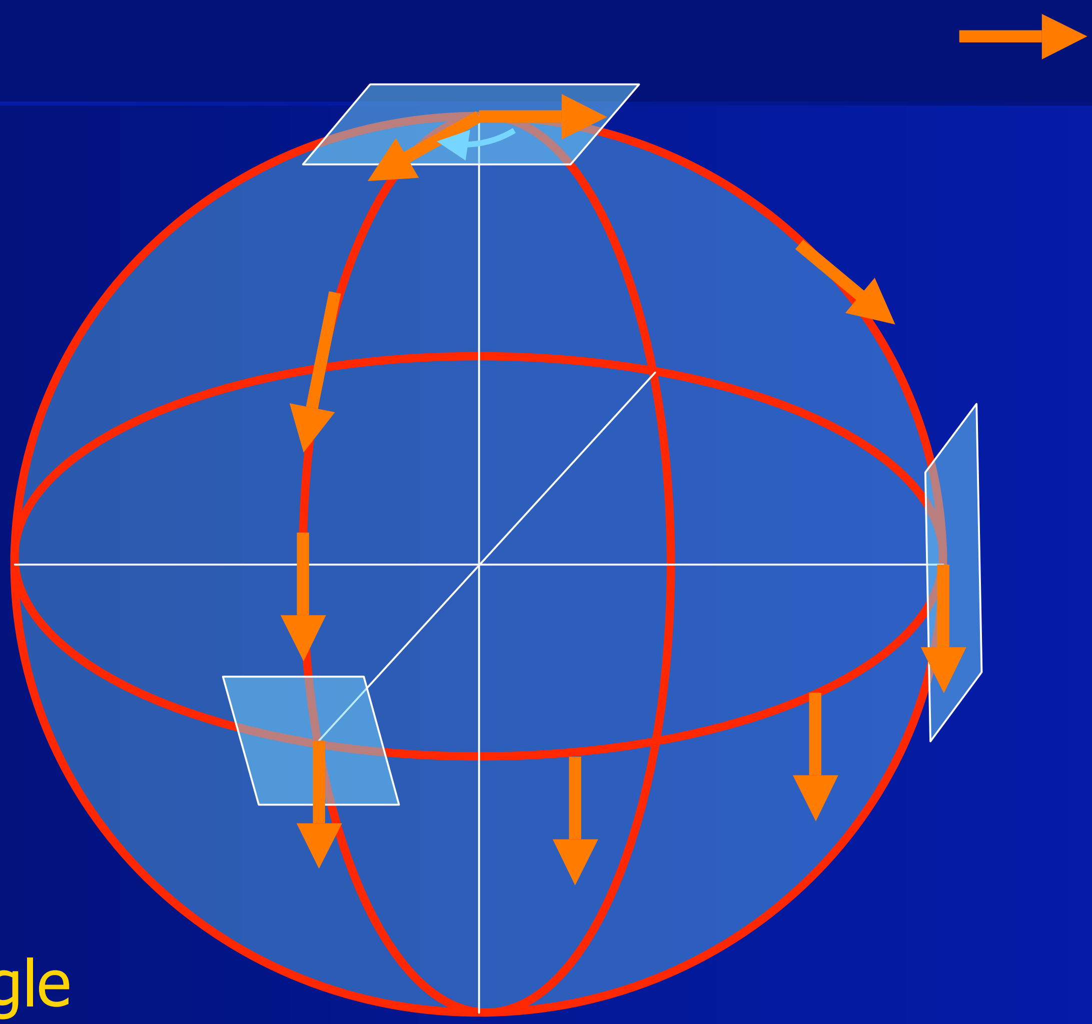
Telephone game



A perfect messenger



Pefect messenger ?
Parallel transport

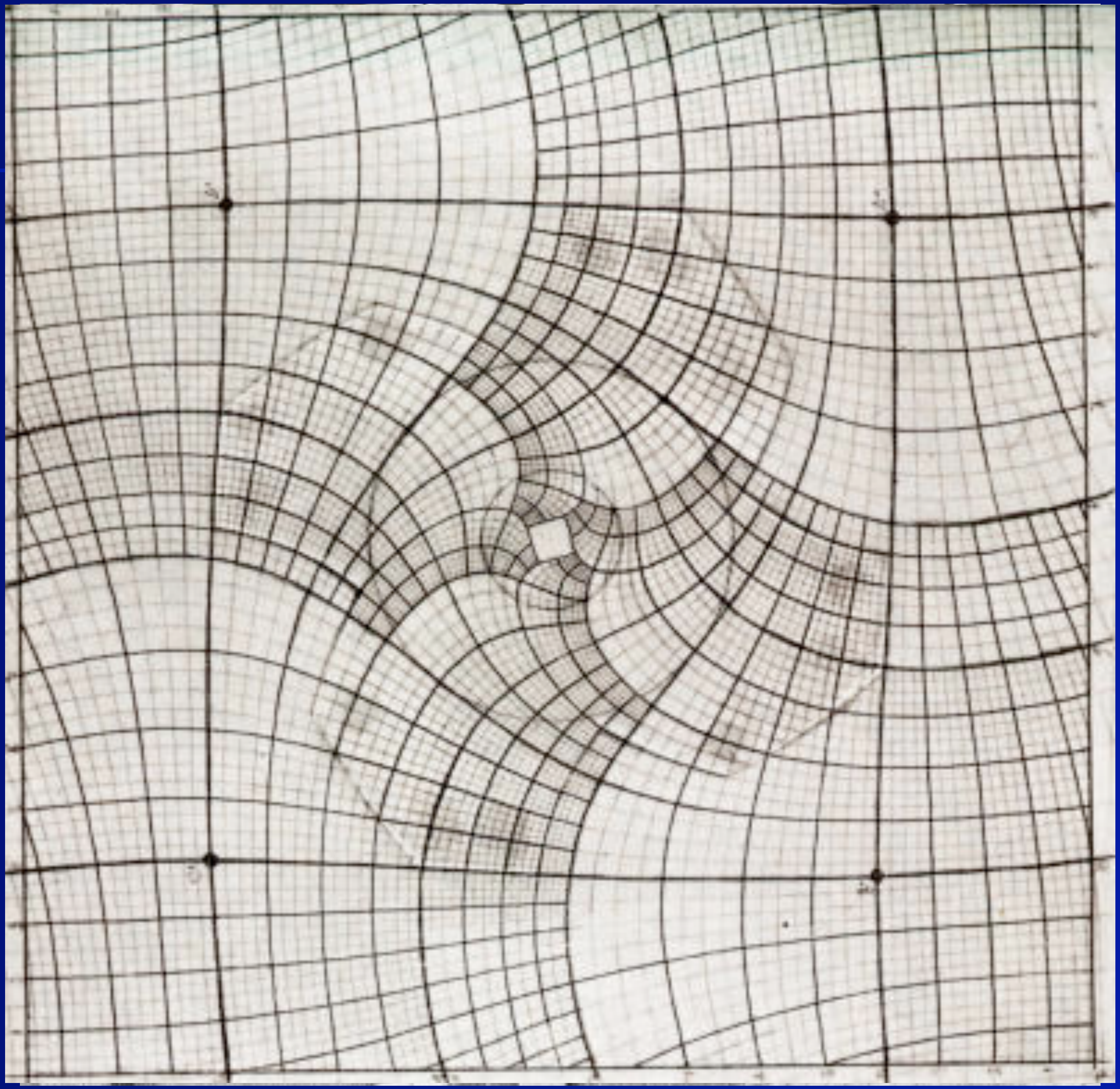


Deficit angle

$\Delta\Phi = \Omega$

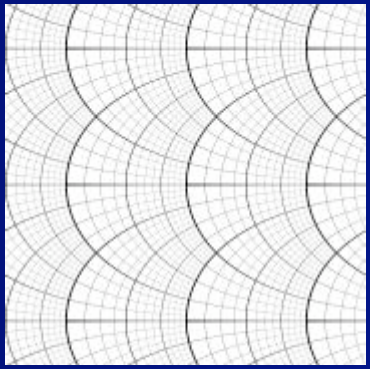
Vortices

A distorted view



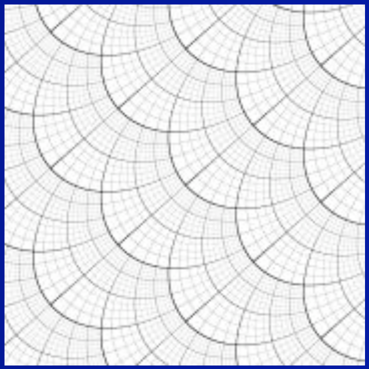
Hendrik Lenstra: A sequence of maps

http://escherdroste.math.leidenuniv.nl/index.php



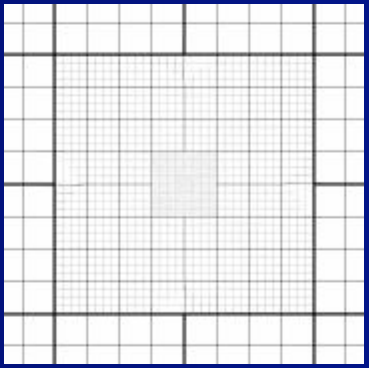
→

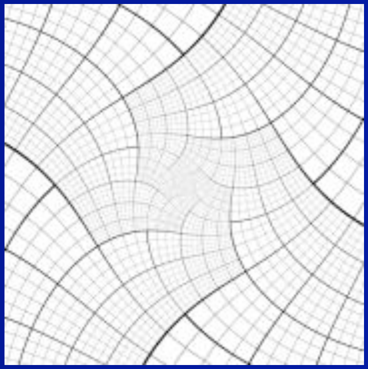
Rotation 41^0
Scaling 0.75



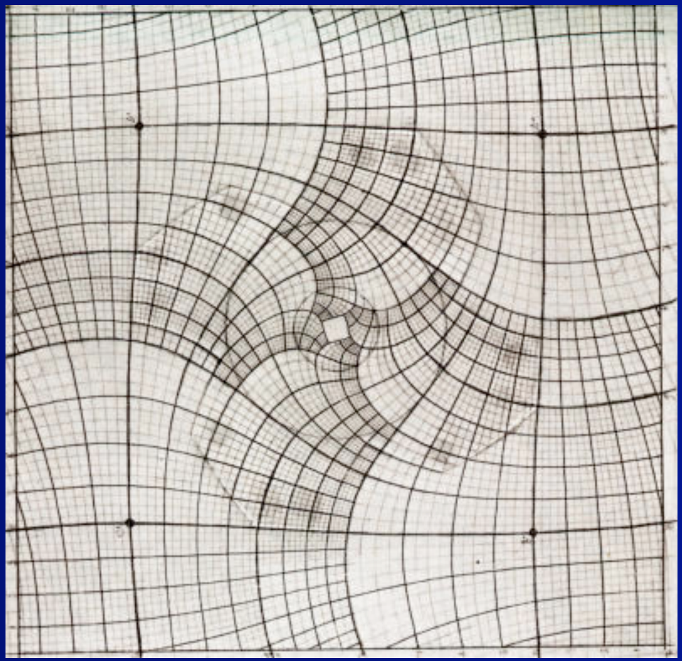
↓ W= exp z

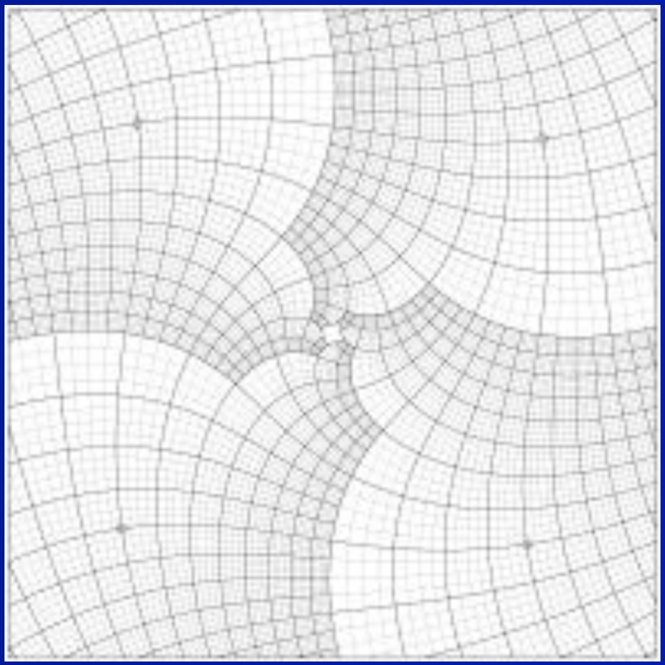
W= exp z ↓





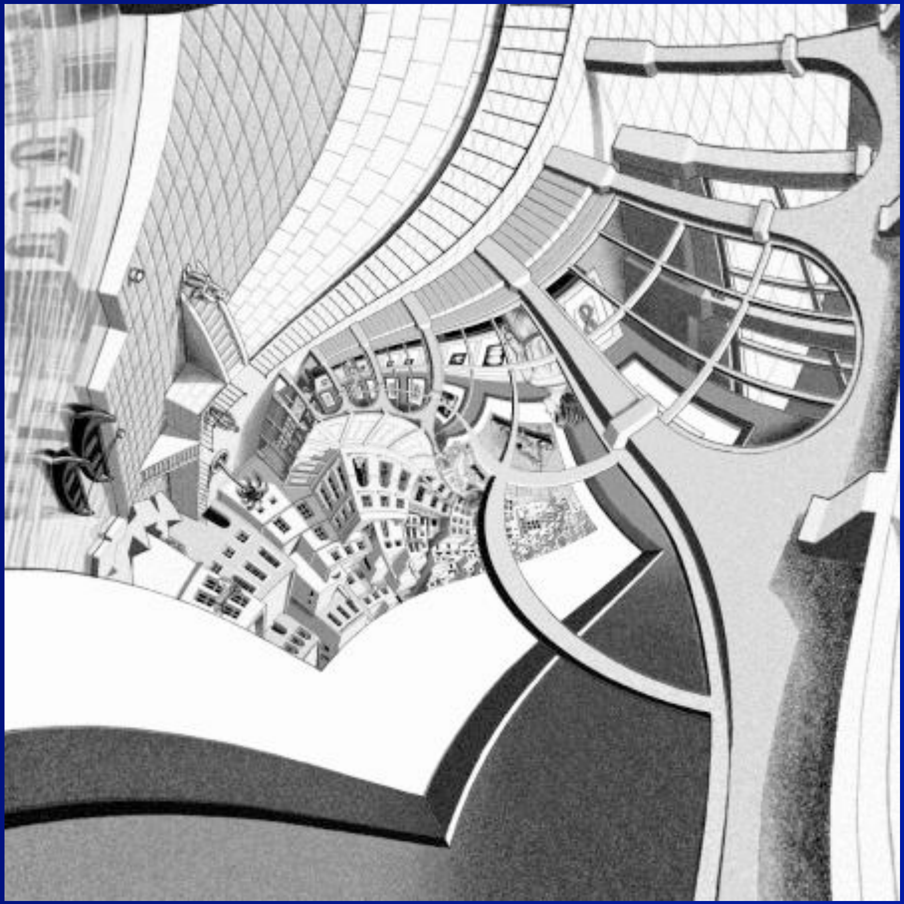
How well did we/Escher do?



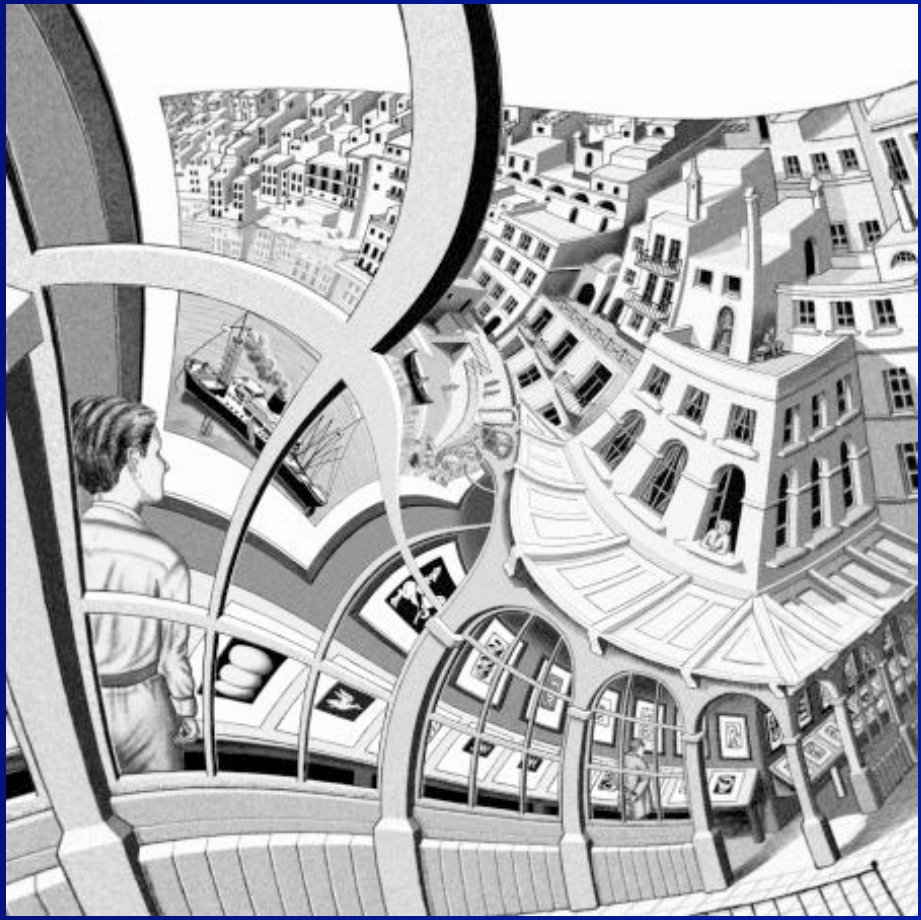


The perfect Escher …

The solution

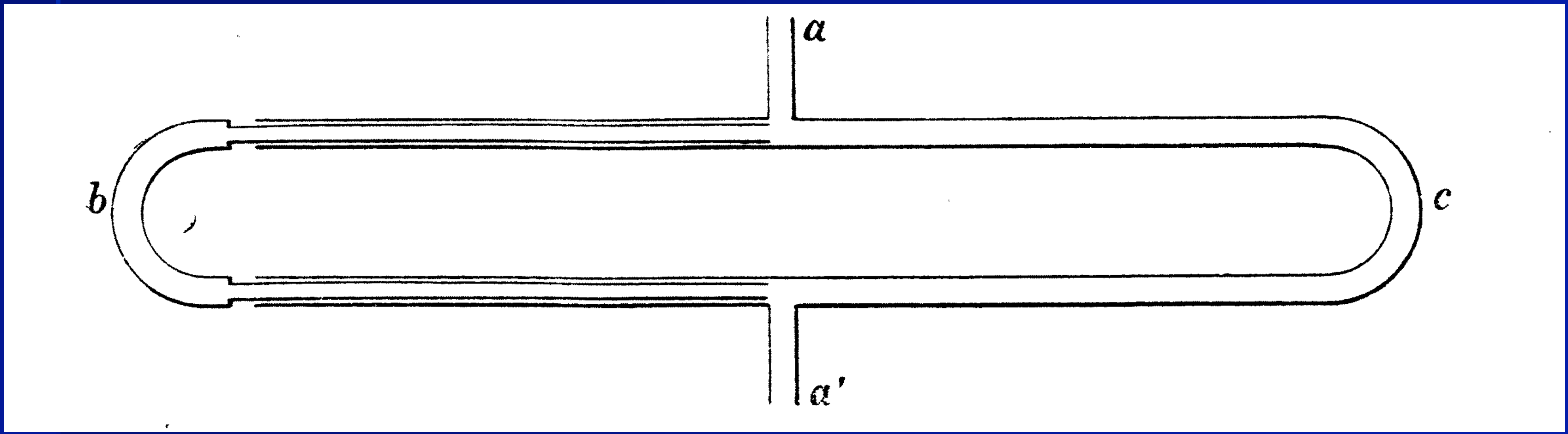


So the story goes on and on and on …



The other Eschers…

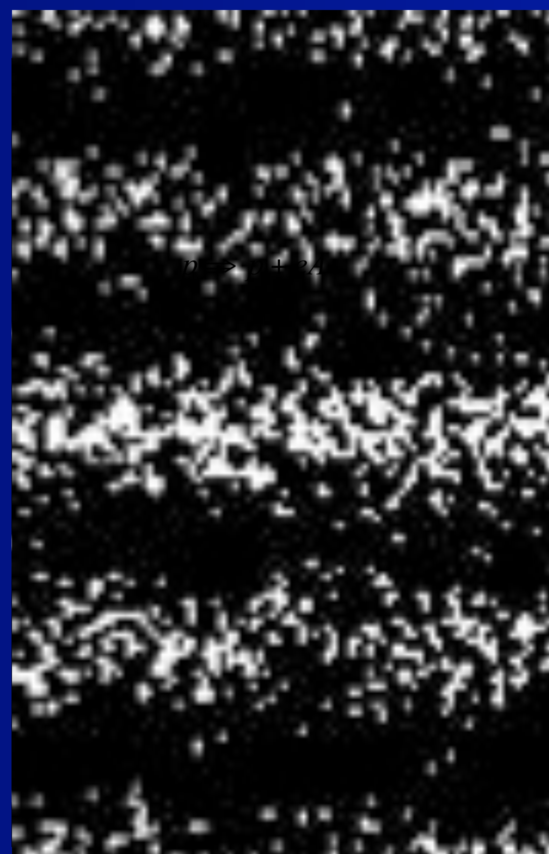
A sound experiment



Interference & topology: $L_b - L_c = n\lambda$

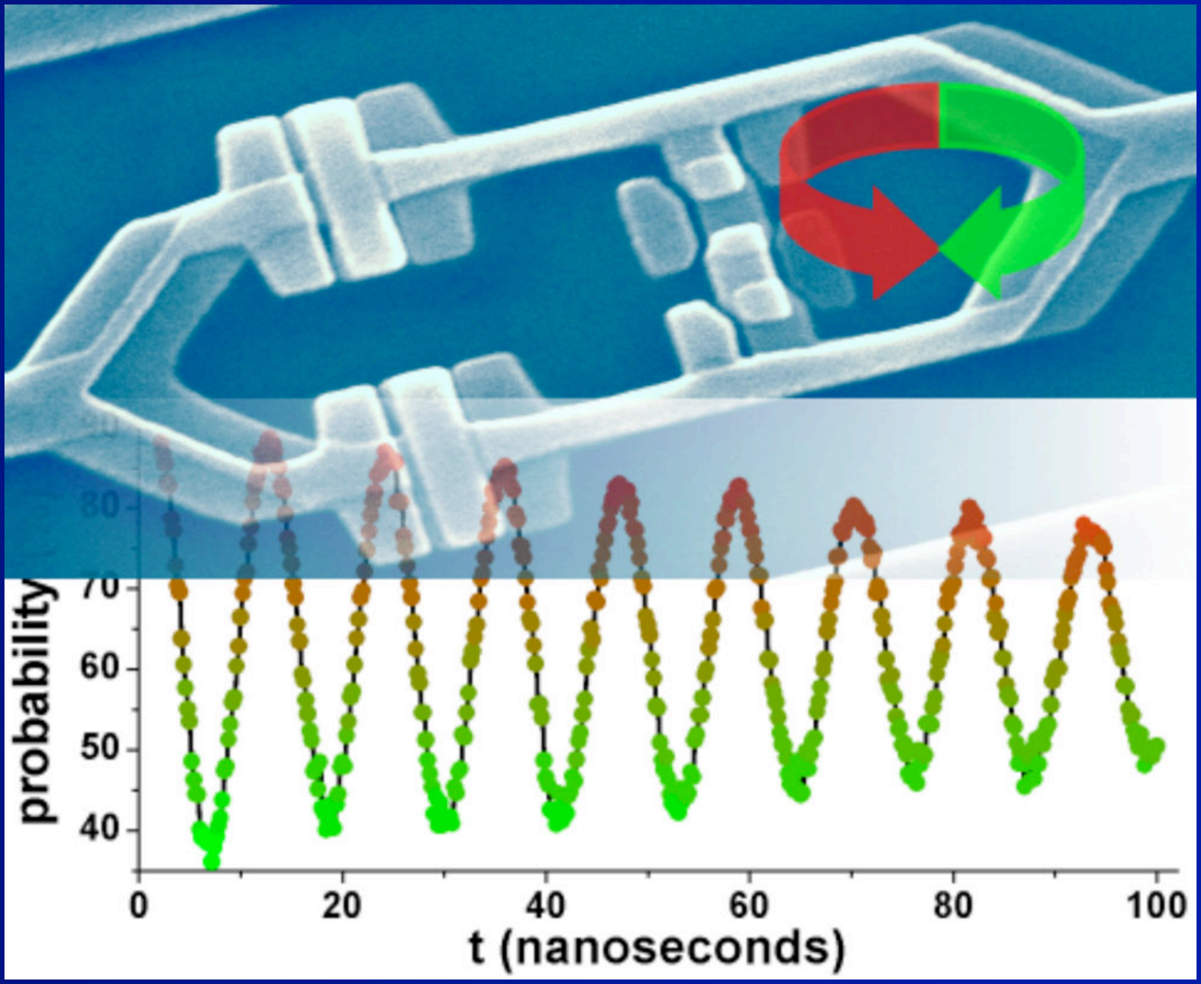
$$\oint \frac{1}{\lambda} dx = n$$

$$\oint p\,dx = \hbar\oint \frac{1}{\lambda}dx = 2\pi n\hbar$$



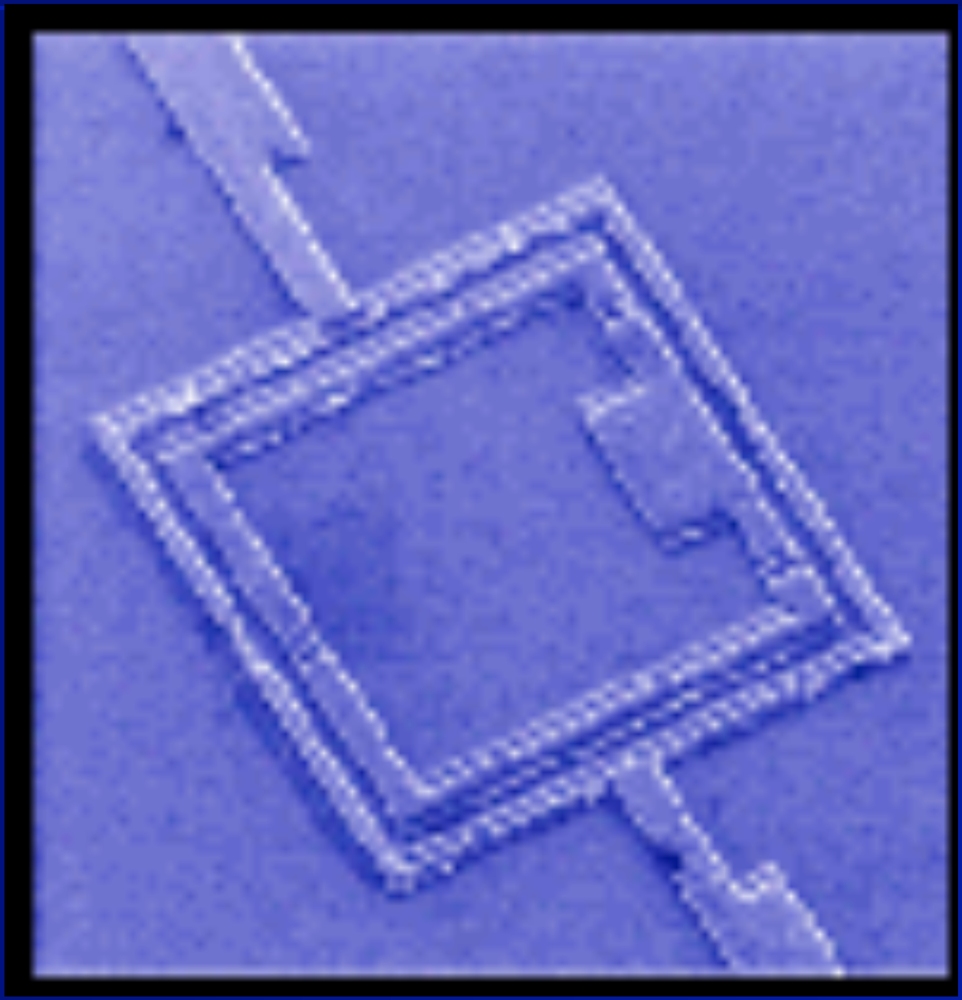
$$p \Rightarrow p + qA$$

Flux qubits

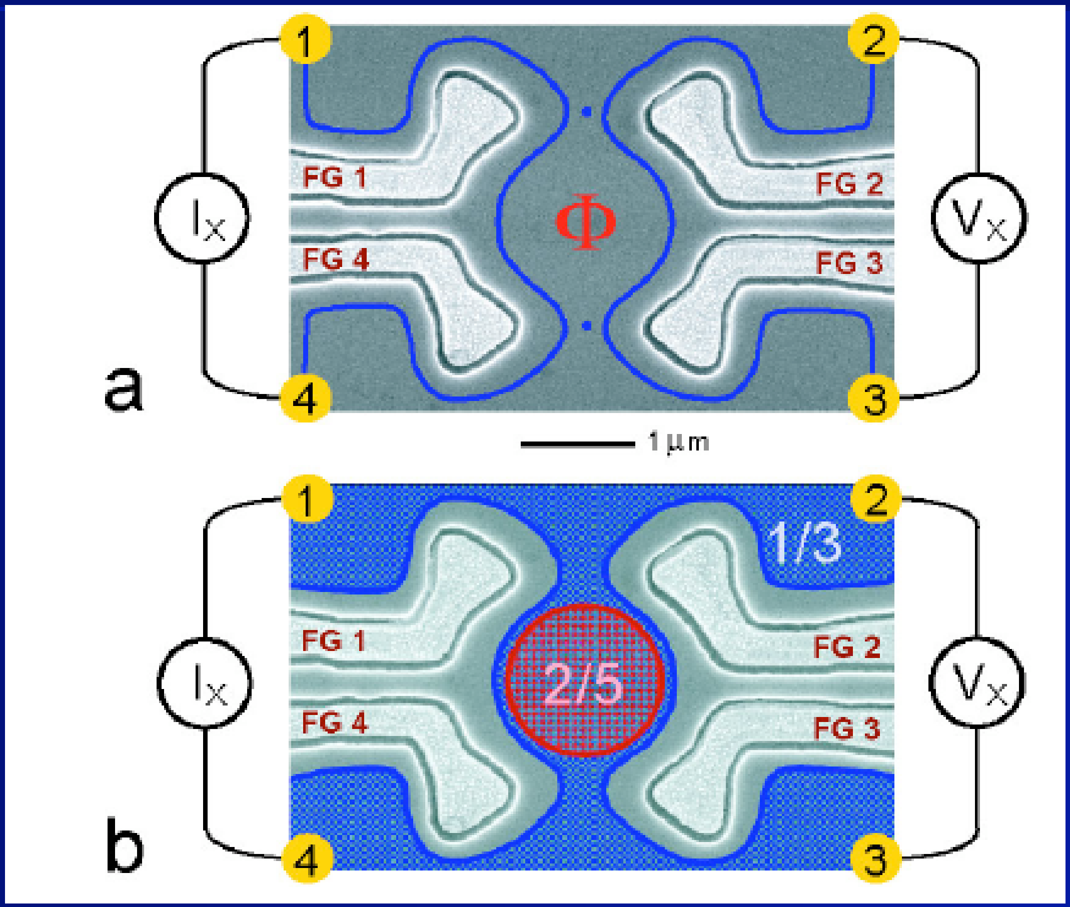


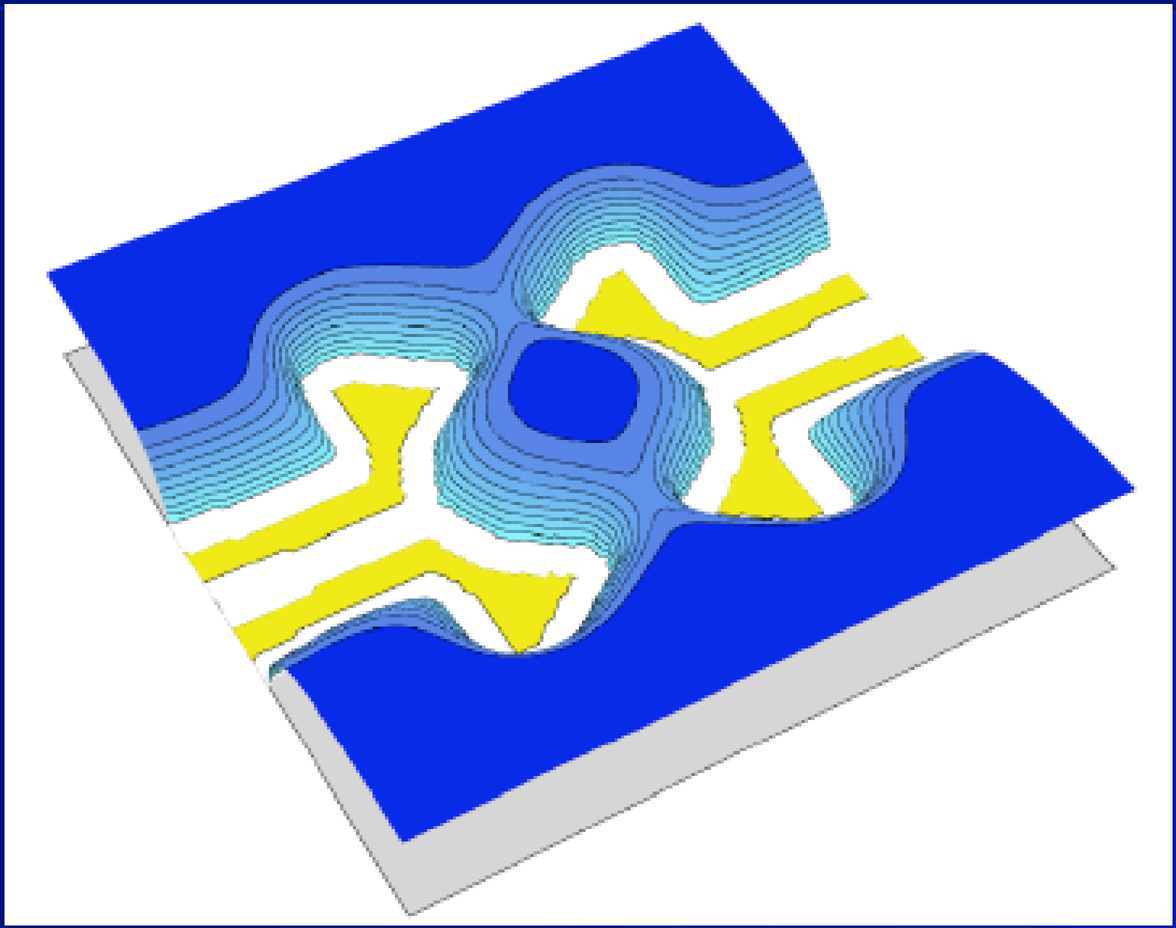
TopQC

Flux qubits

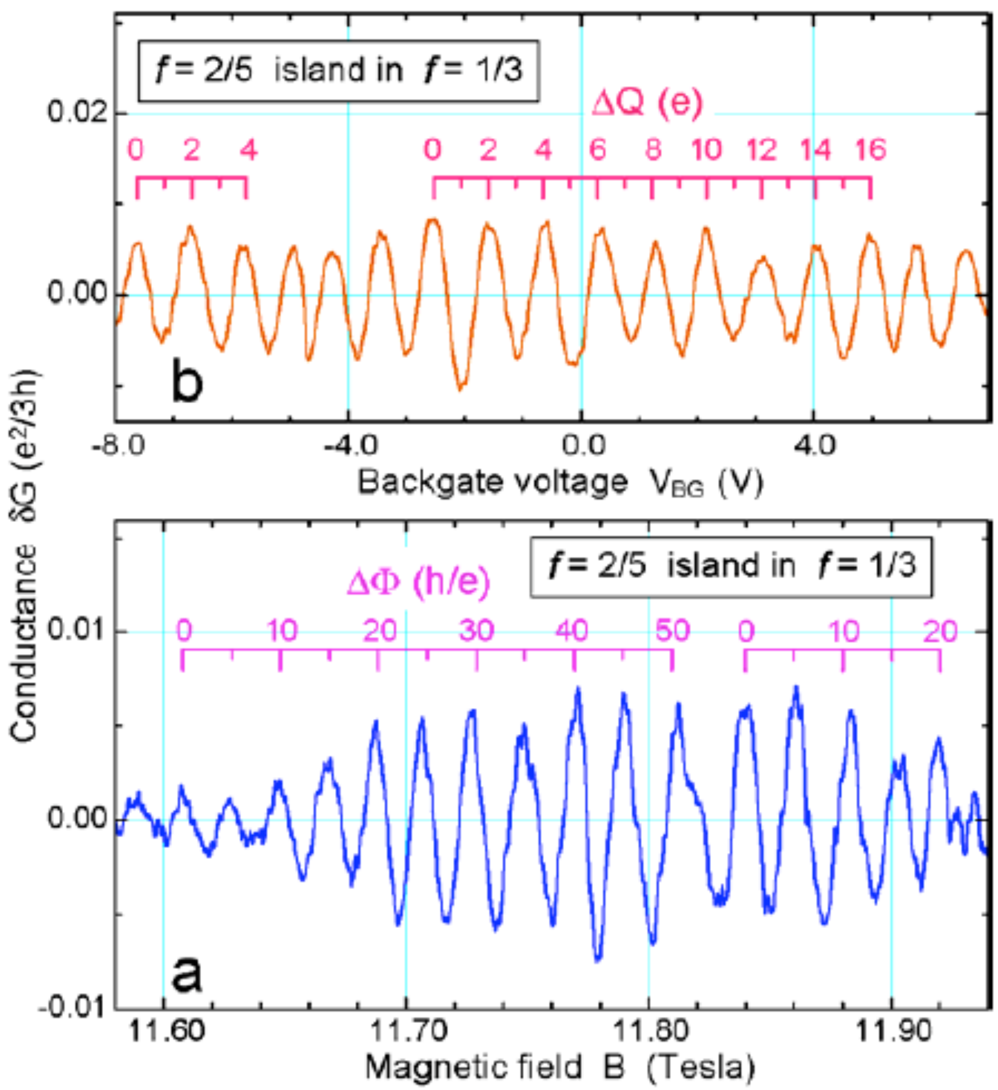


Quasiparticle interferometers (Goldman et al.)





Interference patterns



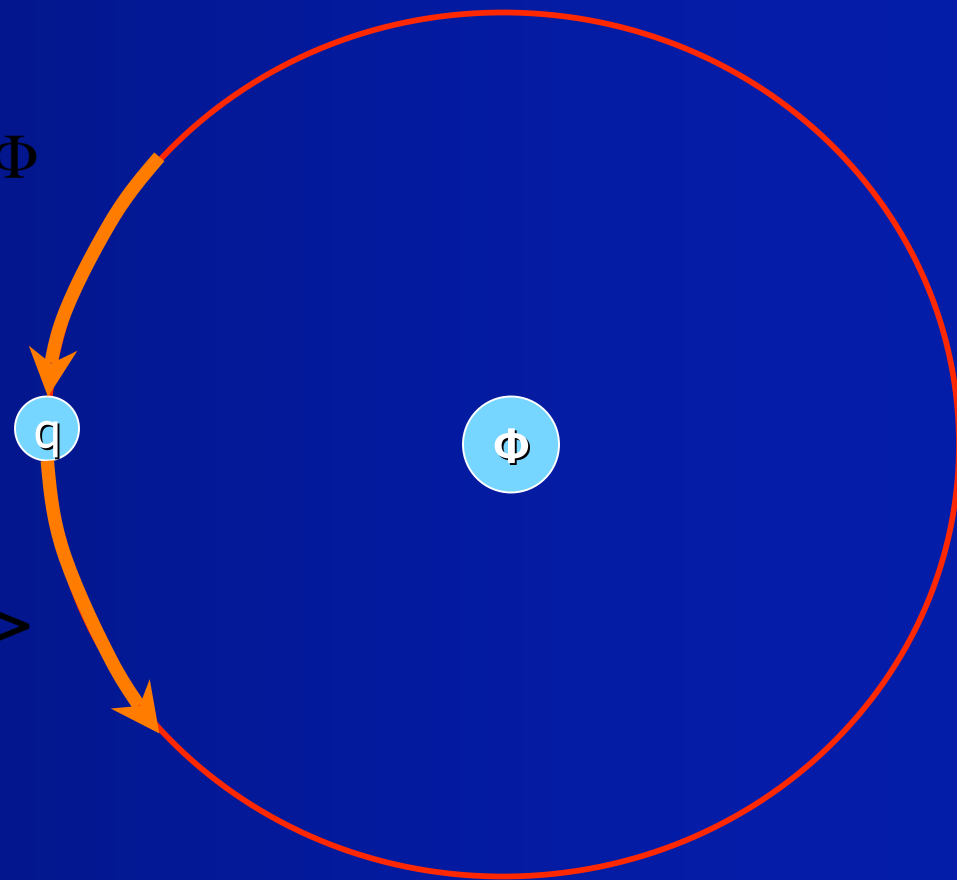
Charge-flux composites: anyons

$$W = e^{iq\oint A\,dx} = e^{iq\Phi}$$

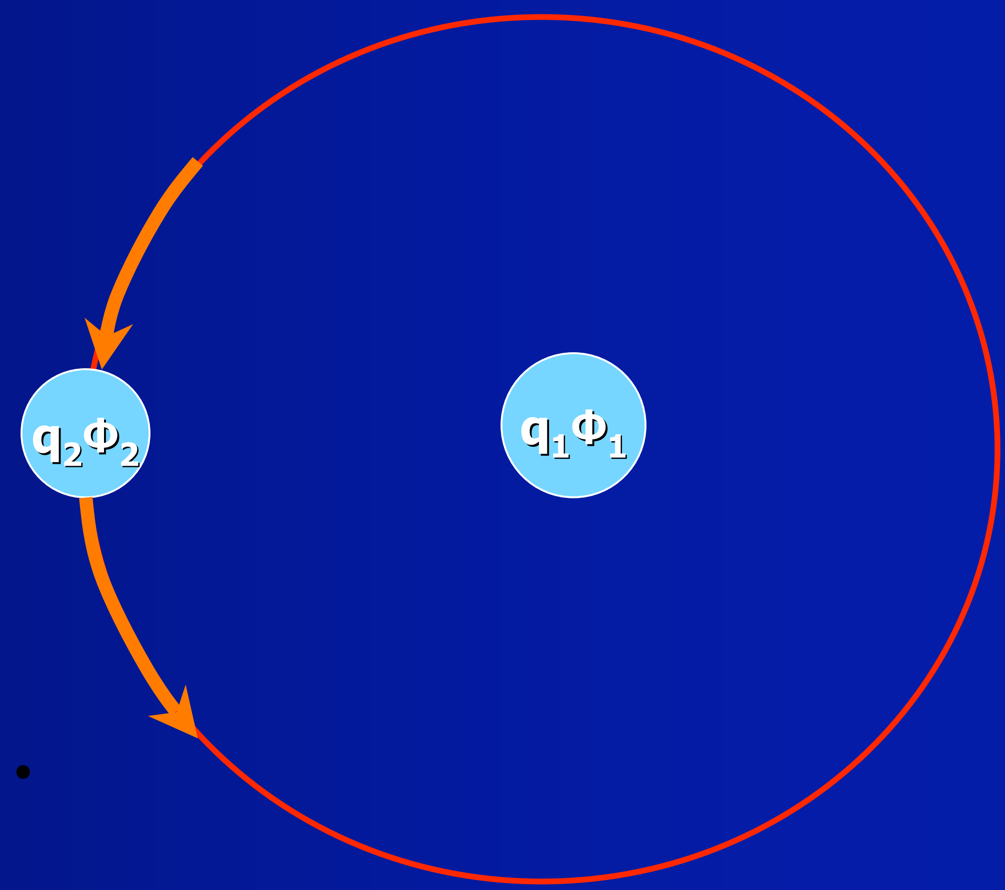
$|\Phi, q\rangle$ →

$$e^{iq\Phi}\,|\,\Phi, q\rangle = e^{2\pi i S}\,|\,\Phi, q\rangle$$

→ $$s = {q\Phi}/{2\pi}$$



Holonomy

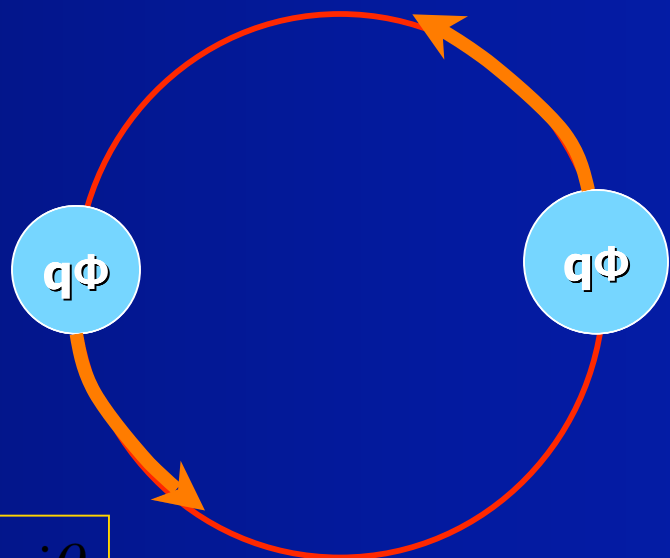


$$e^{q_1\Phi_2 + q_2\Phi_1} = \cdots$$

Quantum statistics:
Exchange of indistinguishable particles

$q_1 = q_2$

$\Phi_1 = \Phi_1$

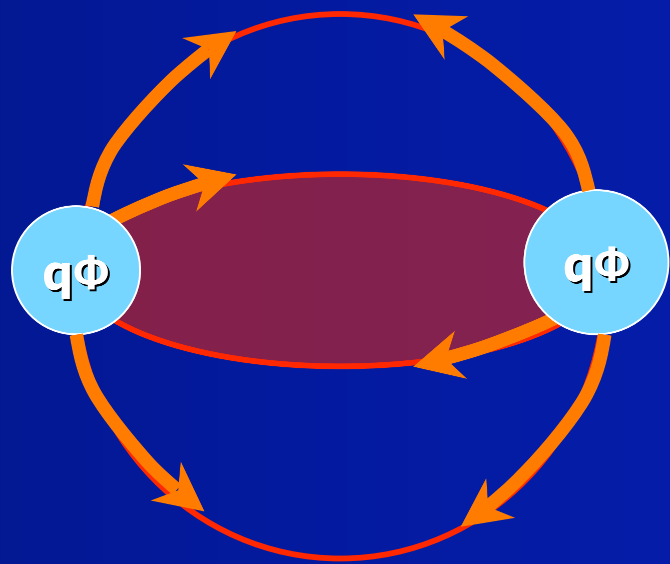


$$e^{2\pi i S} = e^{iq\Phi} = e^{i\theta}$$

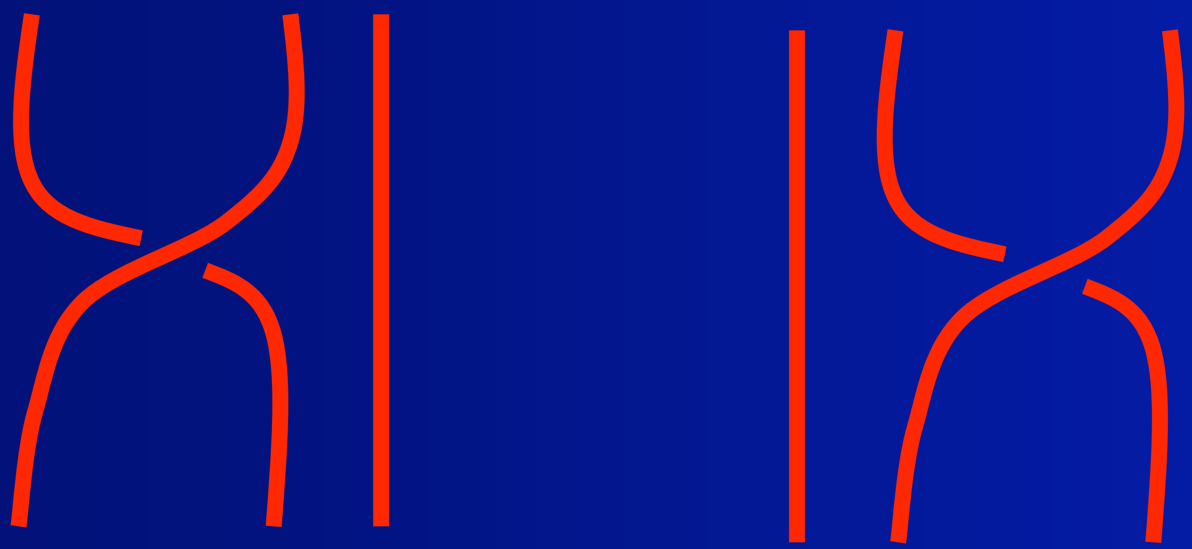
D=3 versus D=2

D= 3: $\tau = \tau^{-1}$

D= 2: $\tau \neq \tau^{-1}$

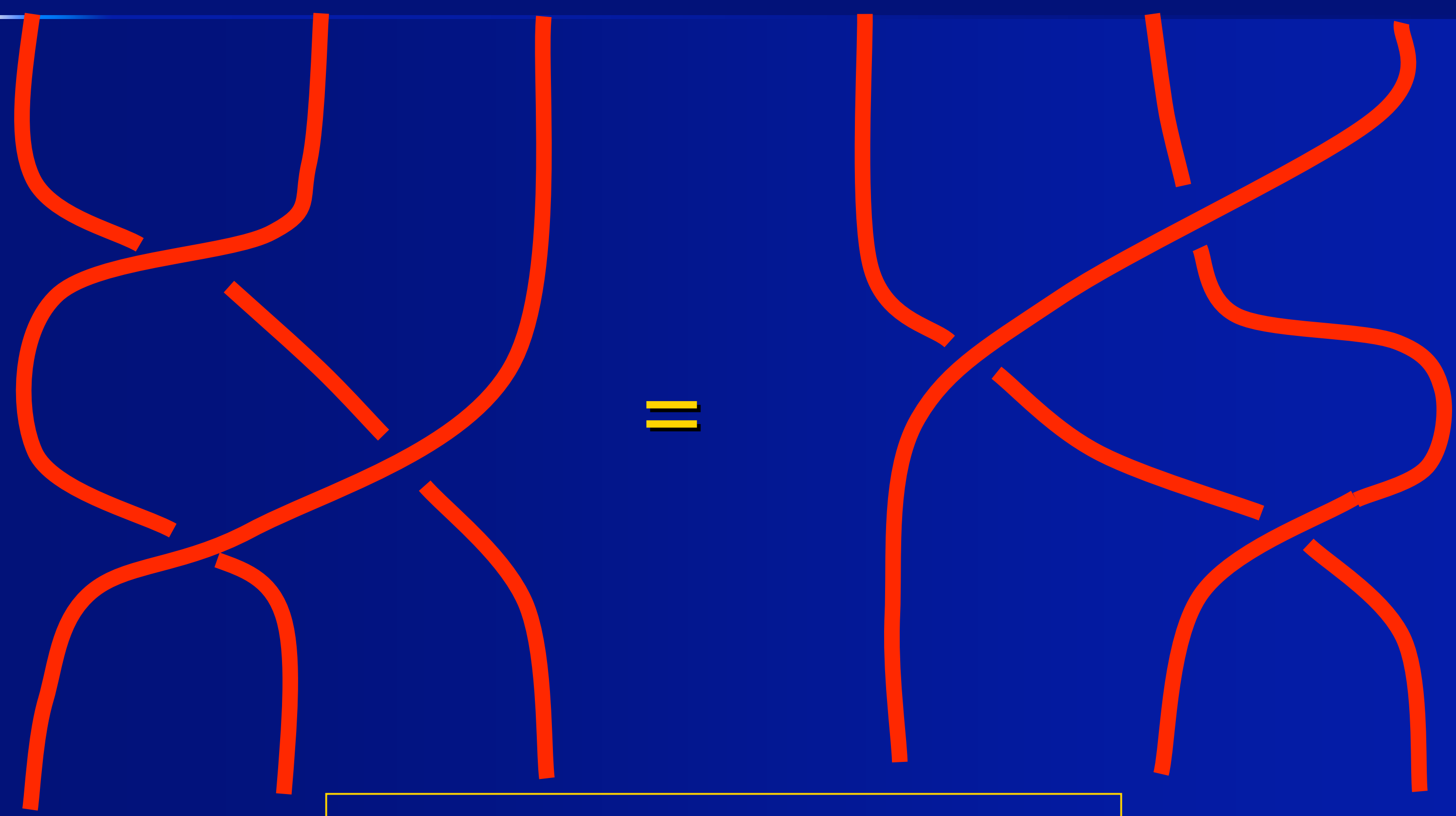


Braid group generators



τ_1 τ_2

Relation between generators (Yang Baxter)



$$\tau_1 \, \tau_2 \, \tau_1 = \tau_2 \, \tau_1 \, \tau_2$$

Defining relations between generators

$D \geq 3$ Permutation group S_n:

$\tau = \tau^{-1} \rightarrow \tau^2 = 1 \rightarrow \tau = \pm 1$ $+1 \rightarrow$ Bosons

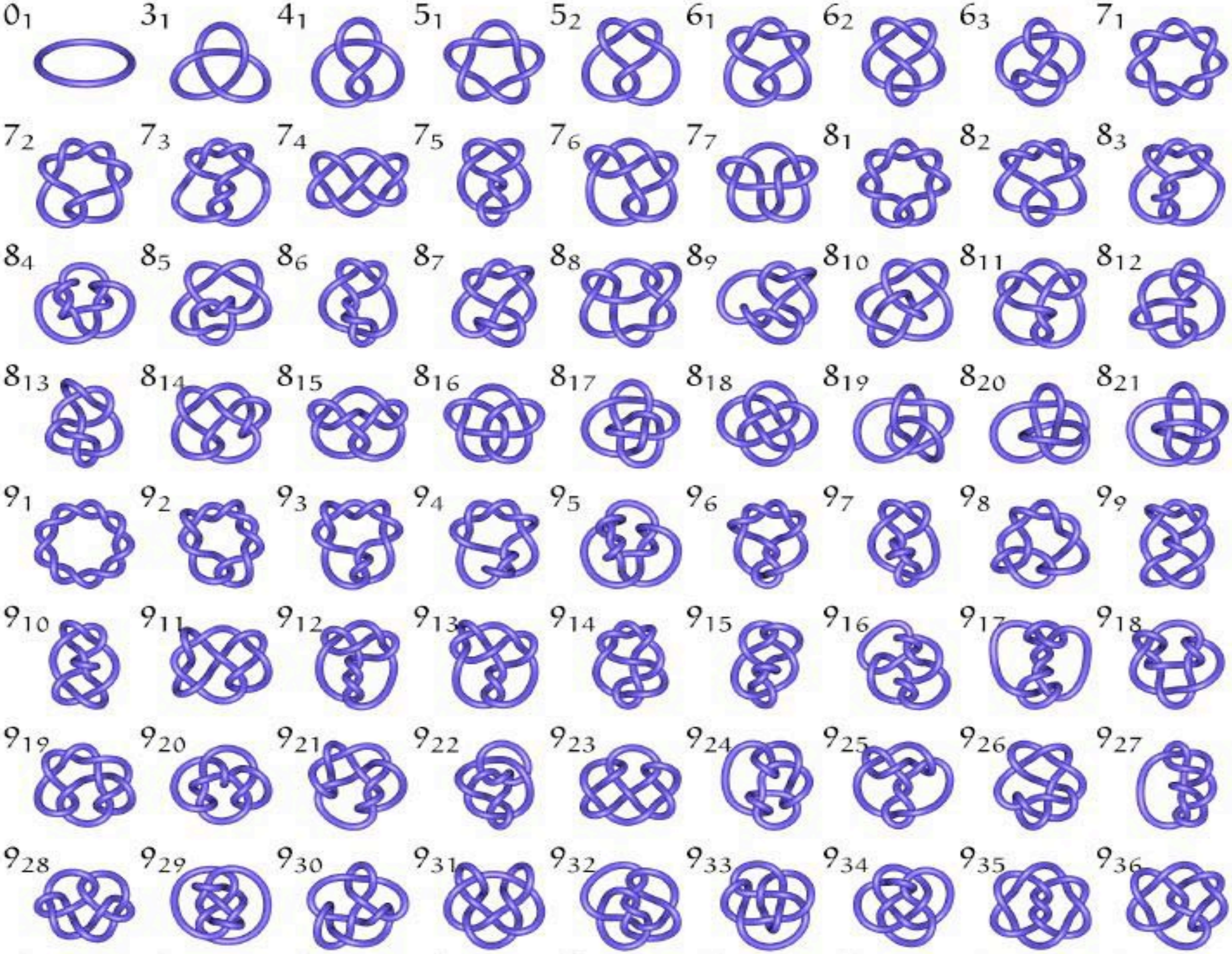
$-1 \rightarrow$ Fermions

$D = 2$ Braid group B_n:

$\tau_1 \tau_2 \tau_1 = \tau_2 \tau_1 \tau_2$ $\rightarrow \tau_j = e^{i\theta}$ Anyons

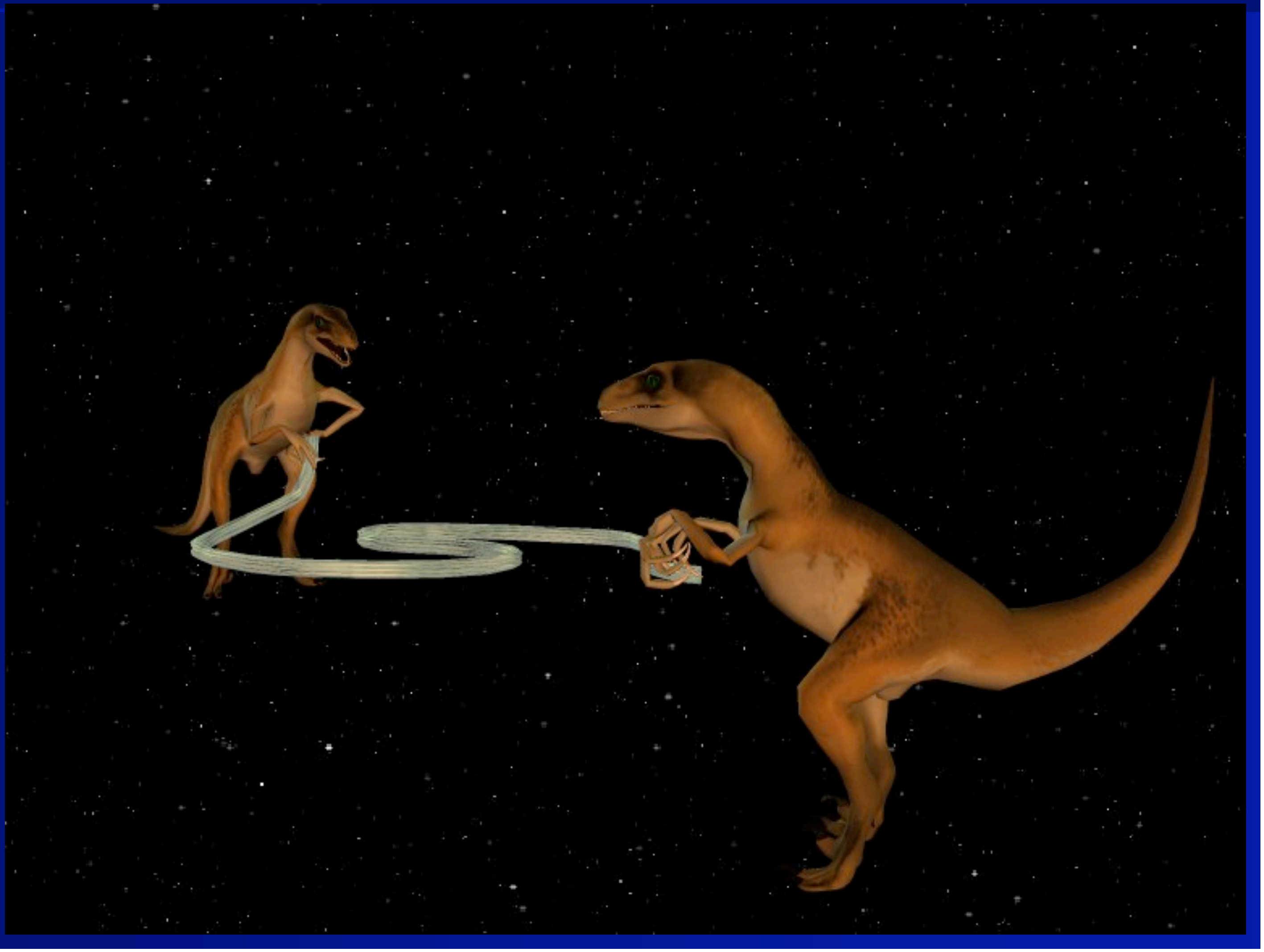
$(\tau \neq \tau^{-1})$ $\rightarrow$ Matrices non-Abelian anyons

Braids & Knots

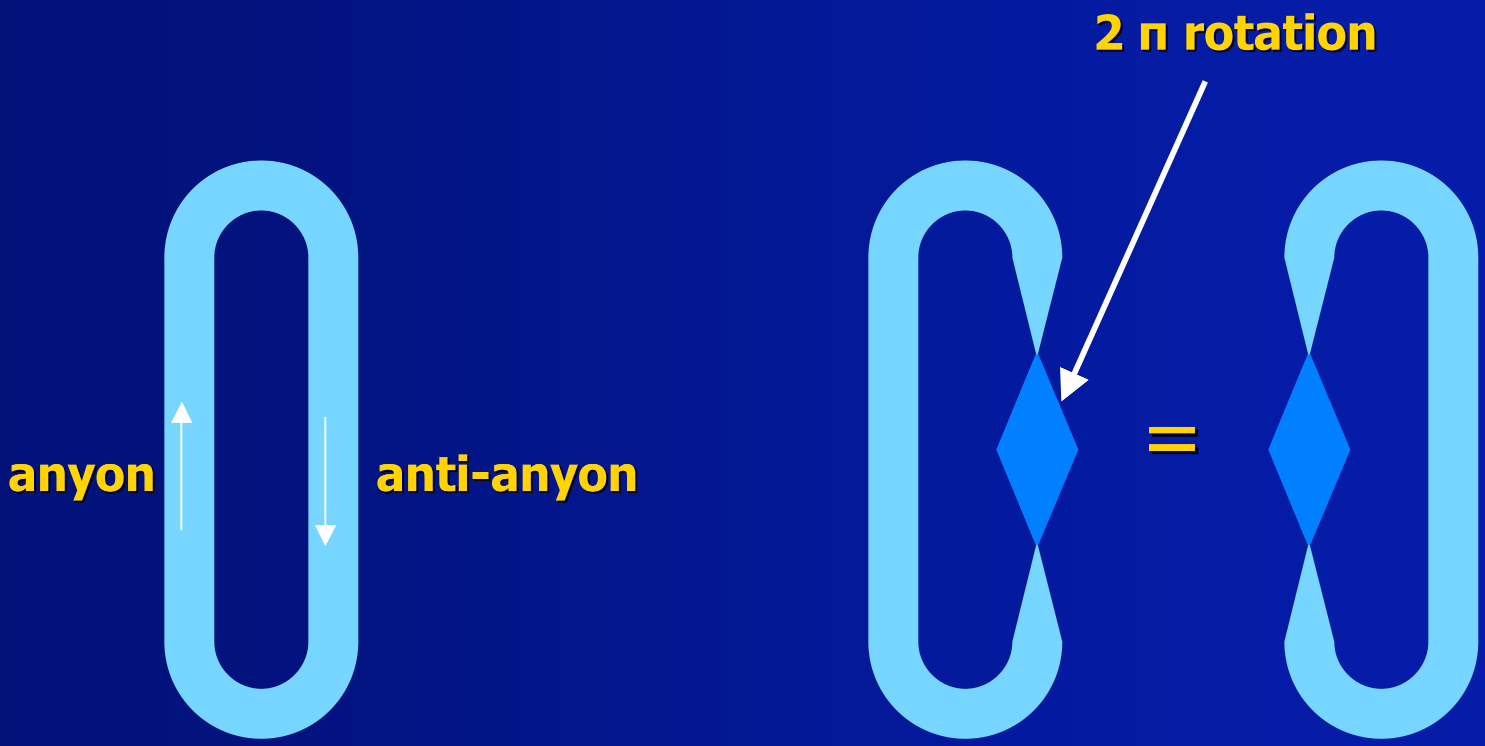


Braids & Knots

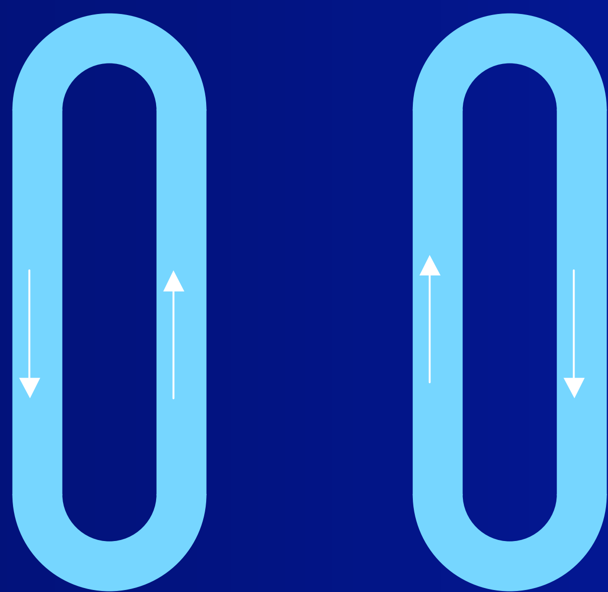
Ribbon algebras



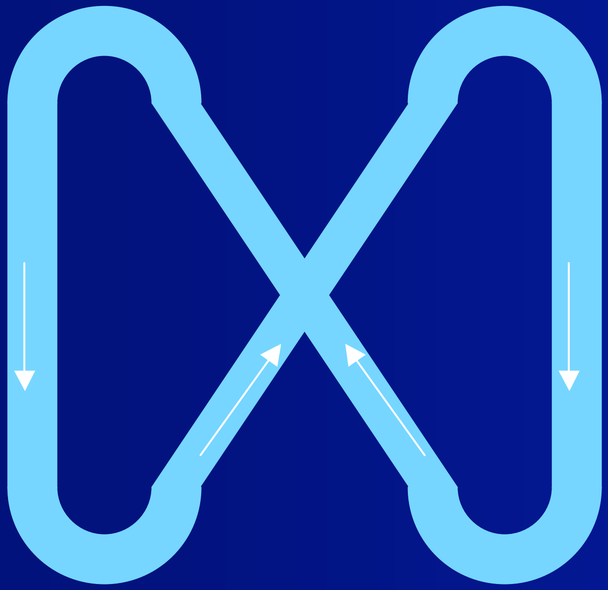
Ribbon diagrams



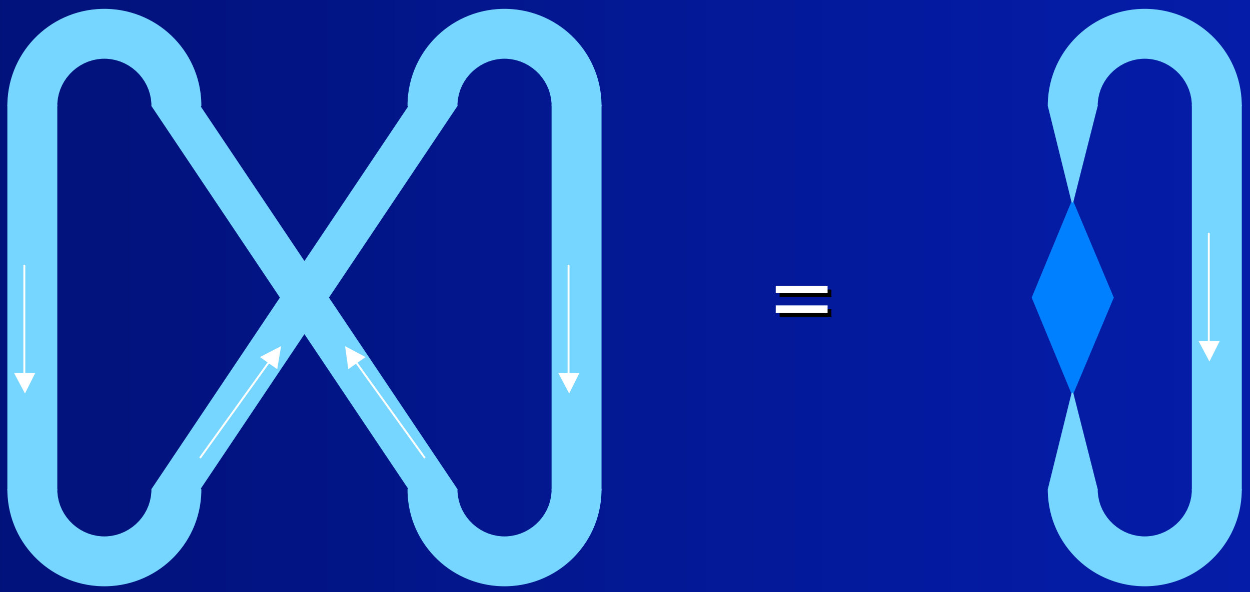
Two anyon—anti-anyon pairs



Anyon interchange



Spin-statistics connection as a topological equivalence



Effect of interchange is equivalent to 2π rotation

Chern-Simons Theory (d=2+1)

The Chern-Simons action:

$$S_{CS} = \frac{k}{4\pi}\int_M \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

With indices:

$$S_{CS} = -\frac{k}{4\pi}\int dt \int_\Sigma \epsilon^{ij}\mathrm{tr}\left(A_i \partial_0 A_j - A_0 F_{ij}\right)$$

Field equations:

$$\varepsilon_{\mu\nu\sigma} F^{\nu\sigma} = j_u \Leftrightarrow \begin{array}{l} E_x = j_y \quad E_y = -j_x \\ \\ B = \rho \end{array}$$

The observables:

$$W_R(C) := \mathrm{tr}_R P \exp \oint_C A$$

CS classification: Group cohomology

- Classification CS theories by $H^4(BG,Z)$
- For finite group H →

  $H^n(BH,Z) = H^n(H,z)$

  $H^n(H,Z) = H^{n-1}(H,U(1))$
- → $H^4(BH,Z) = H^3(H,U(1))$

CS for finite abelian groups

- Consider $H = (Z_N)^k$ then:
- $H^1(H,U(1)) = (Z_N)^k$
- $H^2(H,U(1)) = (Z_N)^{½k(k-1)}$
- $H^3(H,U(1)) = (Z_N)^{k+½k(k-1)+1/6k(k-1)(k-2)}$

Three types of CS actions (?)

- I: (decoupled U(1) theories)

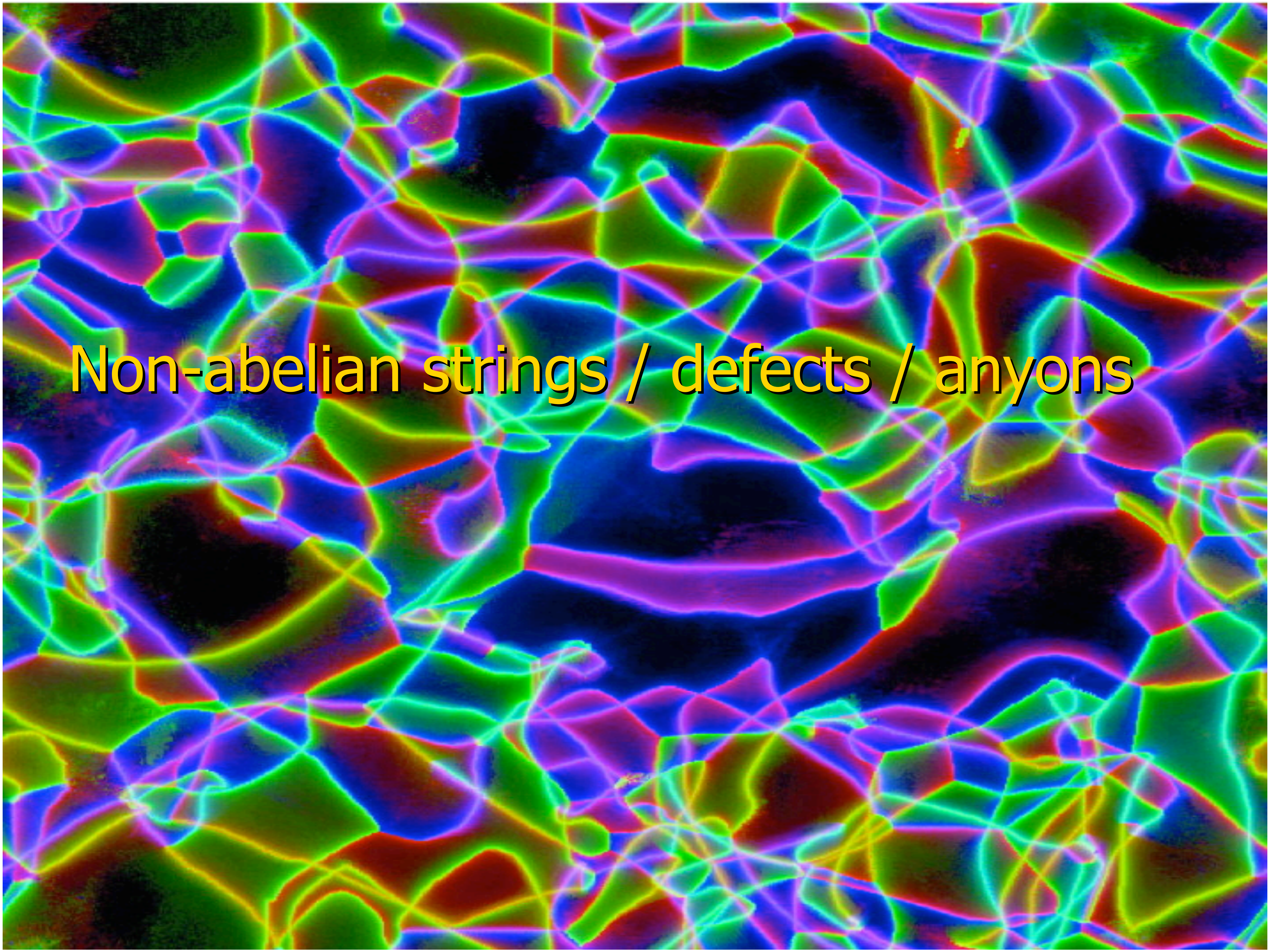
  $L = \Sigma\ \mu_i\ A^{(i)} \cdot F^{(i)}$

- II: (coupled U(1) theories)

  $L = \Sigma\ \mu_{ij}\ A^{(i)} \cdot F^{(j)}$

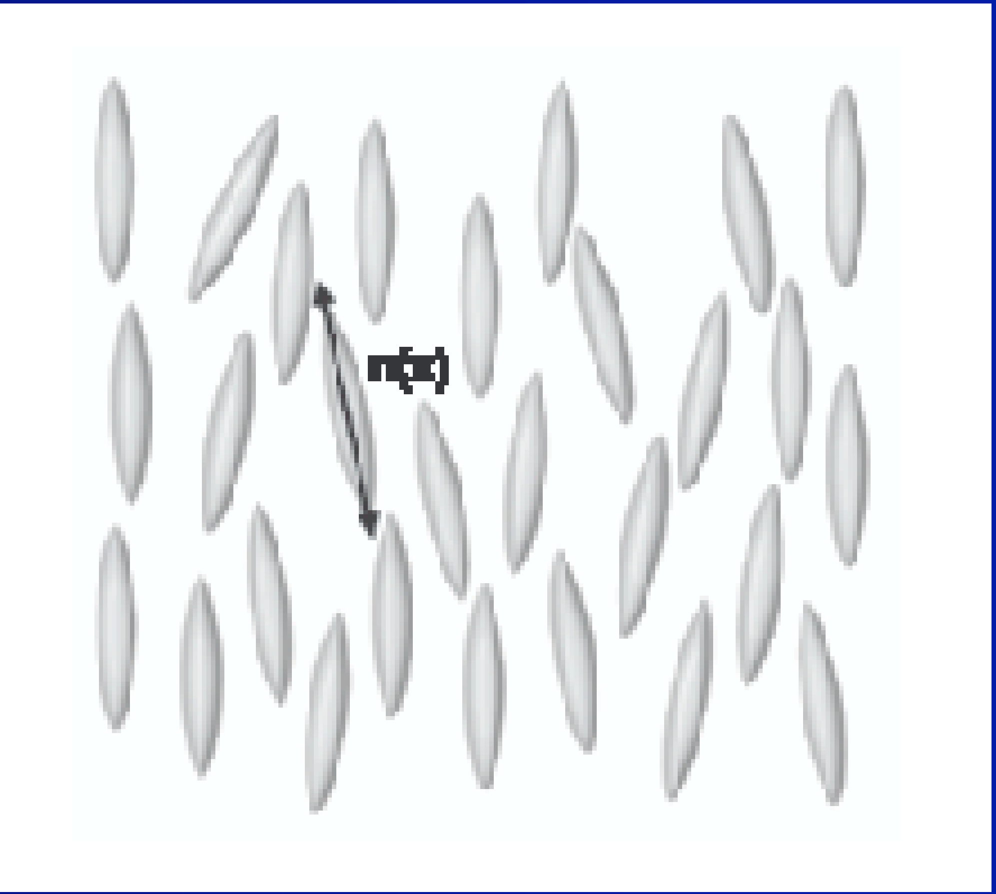
- III:

  Non-abelian theories! E.g. $(Z_2)^3 \leftarrow\rightarrow \underline{D}_2$


Non-abelian strings / defects / anyons

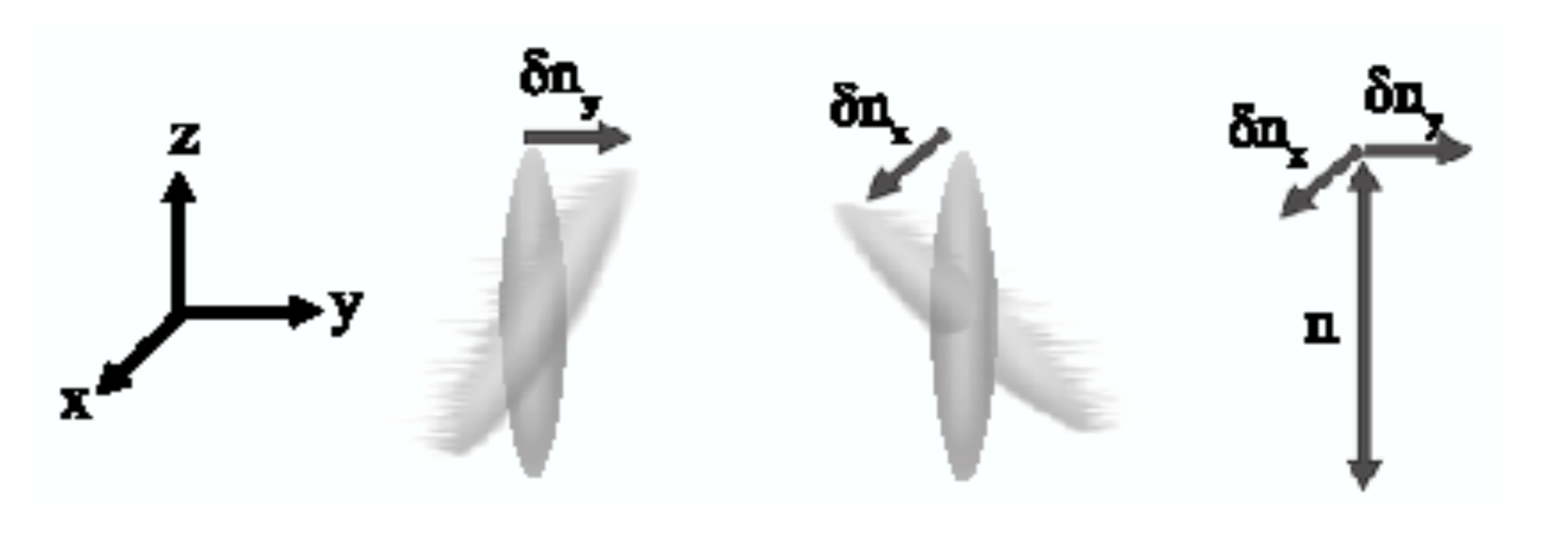
Uniaxial nematic

Uniaxial nematic ←→ Alice electrodynamics



G=S0(3)
H = O(2) = U(1) x| Z_2

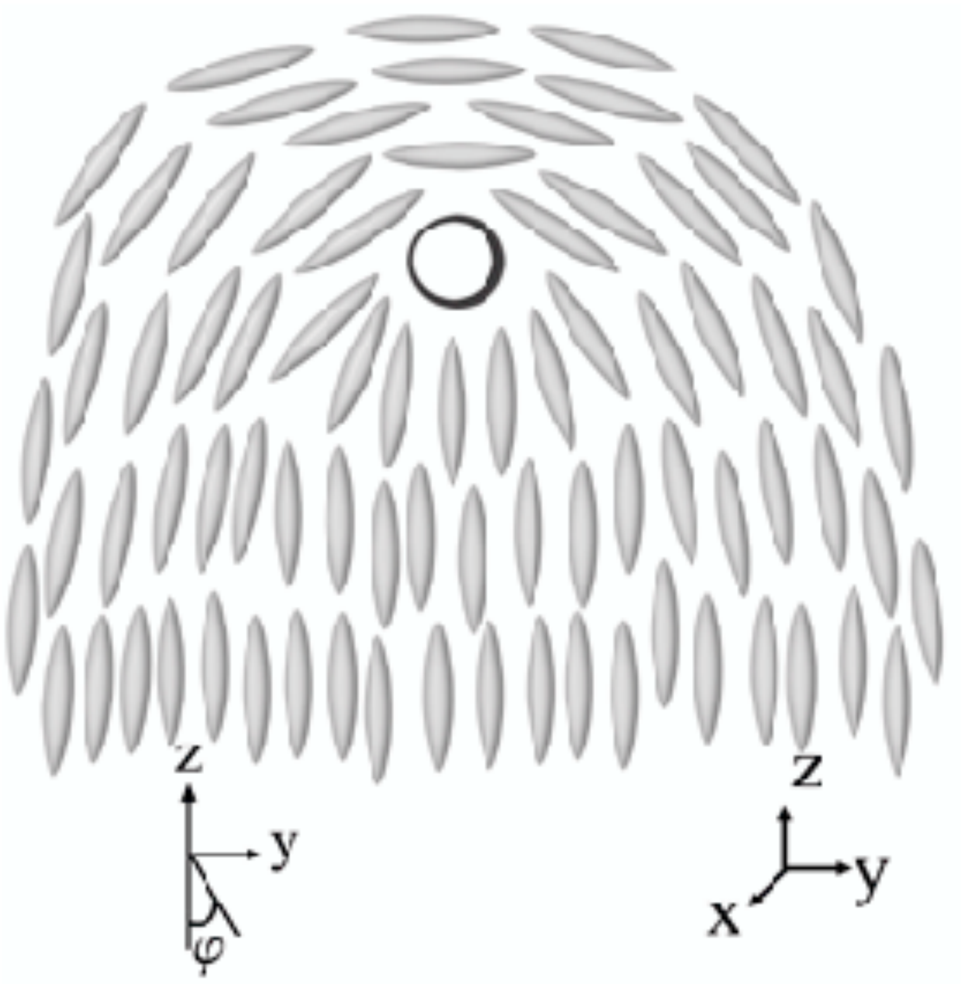
Goldstone modes



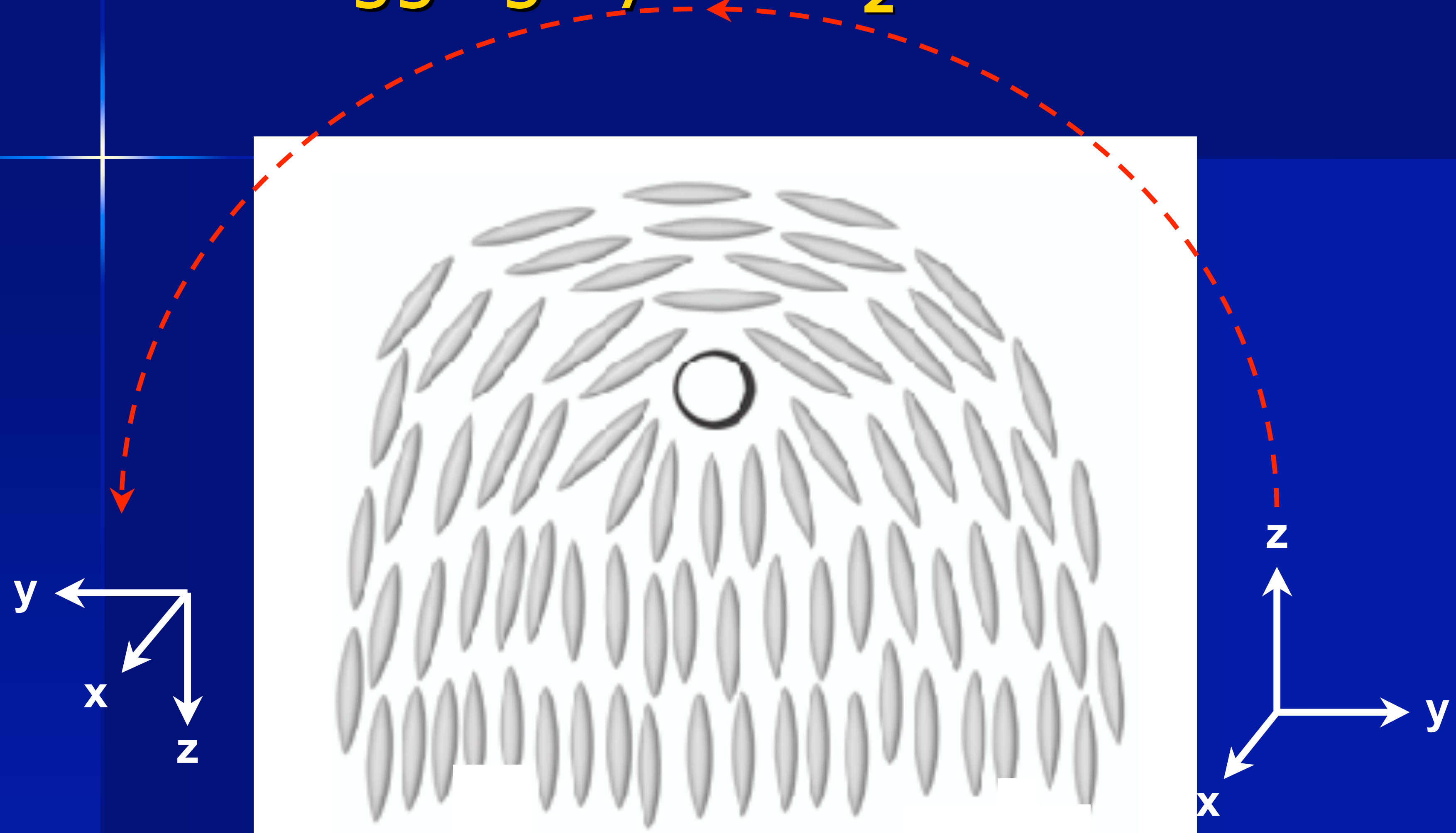
Generator T_x T_y

The Z_2 defect

$\Pi_1 (G/H) = \Pi_0 (H) = Z_2$

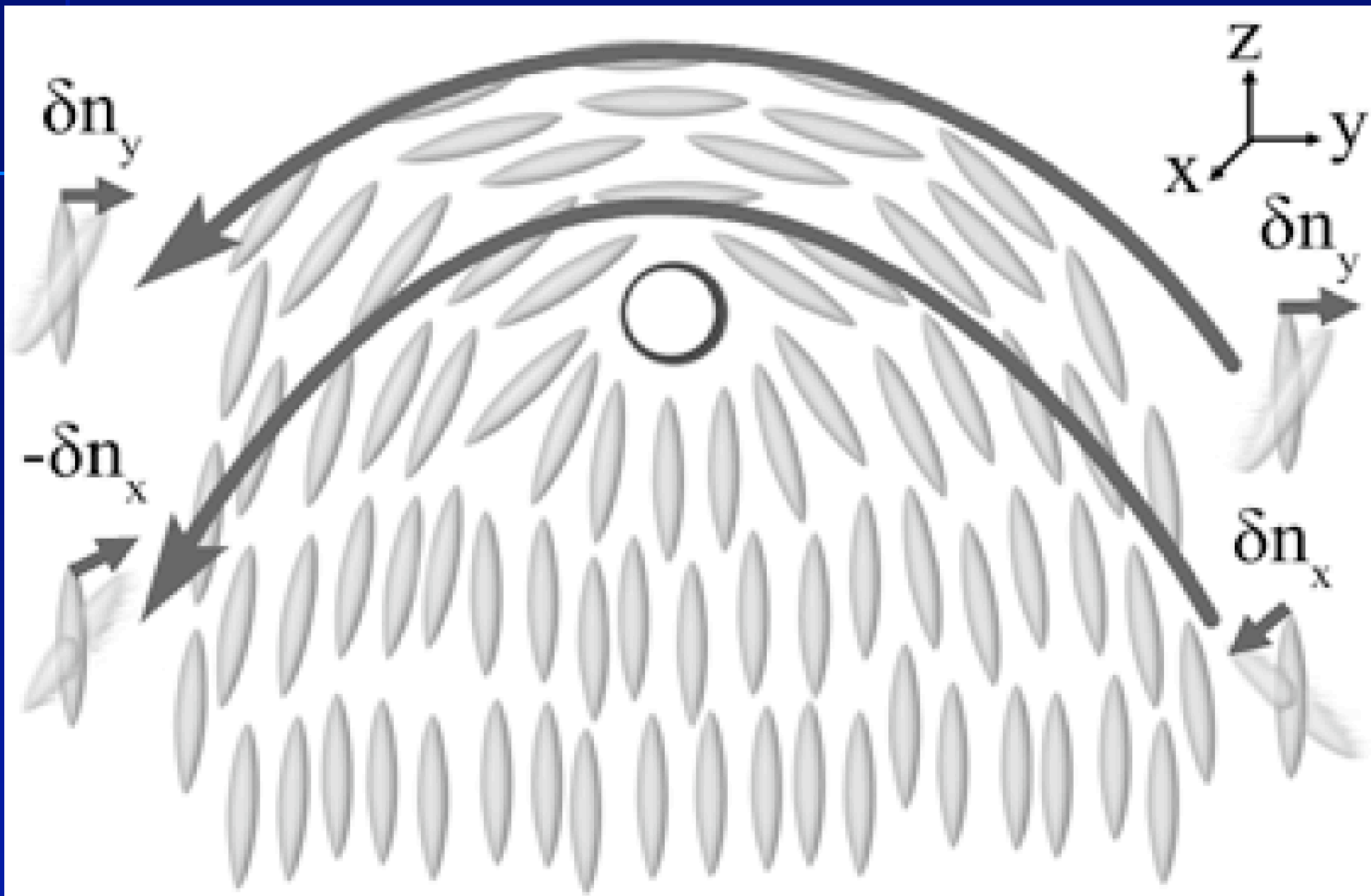


Frame dragging by the Z_2 defect



Frame gets rotated by angle π around x-axis

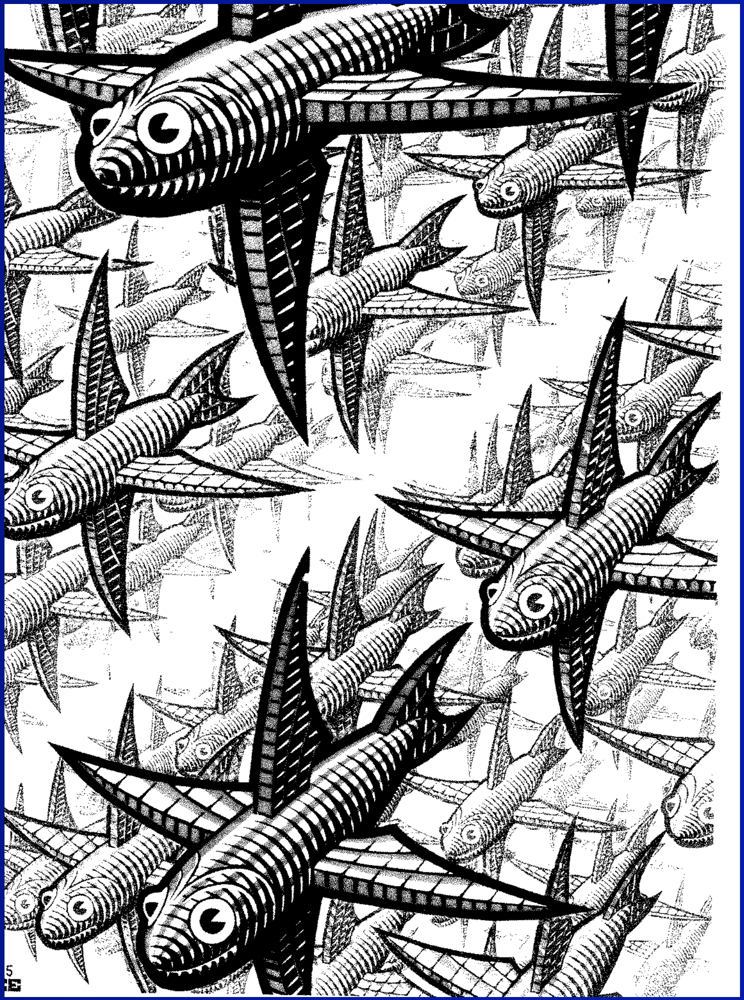
Topological interaction of modes with defect



Presence of defect
→ Obstruction to global implementation of certain symmetries

Lattice

Lattice with spins



Breaking the Euclidean group

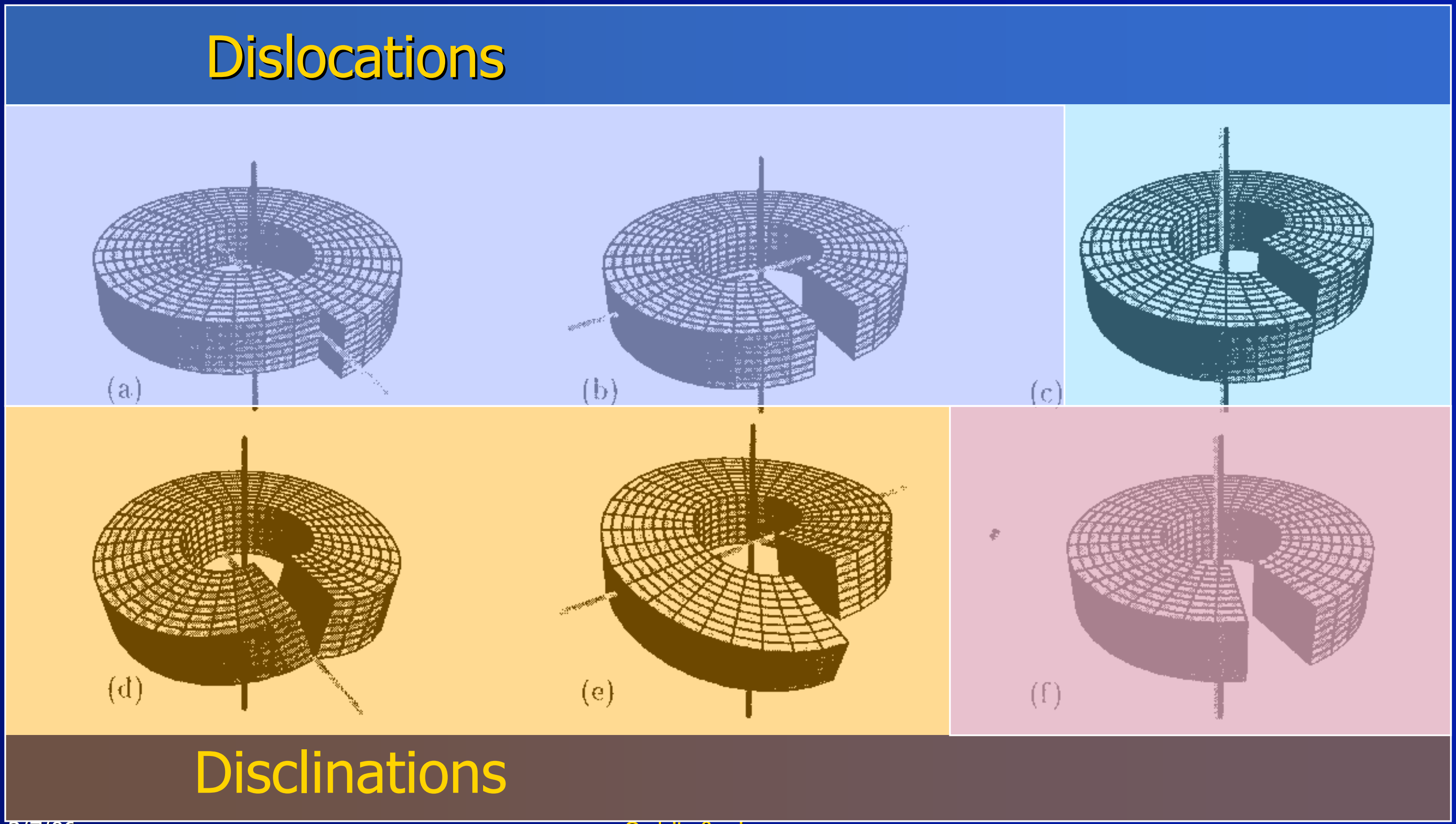
G = Euclidean group of continuous rotations and translations
(E_3 = 6 parameter goup , E_2 is three parameter group)
$(R_1, a_1)\,(R_2, a_2) = (R_1R_2, a_1 + R_1a_2)$

H = Discrete symmetry group of crystal lattice
(Square lattice in d=2: $H = Z_4 \underline{X} (Z \times Z)$

Excitations:
1. Goldstone modes → Phonons G/H (fundamental)
2. Solitons (defects) → $\pi_1(G/H) = \pi_0\,(H) = H$ (topological)

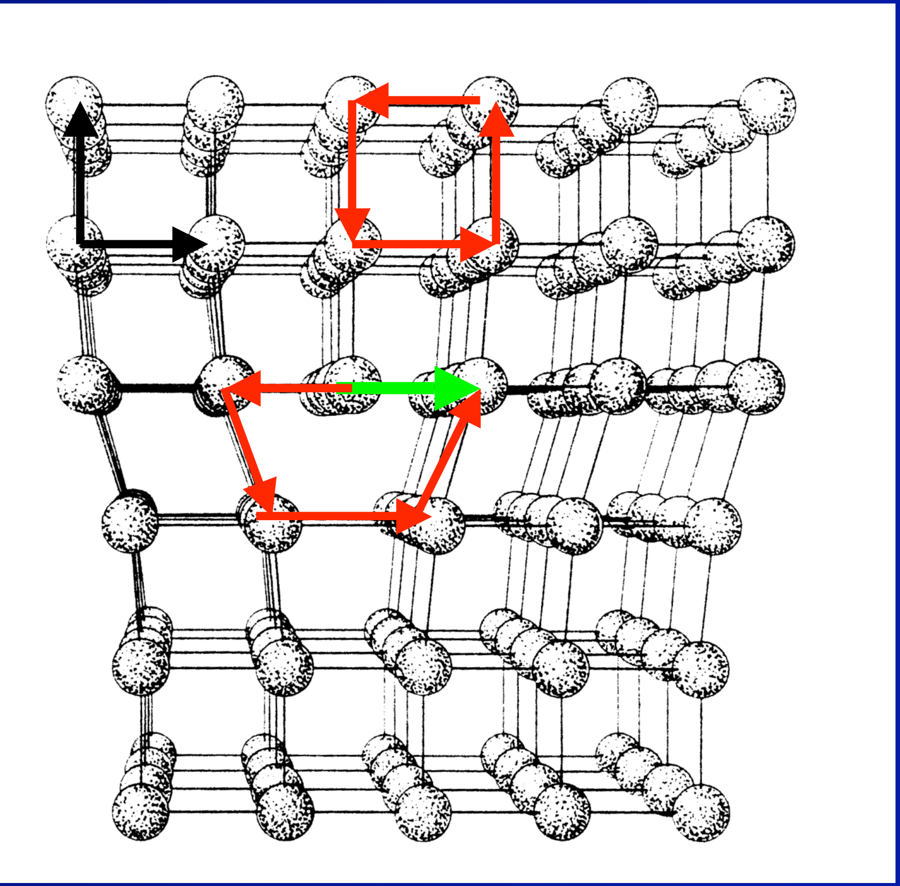
→ group H classifies line/point defects

Volterra moves: a defect classification

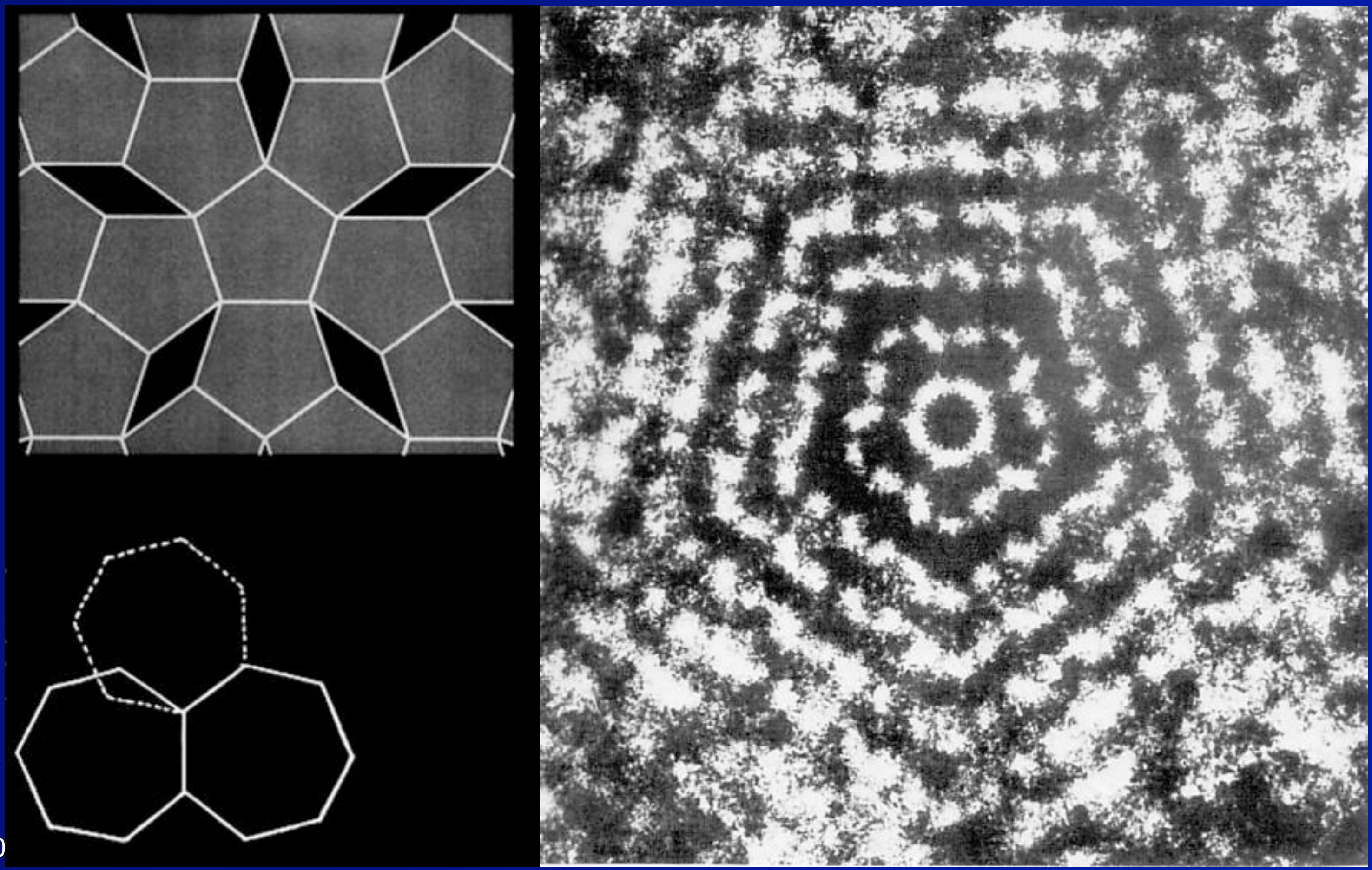


Dislocation: translational defect

- Defect (1, a)
- Burgersvector a

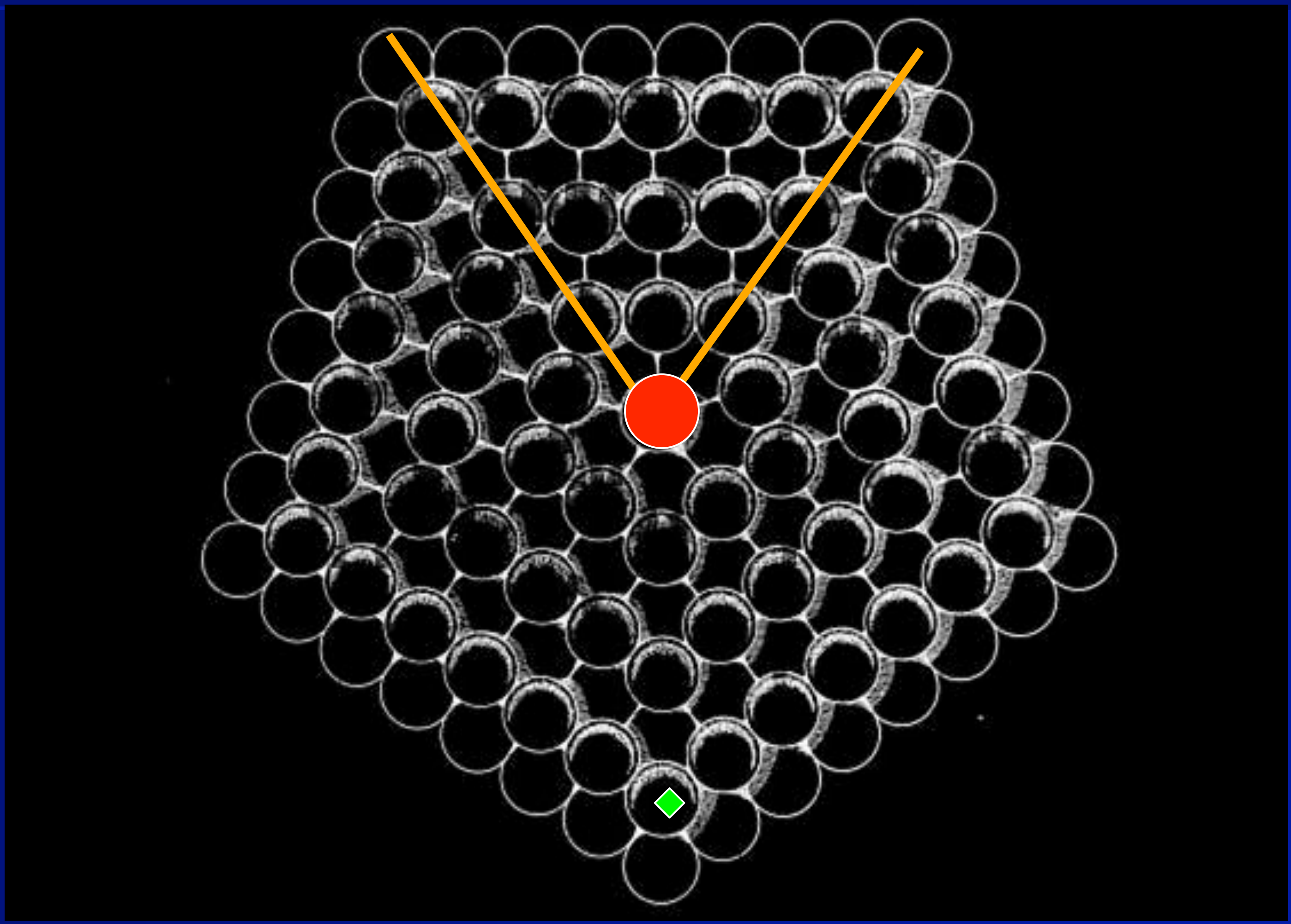


5-fold symmetry?



R17 virus

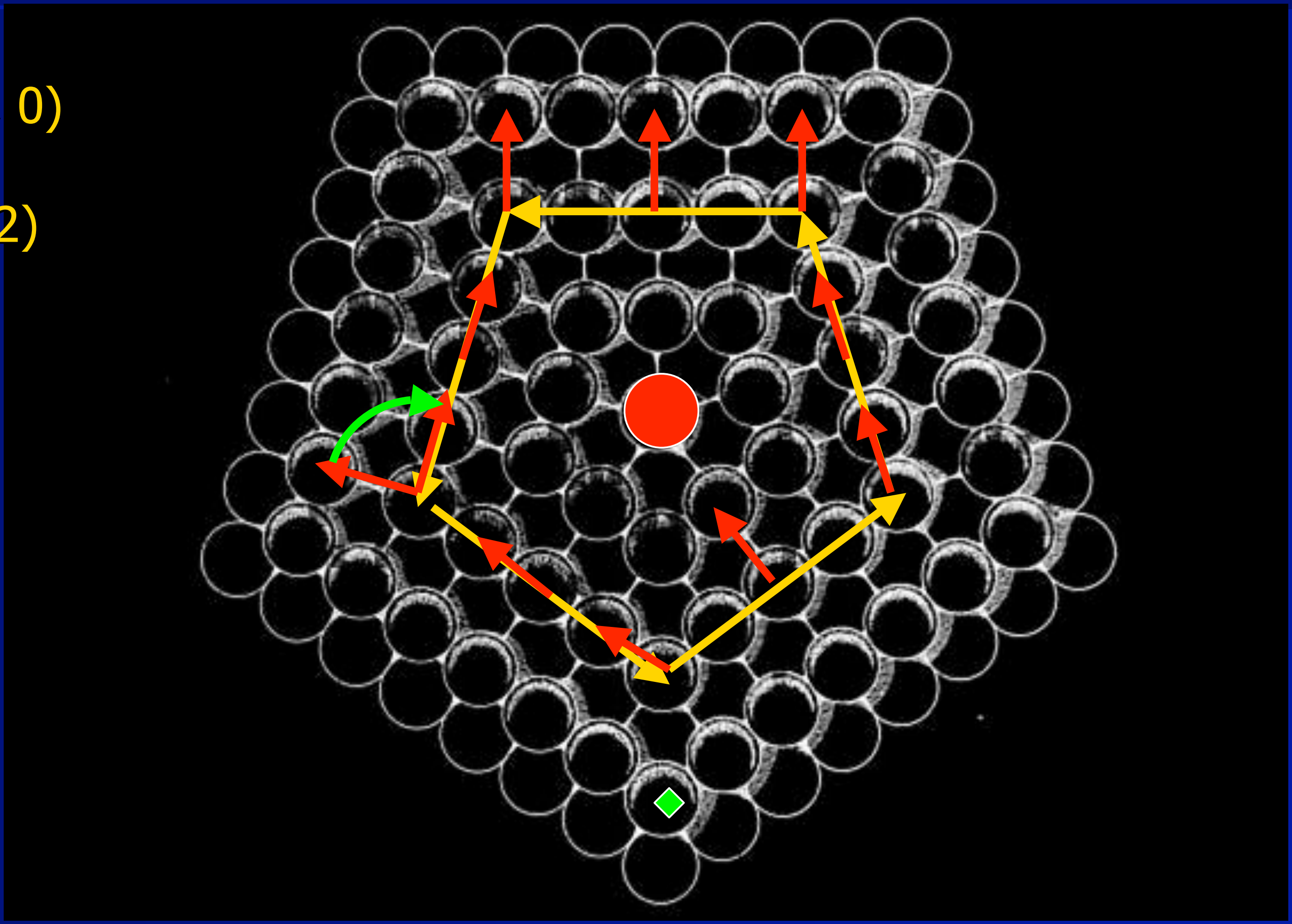
Disclination



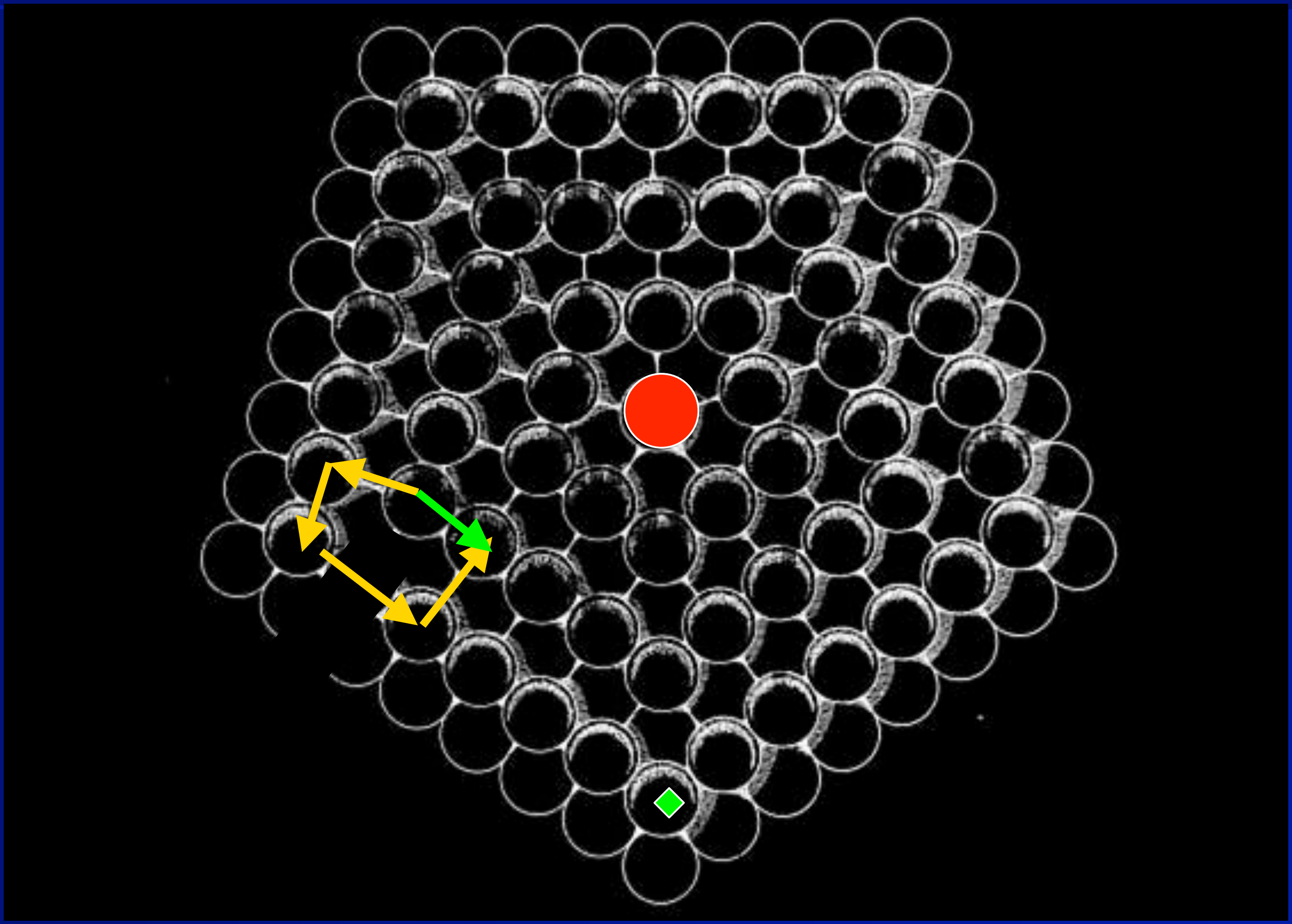
Disclination: Rotational defect

Defect (R, 0)

R=R(-π/2)



Two non-commuting defects



Braiding of noncommuting defects

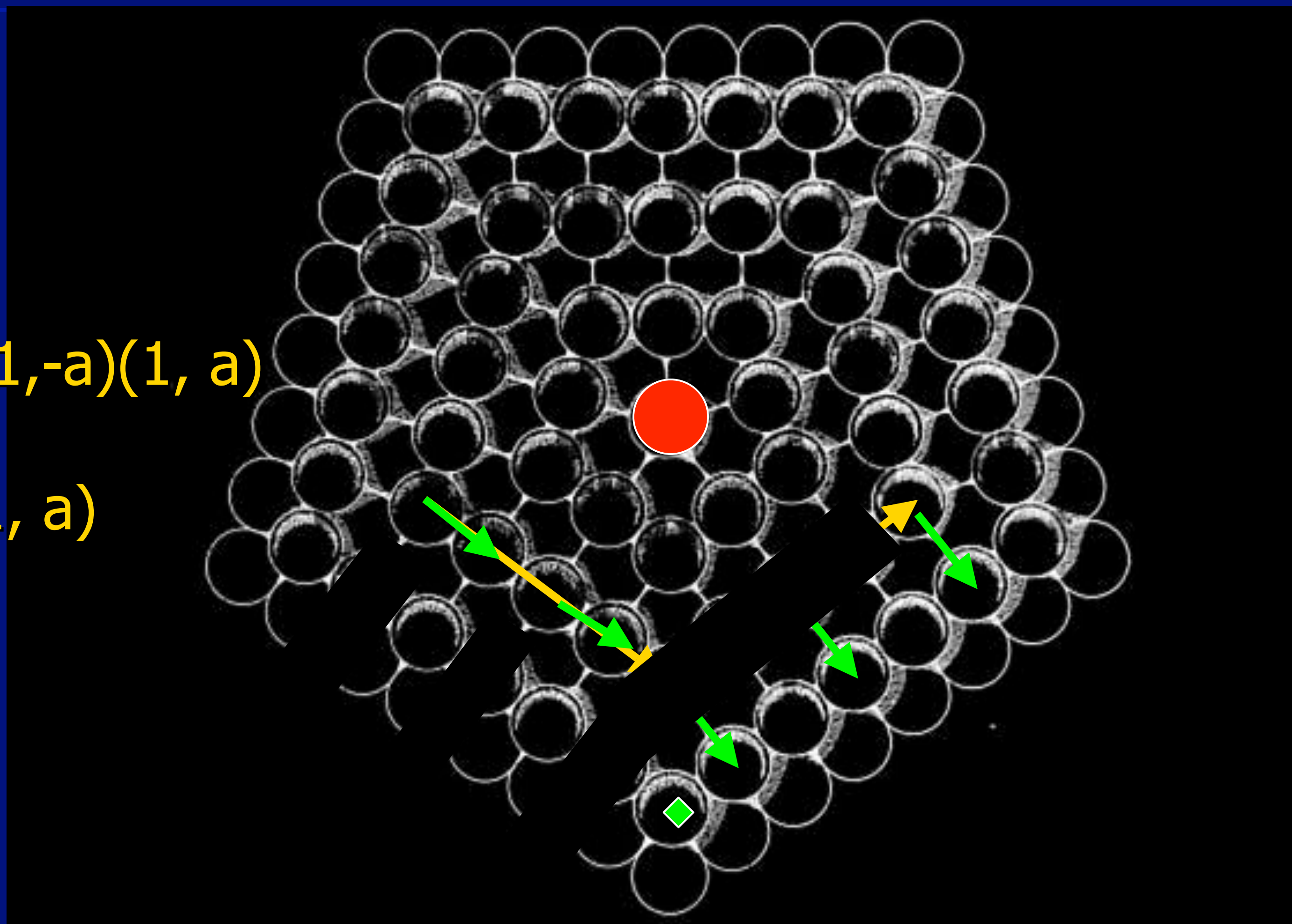
T :

(1,a)(R,0) →

→ (1, a)(R,0) (1,-a)(1, a)

= (R, a - Ra) (1, a)

= (R, a)



Braiding of noncommuting defects

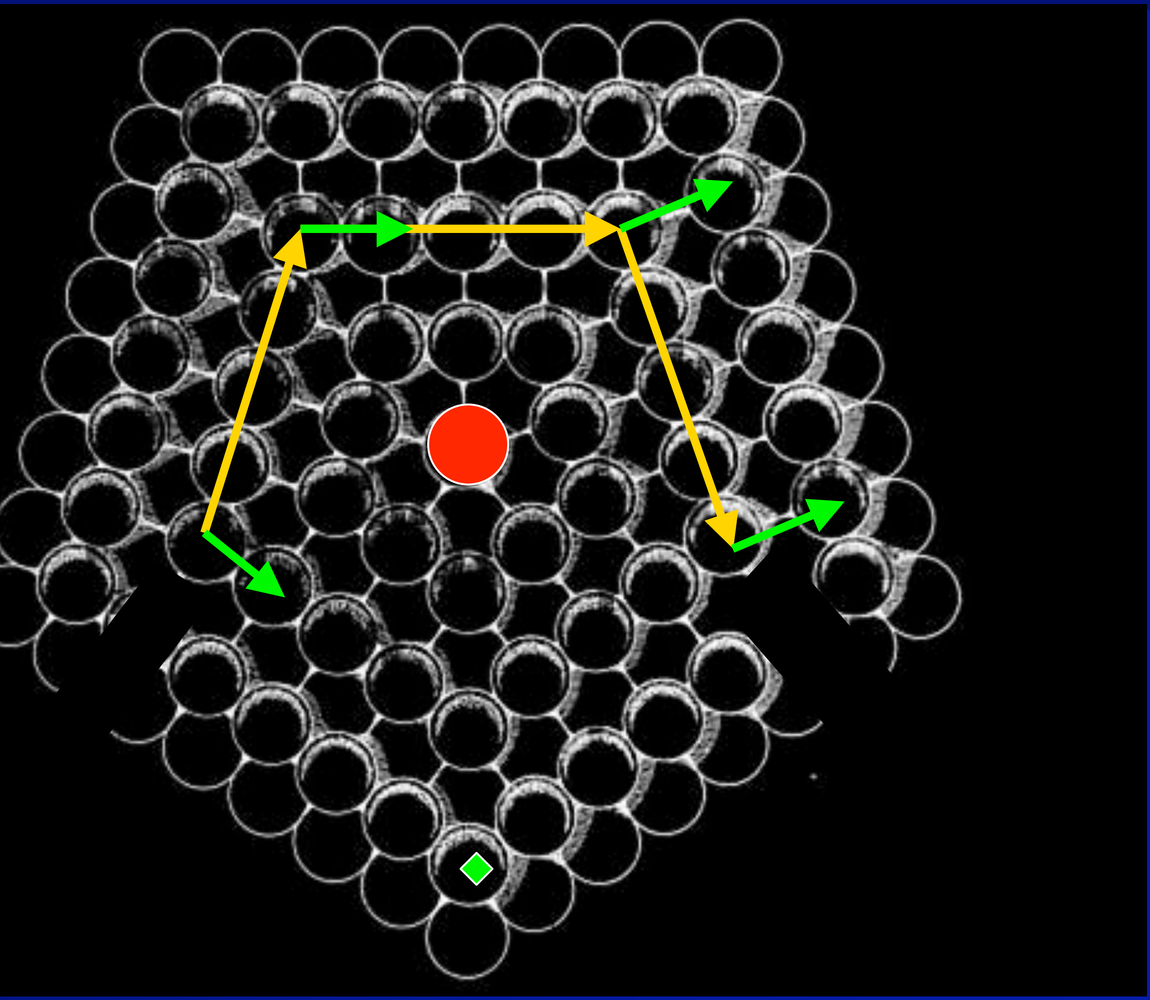
T^{-1} :

$(1,a)(R,0) \rightarrow$

$\rightarrow (R,0)(R^{-1},0)(1,a)(R,0)$

$\rightarrow (R,0)(1,R^{-1} a)$

$= (R,a)$



Non-abelian flux in discrete gauge theories

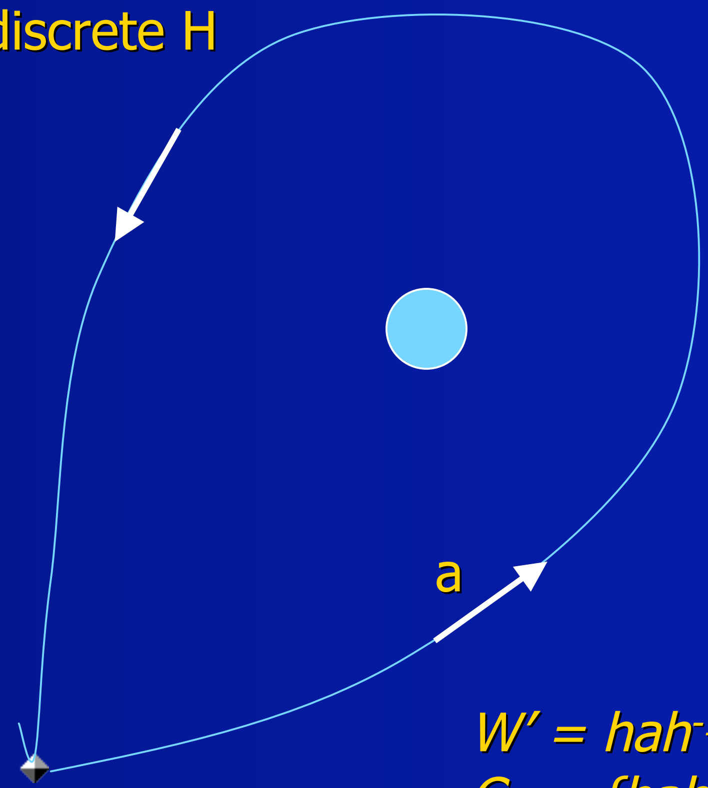
Setting: continuous G breaks to discrete H

$$W = Pe^{iq\oint A dl}$$

group element of *H*

algebra

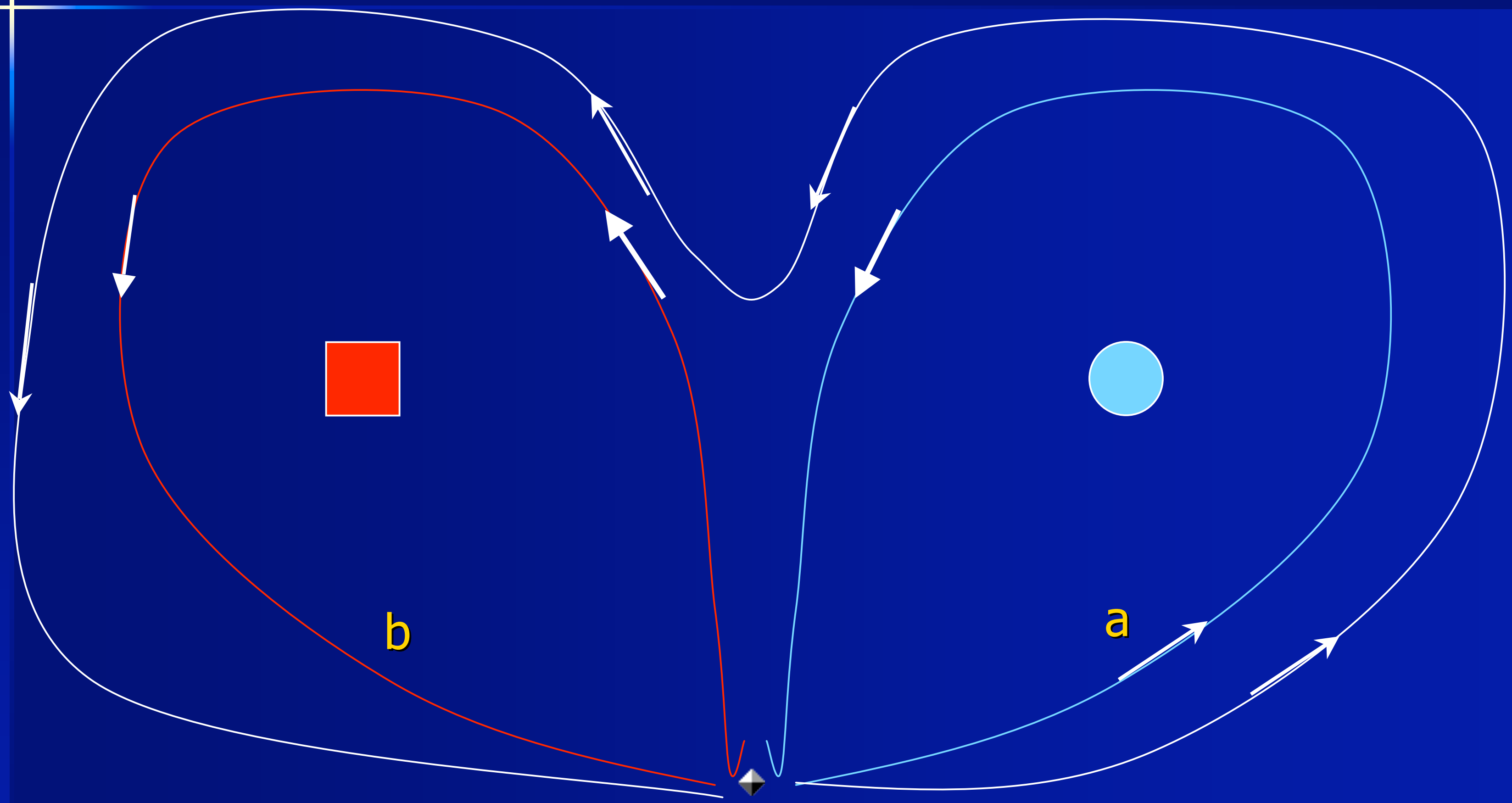
$W = a$



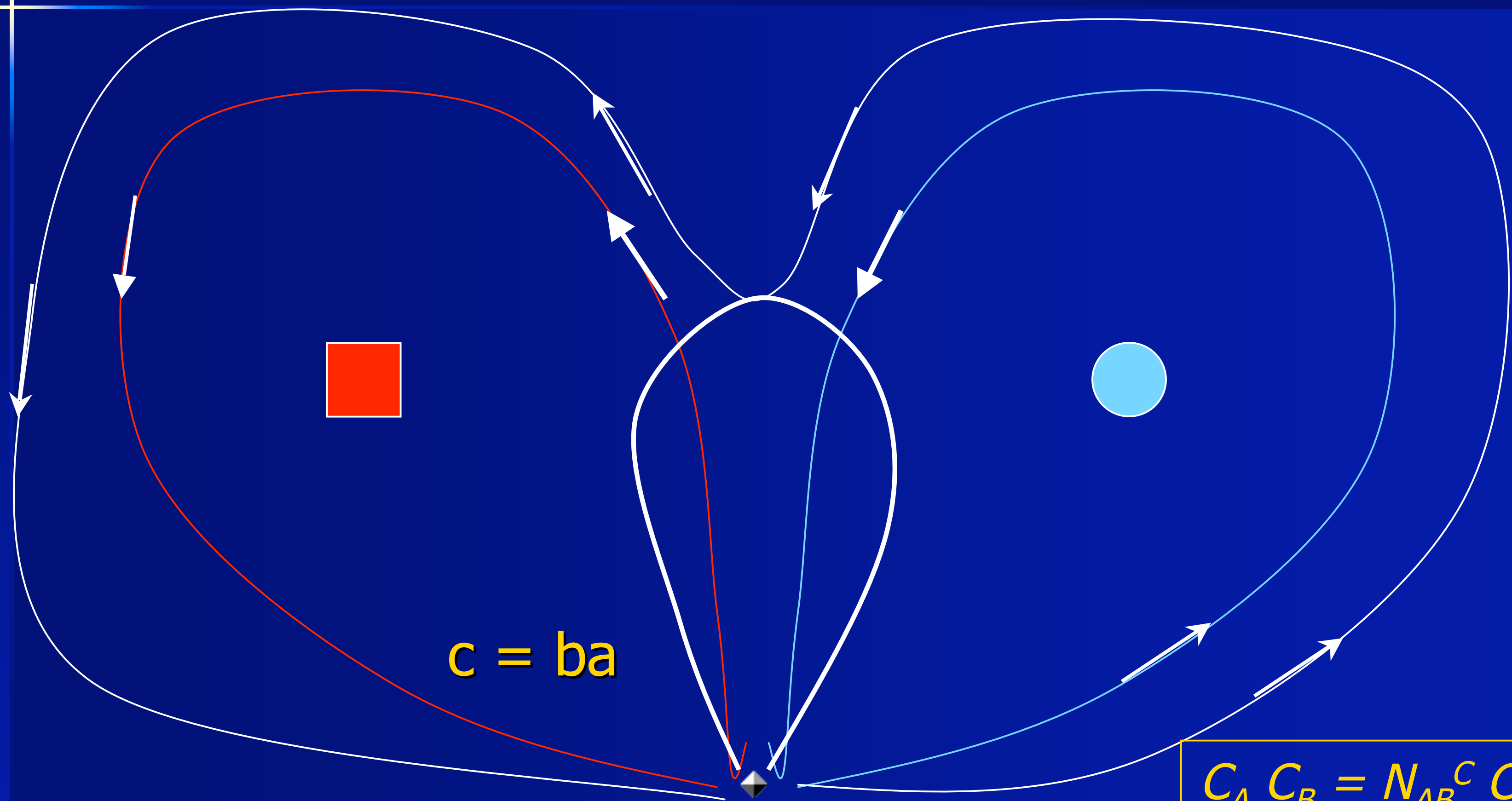
$W' = hah^{-1}$

$C_A = \{hah^{-1}\}$

Composition rules: Fusion of defects

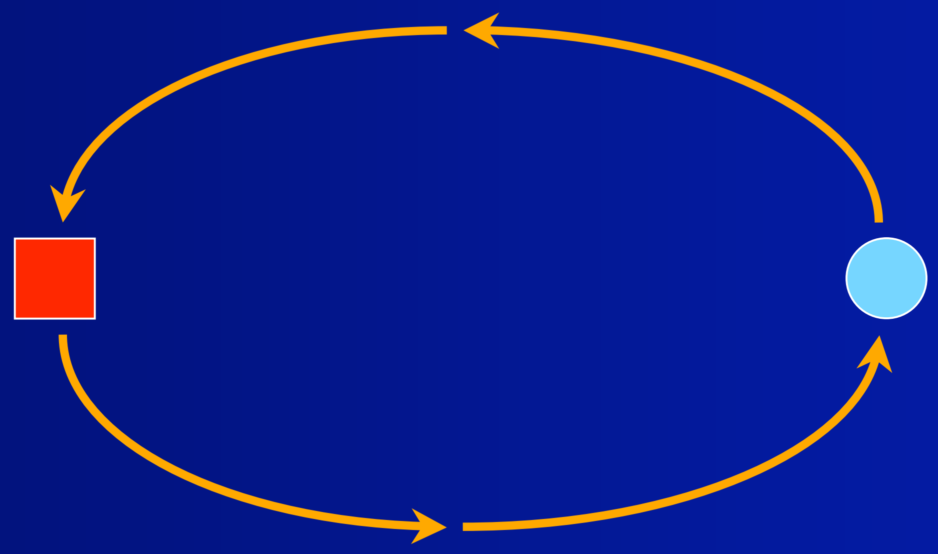


Composition rules: Fusion of defects

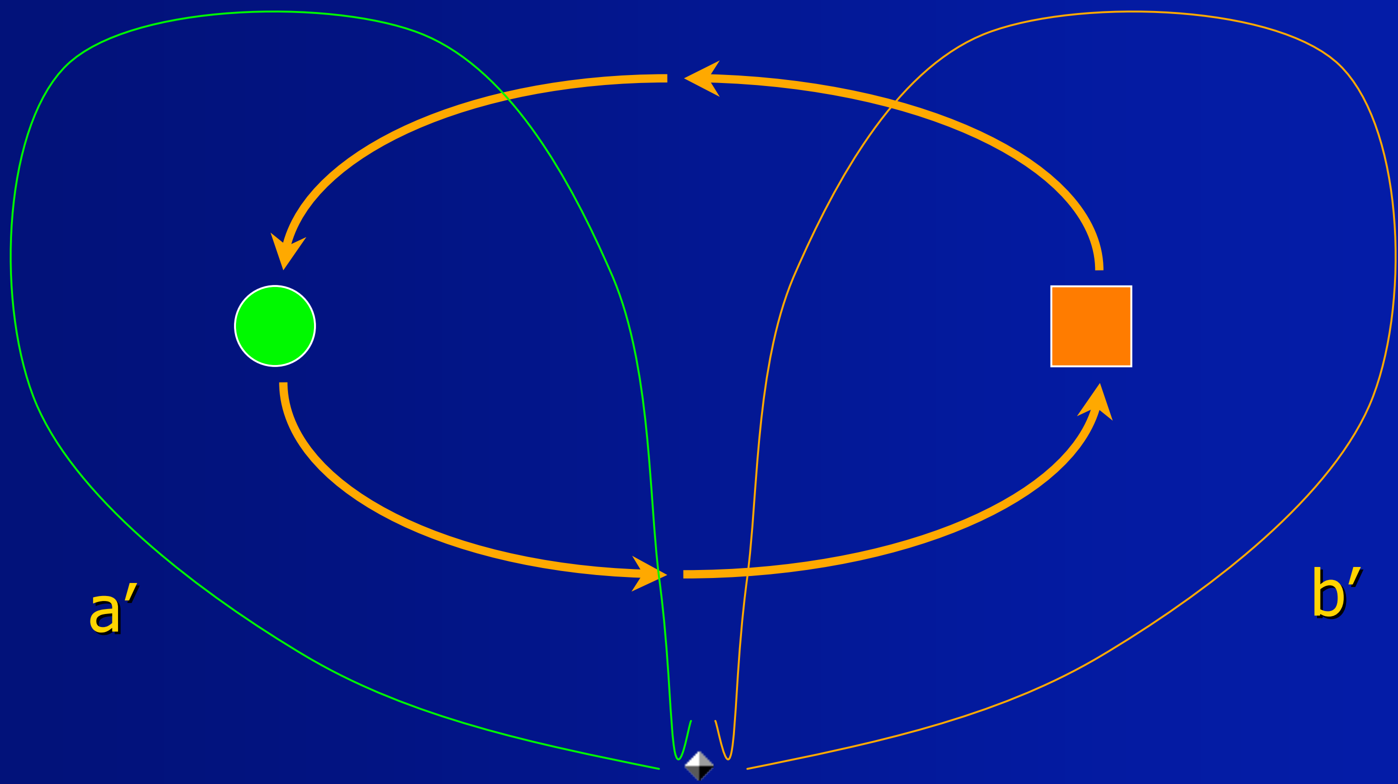


$C_A\ C_B = N_{AB}{}^C\ C_C$

Interchange: braiding of defects

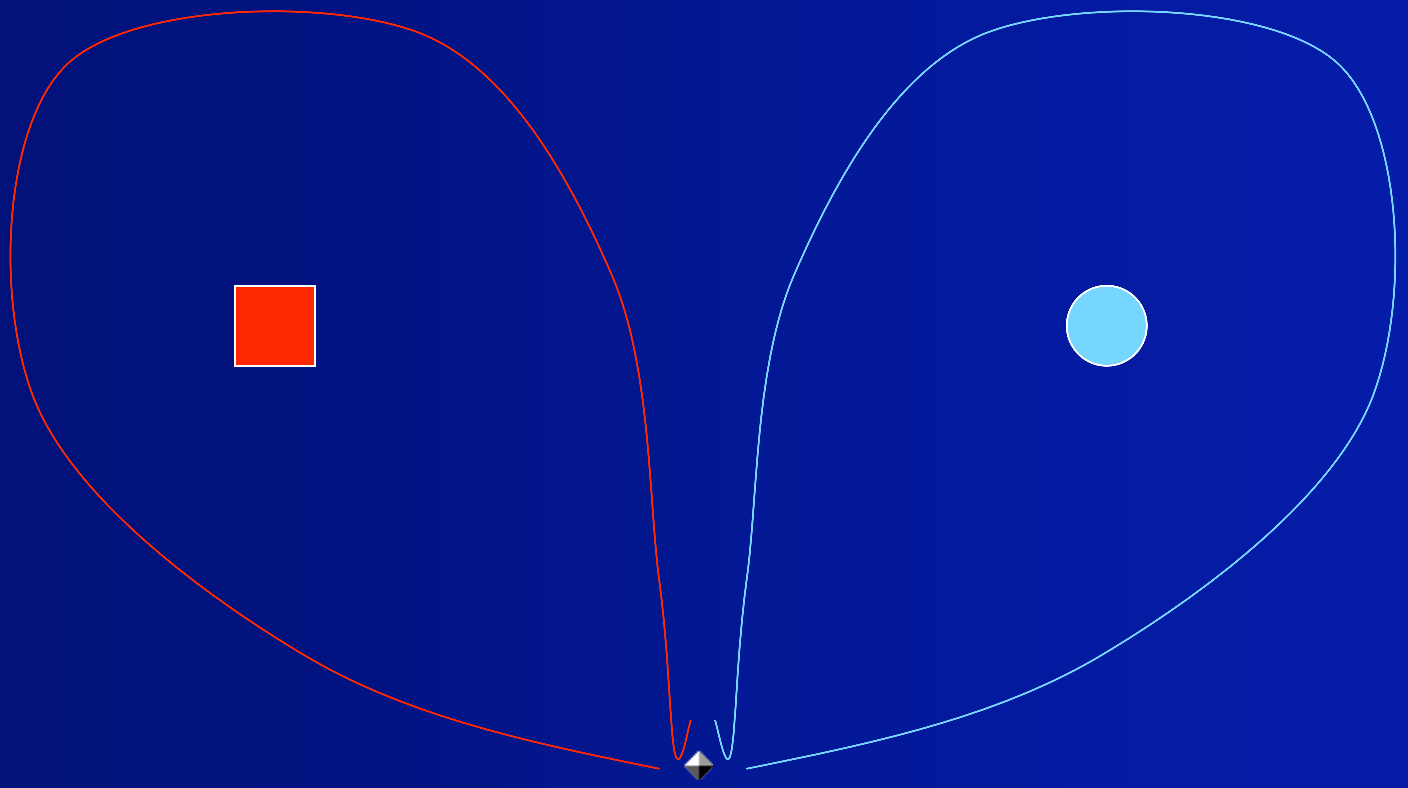


Interchange: braiding of defects

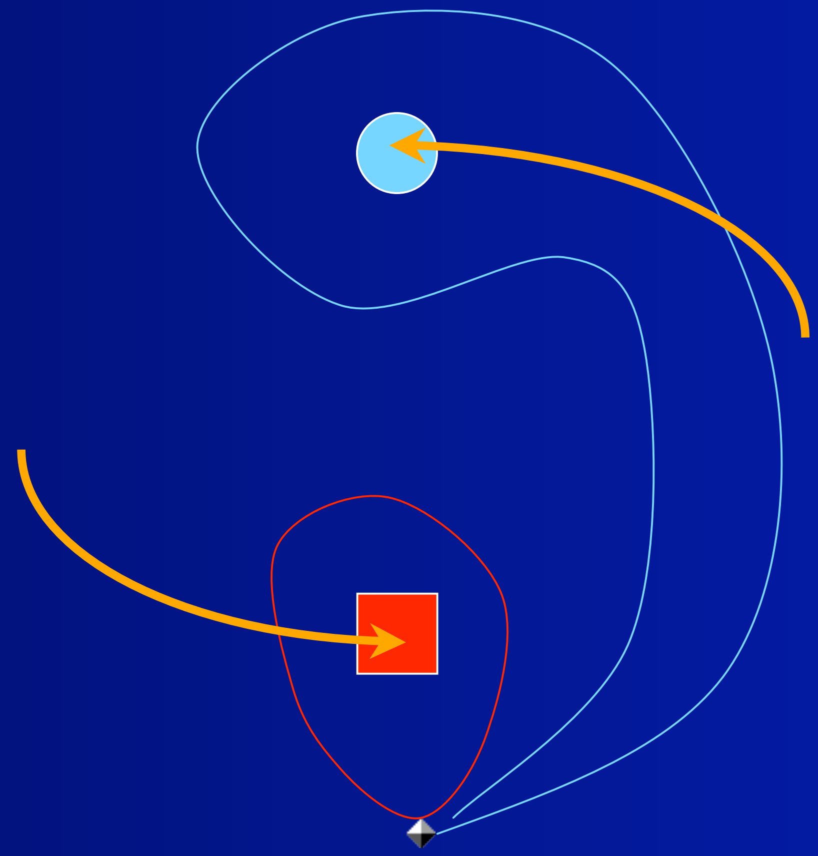


R(ba) = a′b′

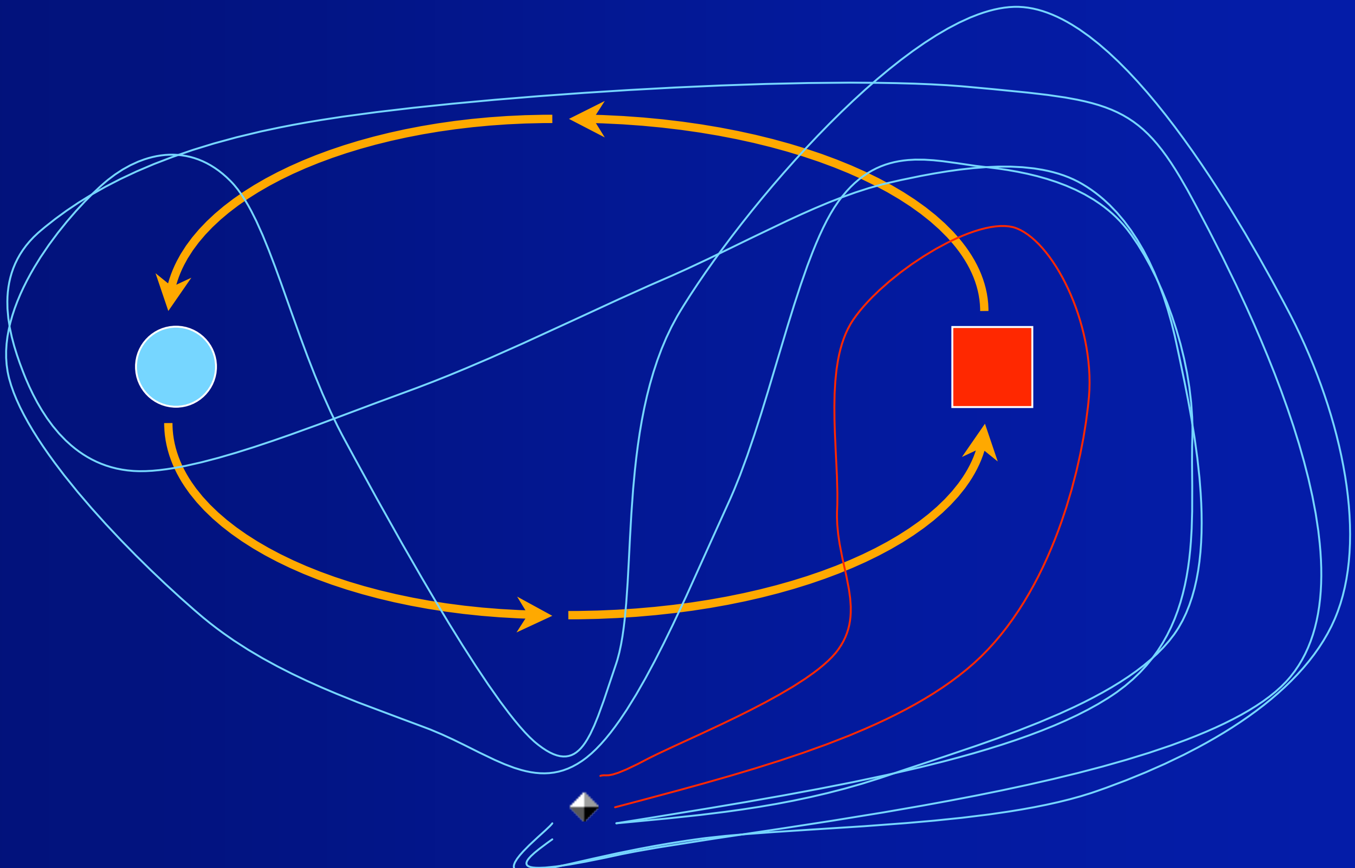
Interchange: braiding of defects



Interchange: braiding of defects



Interchange: braiding of defects

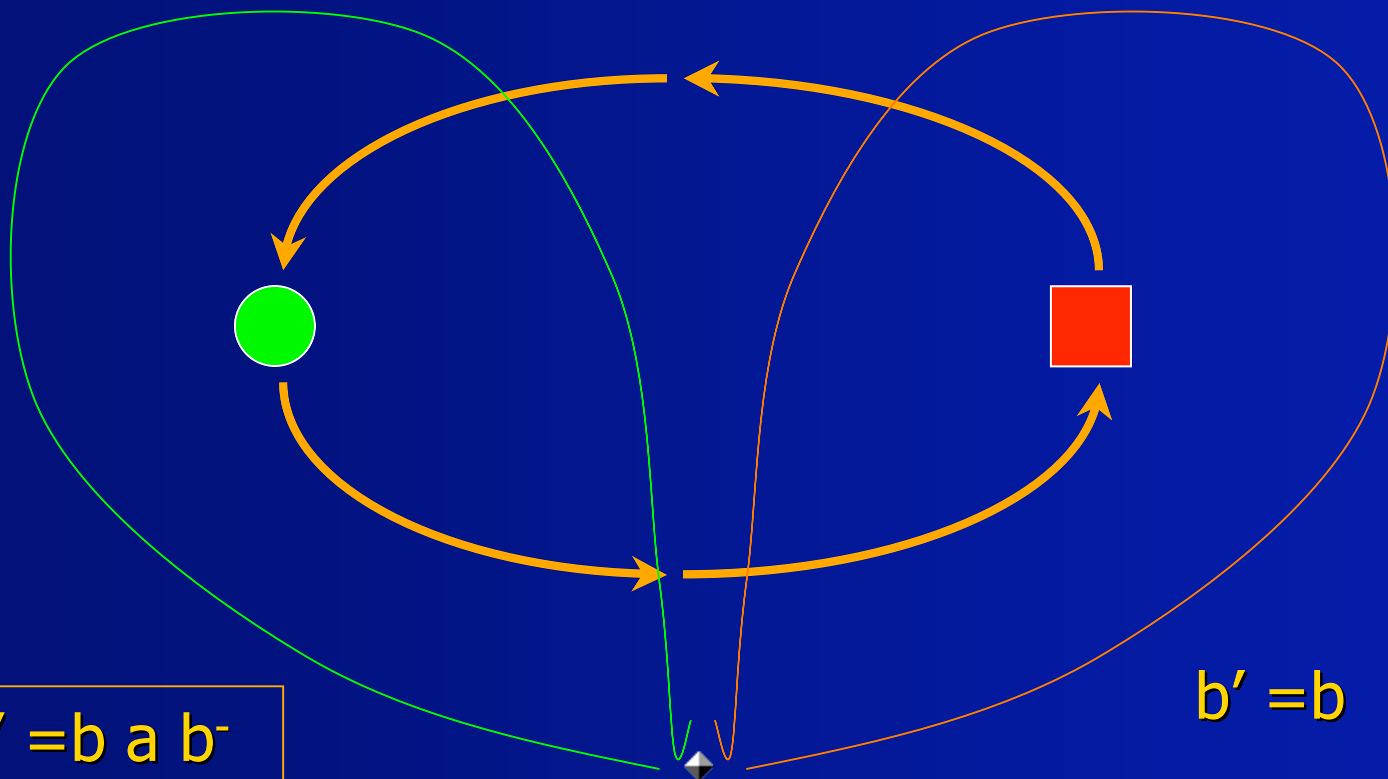


Interchange: braiding of defects

$a = b^{-1} a' \, b$

→ $a' = b \, a \, b^{-1}$

Interchange: braiding of defects

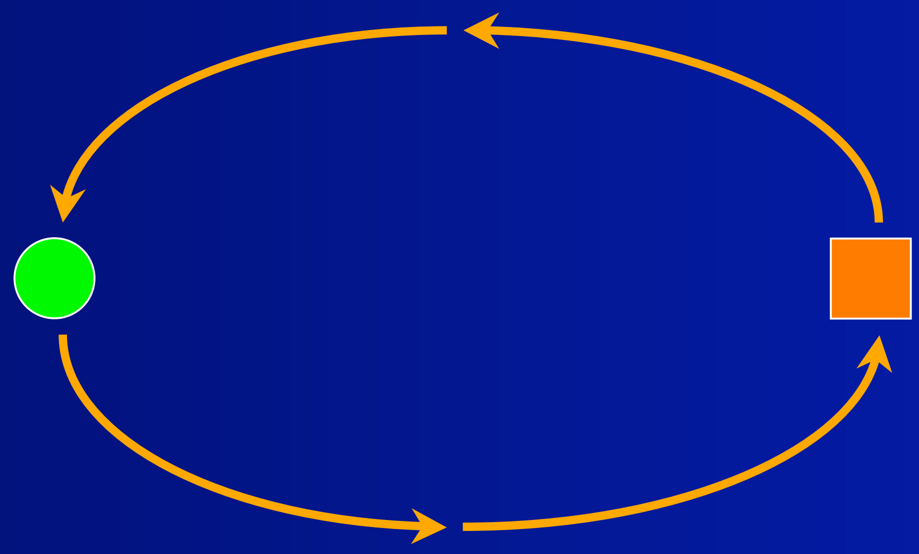


$a' = b\,a\,b^{-1}$

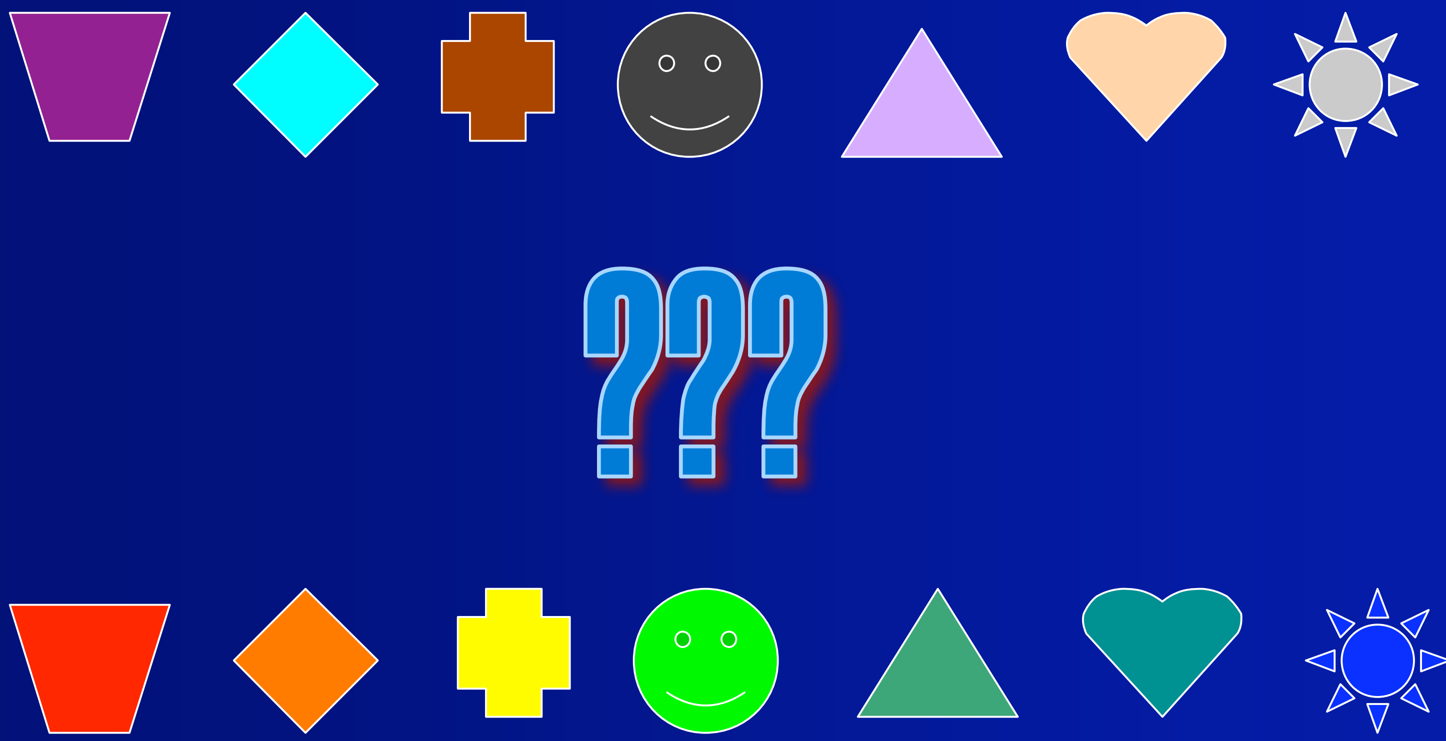
$b' = b$

Algebraic argument

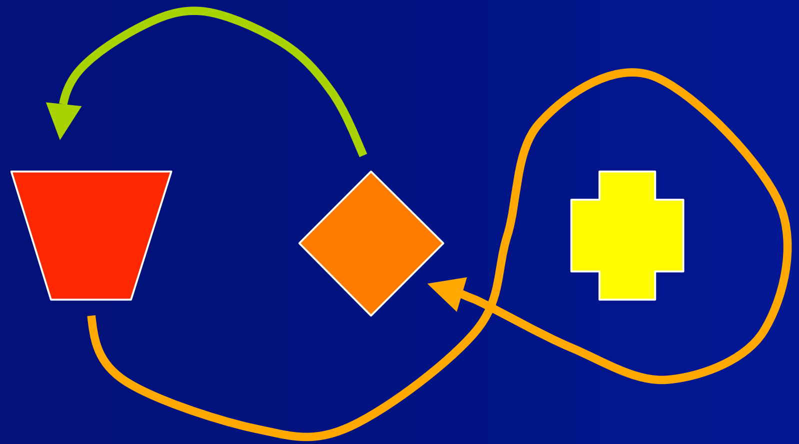
- R(ba) = a'b'
- in fact ba = a'b'   (c=c)
- we note that b'=b
- → ba = a'b
- → $a' = bab^{-1}$

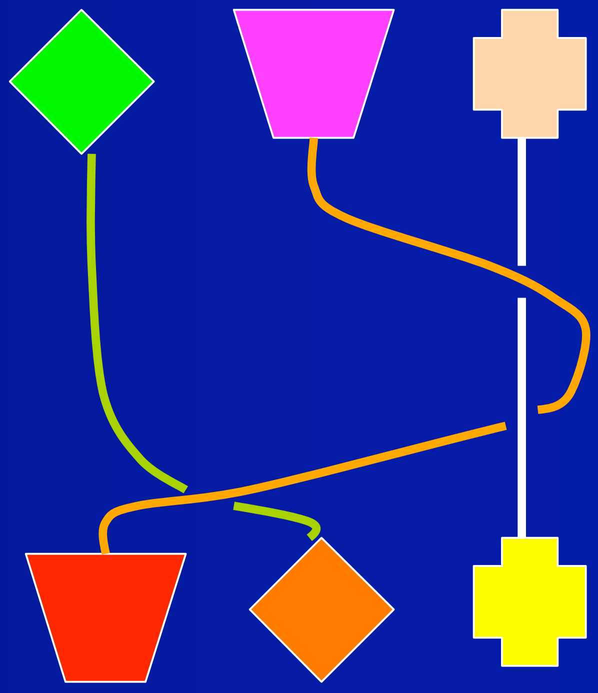


Multiparticle braid relations

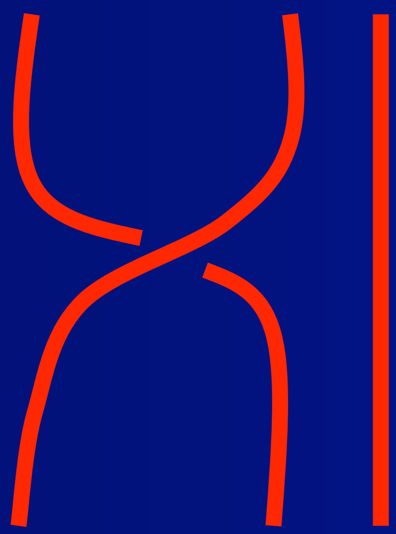


Braid group B_n on n strands

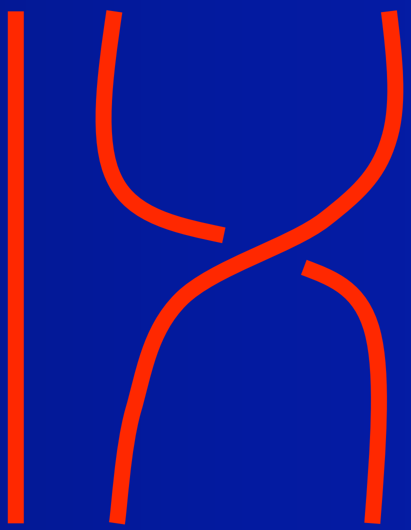




Braid group generators

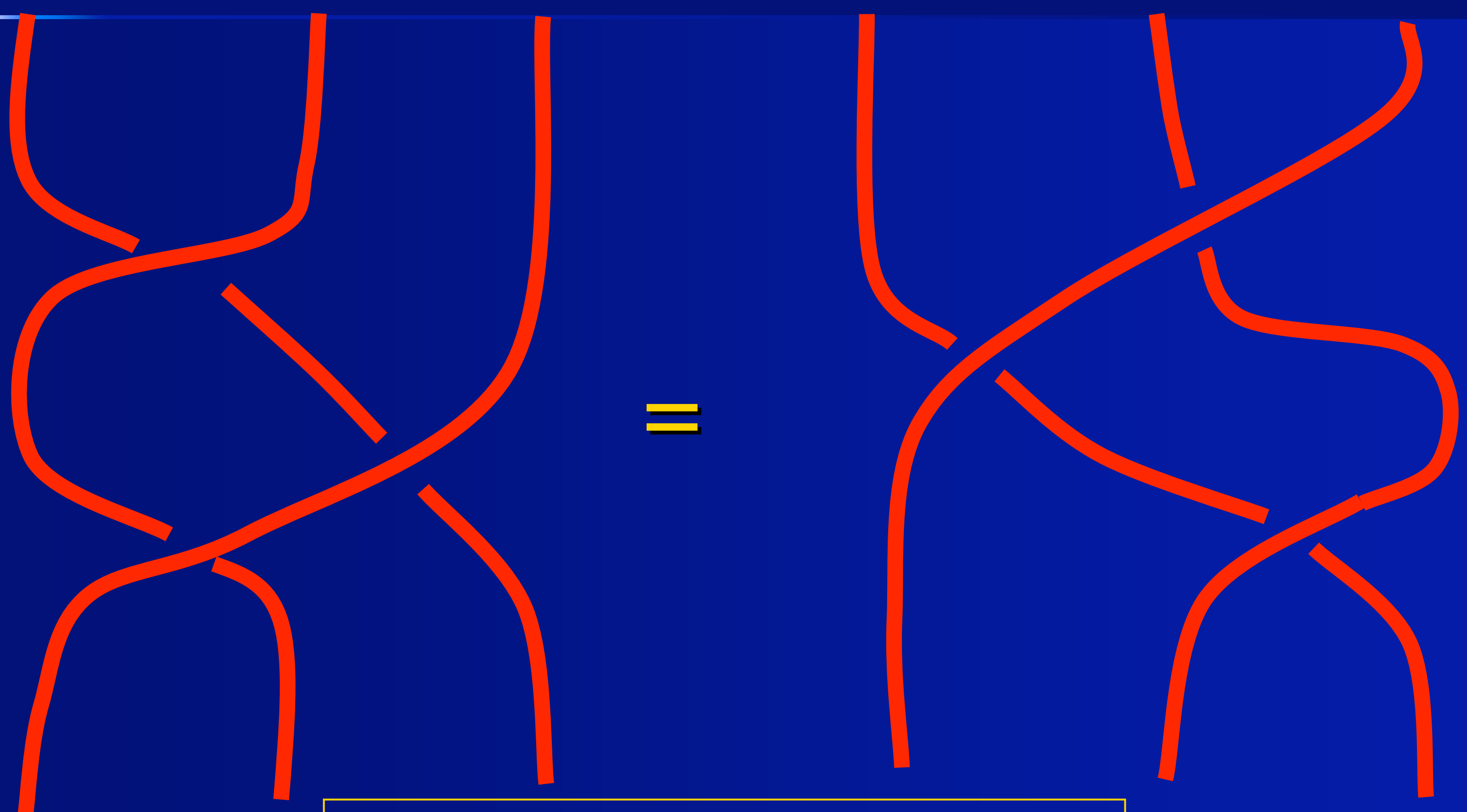


τ_1



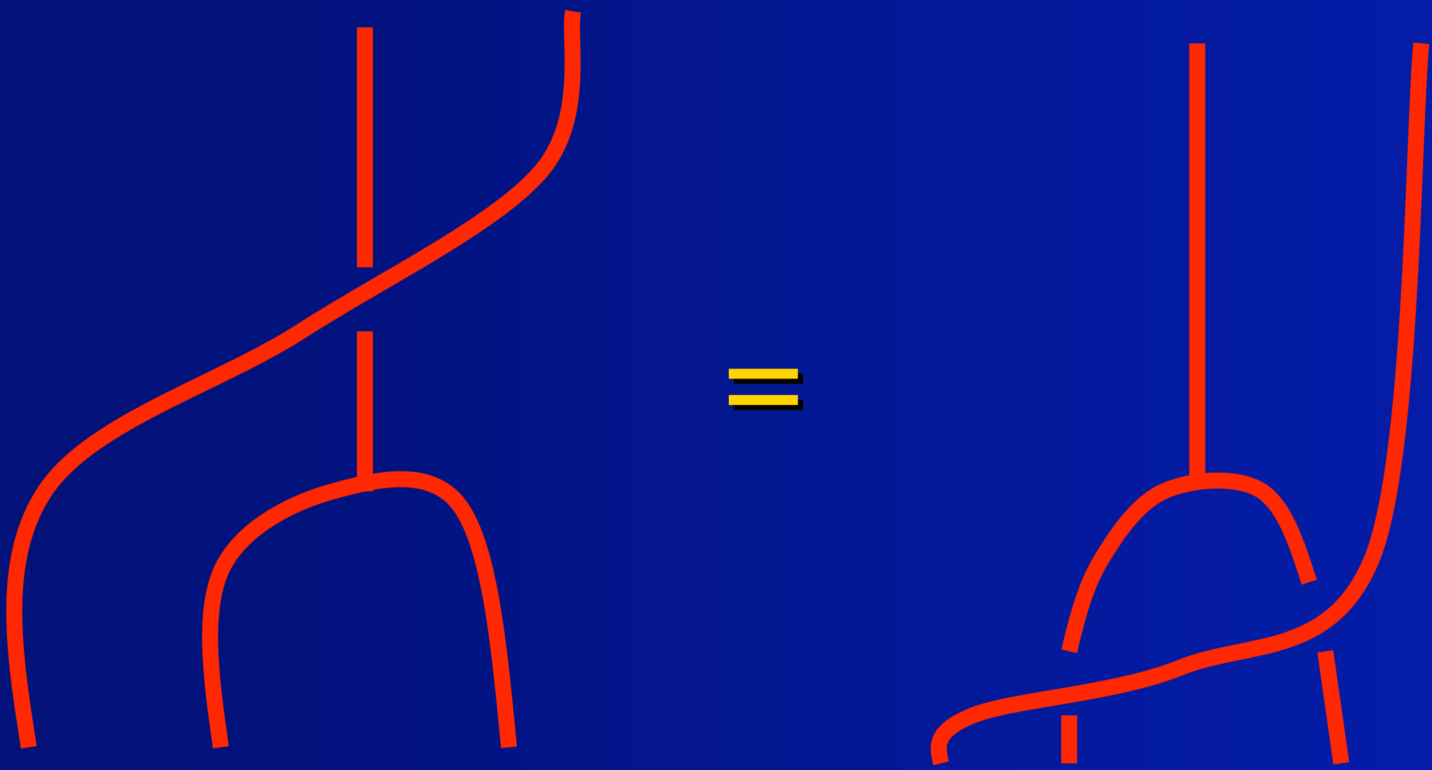
τ_2

Relation between generators (Yang Baxter)

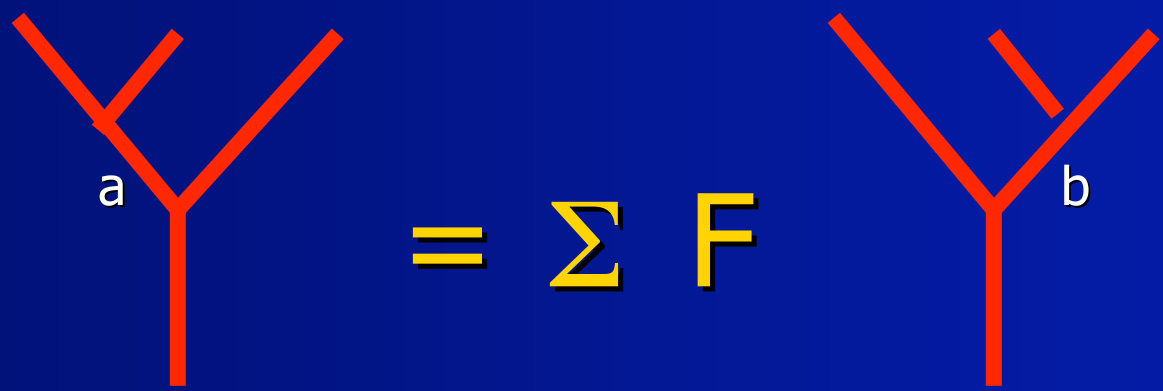


$$\tau_1 \, \tau_2 \, \tau_1 = \tau_2 \, \tau_1 \, \tau_2$$

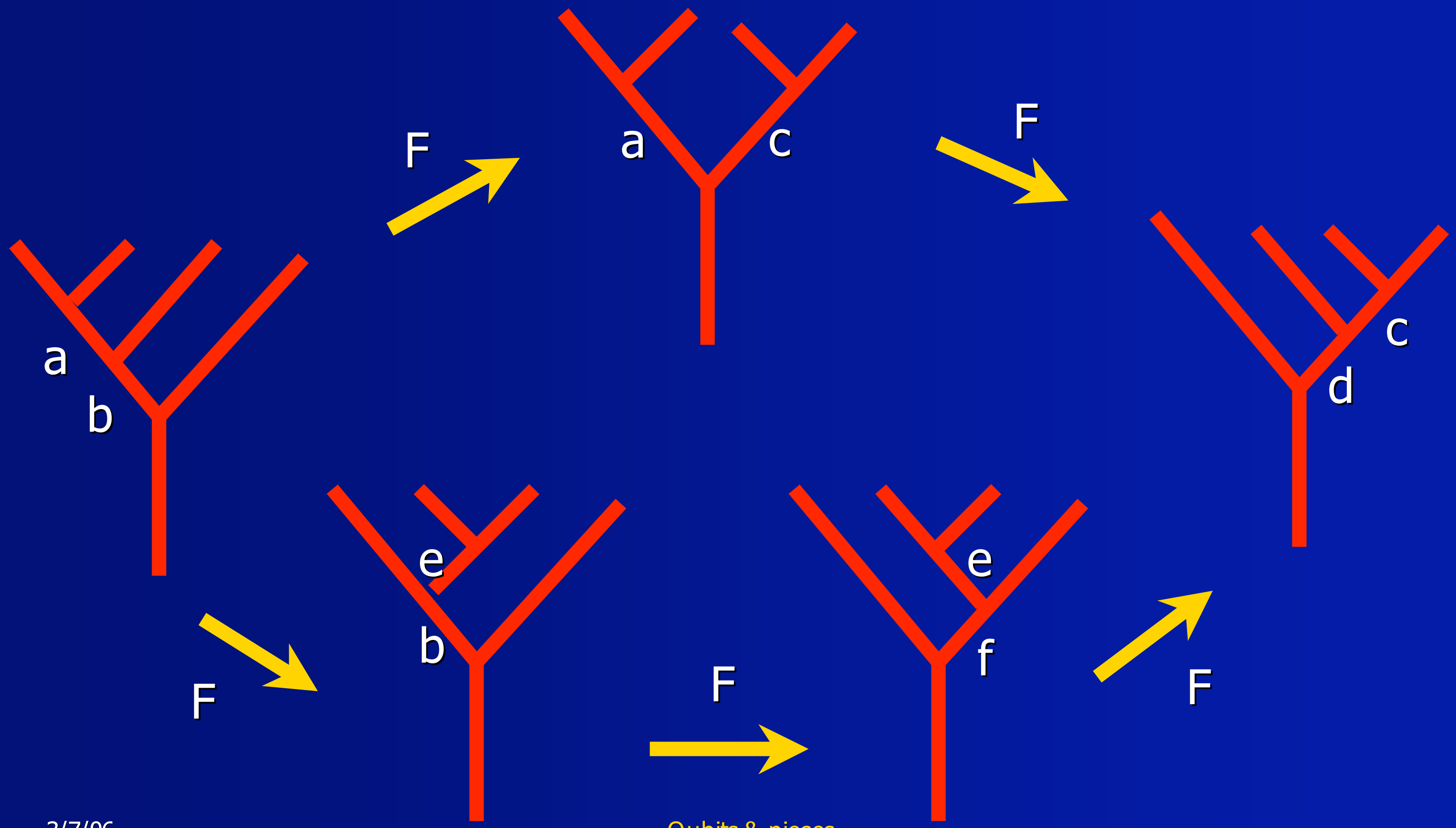
Consistency of braiding and fusion



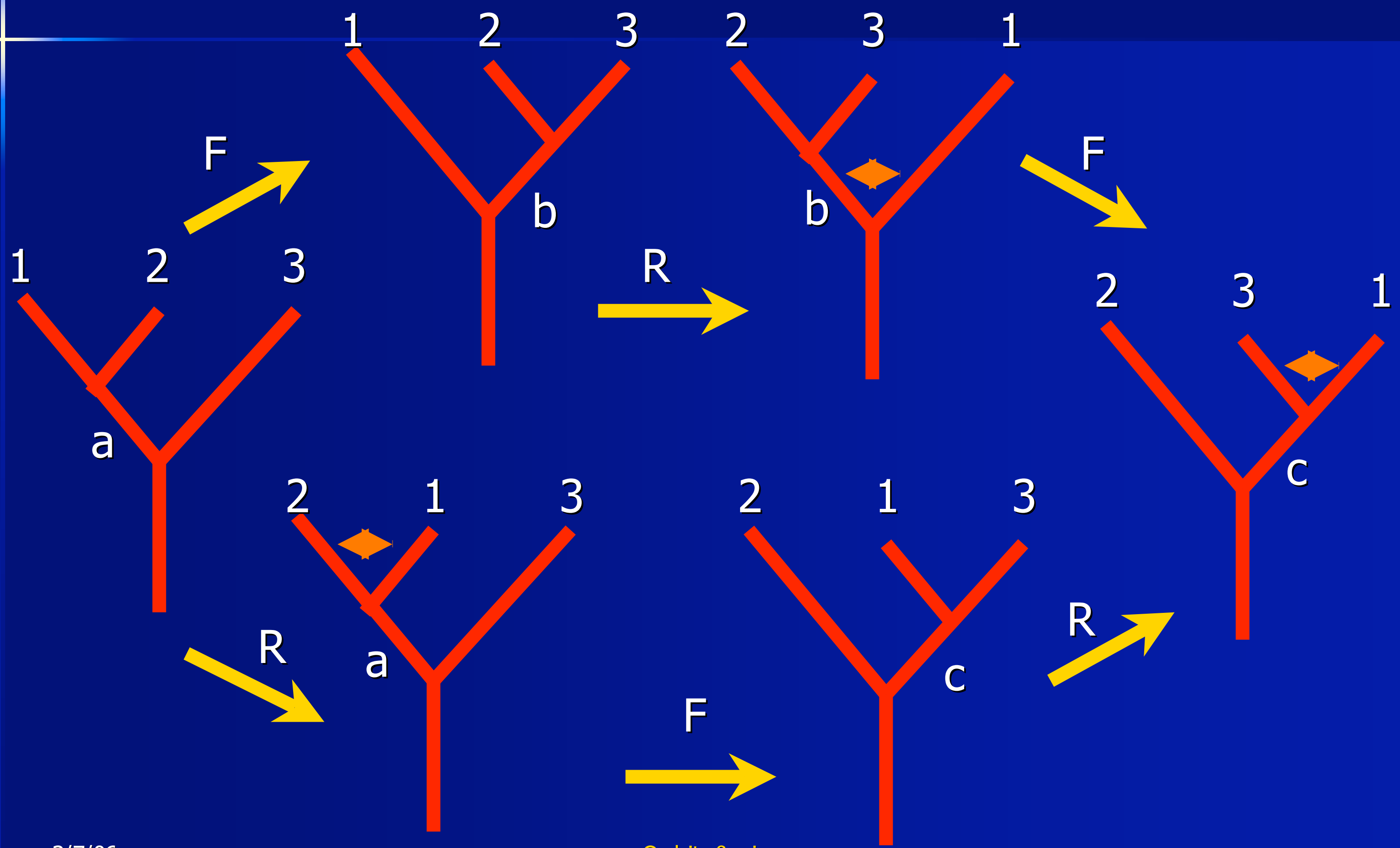
Associativety of fusion rules



Pentagon relation



Hexagon relation



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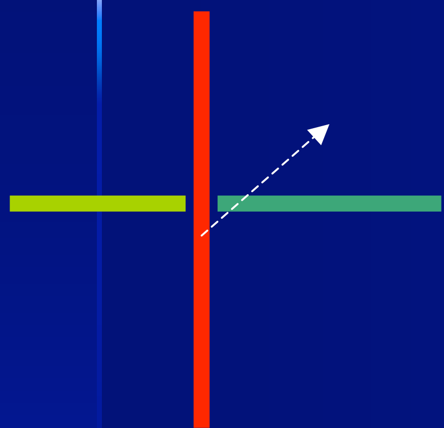
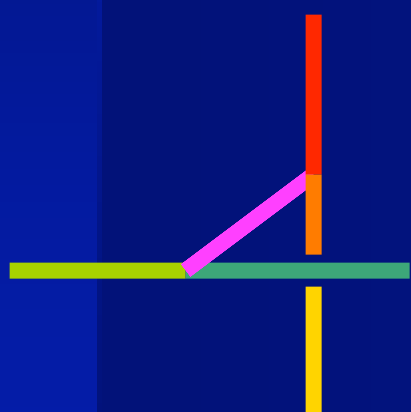
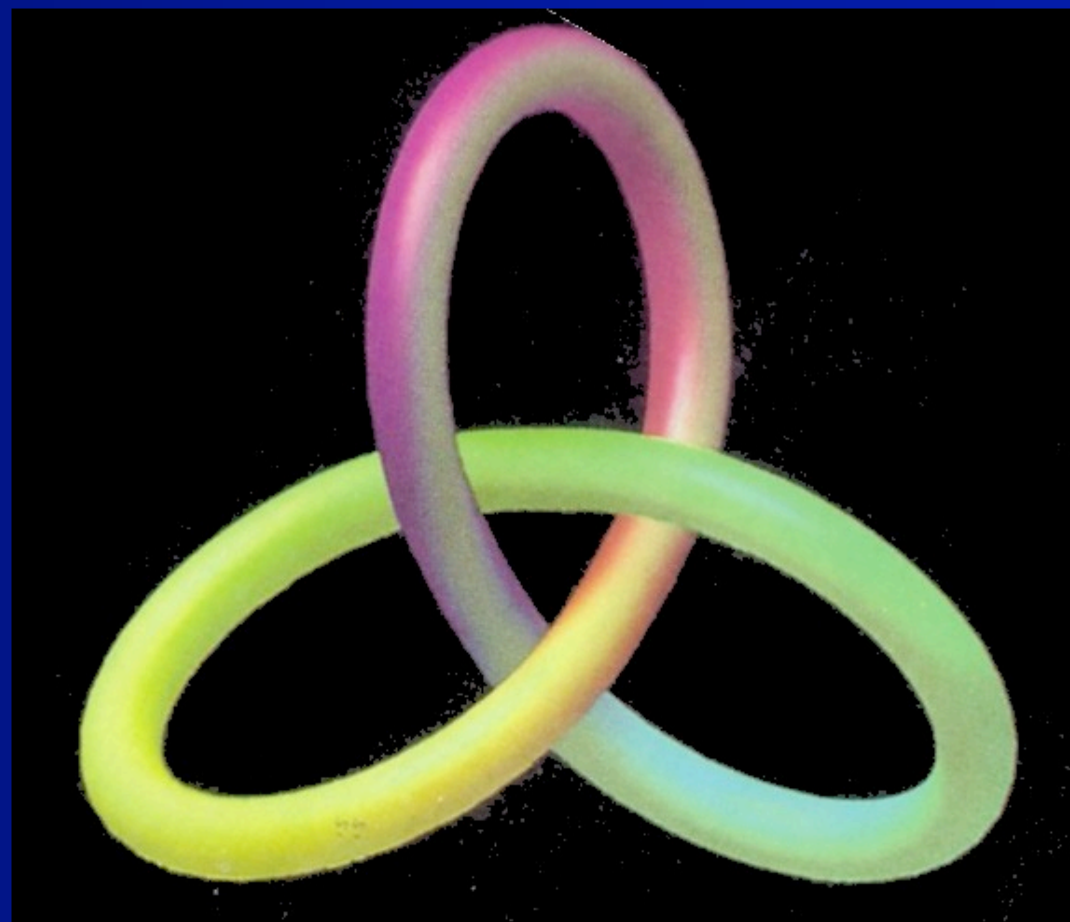
Knots in d=3 etc.







Hopf-algebra's

Hopf algebra $\mathcal{A}$ Ex: Group algebra $\mathbb{C}H$ Basis: $\{h_i\}\quad h_i \in H$			Dual Hopf algebra $\mathcal{A}^*$ Functions on the group $F(H)$ $\{f_i\}\quad f_i = f_{h_i} = P_{h_i}\quad f_i(x) = \delta_{h_i,x}$	
	Algebra		Dual algebra	
product	$\cdot$	$h_1 \cdot h_2 = h_1 h_2$	$\star$	$f_1 \star f_2(x) = f_1 \otimes f_2 \Delta(x)$
unit	e	$eh = he = h$	e^*	$e^* f(x) = \varepsilon(x) = 1$
	Co-algebra		Dual co-algebra	
co-product	Δ	$\Delta(h) = h \otimes h$	Δ^*	$\Delta^*(f)\ (x, y) = f(x \cdot y)$
co-unit	ε	$\varepsilon(h) = 1$	ε^*	$\varepsilon^*\ (f) = f(e)$
antipode	S	$S(h) = h^{-1}$	S^*	$S^*(f)\ (x) = f(S(x)) = f(x^{-1})$

The Quantum Double D(H) (Drinfeld, DPR)

Double algebra $\mathcal{D} = \mathcal{A}^* \times \mathcal{A}$
Ex: Hopf double algebra $D(H) = F(H) \times \mathbb{C}H$
Basis: $\{f_i \times h\}$ $h \in H$

	Algebra	
product unit	$\cdot$ e	$(f_1 \times h_1) \cdot (f_2 \times h_2)(x) = f_1(x) f_2(h_1 x h_1^{-1}) \times h_1 h_2$ $(1 \times e)(x) = e$
	Co-algebra	
co-product co-unit antipode	Δ ε S	$\Delta(f \times h)(x, y) = f(xy) h \otimes h$ $\varepsilon(f \times h)(x) = f(e)$ $S(f \times h)(x) = f(h^{-1} x^{-1} h) h^{-1}$
Central (ribbon) element R-element $R \in \mathcal{D} \otimes \mathcal{D}$	c R	$c = \sum_h (f_h \times h)$ $R = \sum_h (f_h \times e) \otimes (1 \times h)$

Representation Theory

Representations Π_α^A of $D(H) = F(H) \times \mathbb{C}H$		
representation	Π_α^A A α	$A \sim$ defect/magnetic label, $\alpha \sim$ ordinary/electric label $C_A \sim$ Conjugacy class (orbit of representative element h_A). $\alpha \sim$ is a representation of the normalizer N_A of h_A in H.
carrier space	V_α^A	$\|v>: H \to V_\alpha \ \{\|v(x)> \ \| \ \|v(xn)>= \alpha(n^{-1}) \ \|v(x)>, \ n \in N_A\}$
action of $D(H)$ on V_α^A		$\pi_\alpha^A(f \times h)\|v(x)>= f(xhx^{-1}) \ \|v(h^{-1}x)>$
central element spin factor	c s_α^A	$\Pi_\alpha^A(c)\|v(x)>= \alpha(h_A^{-1})\|v(x)>$ $s_\alpha^A \equiv \alpha(h_A^{-1})$
tensor products	$\Pi_\alpha^A \otimes \Pi_\beta^B$	$\Pi_\alpha^A \otimes \Pi_\beta^B(f \times h)V \otimes W \equiv \Pi_\alpha^A \otimes \Pi_\beta^B \Delta(f \times h)V \otimes W$ Clebsch Gordon series: $\Pi_\alpha^A \otimes \Pi_\beta^B = \sum_{C,\gamma} N_{\alpha\beta C}^{AB\gamma} \ \Pi_\gamma^C$

Action of braidgroup on two particle state

$$\mathcal{R}^{AB}_{\alpha\beta} \;:=\; \sigma \circ (\Pi^A_\alpha \otimes \Pi^B_\beta)(\,R\,)$$

$$\mathcal{R}\,|\,{}^{A}h_i,\,{}^{\alpha}v_j\rangle|\,{}^{B}h_m,\,{}^{\beta}v_n\rangle \;=\; |\,{}^{A}h_i\,{}^{B}h_m\,{}^{A}h_i^{-1},\,\beta(\,{}^{A}\tilde{h}_i\,)_{ln}\,{}^{\beta}v_l\rangle|\,{}^{A}h_i,\,{}^{\alpha}v_j\rangle$$

Braiding etc.

Braiding relations:

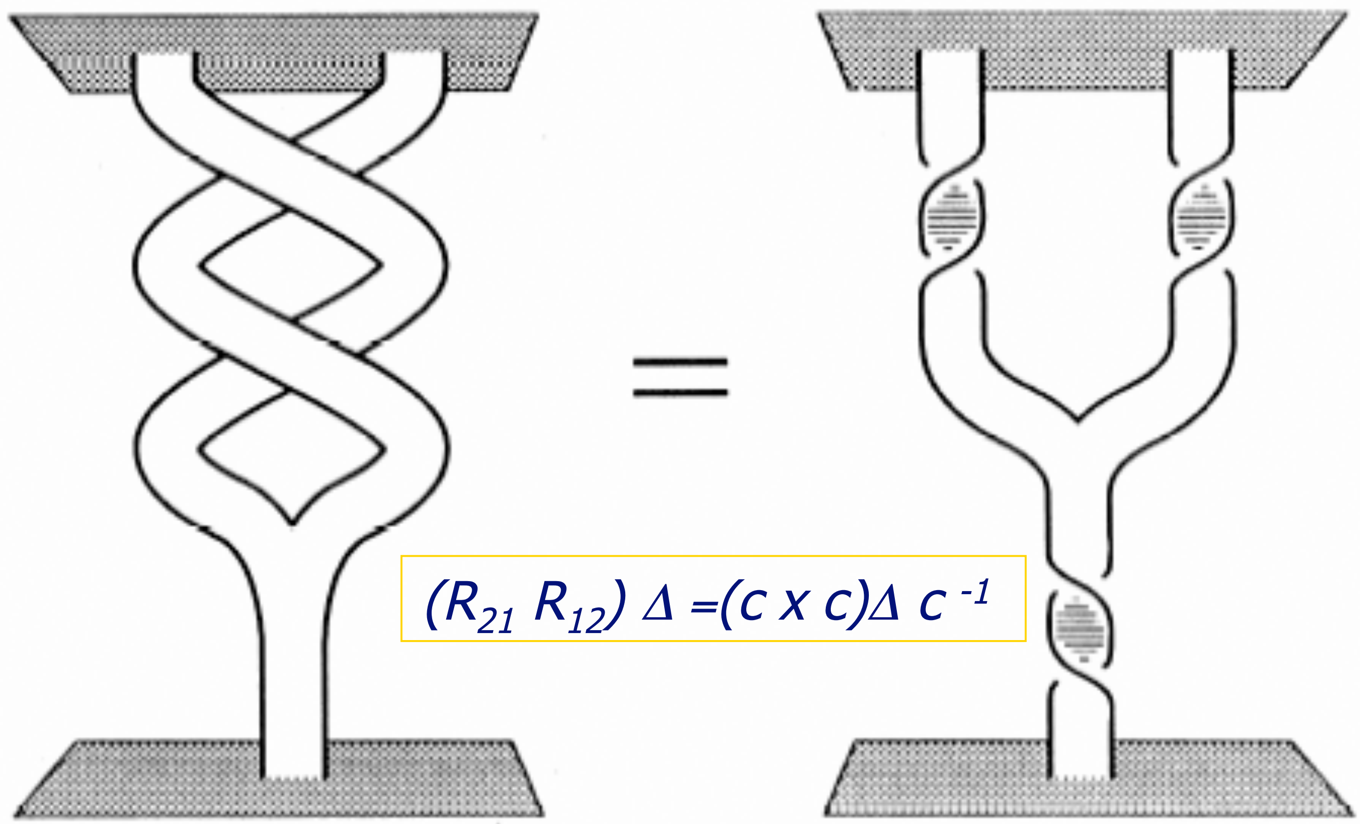
- Braiding R commutes with action of $D(H)$:

$$\Delta^{op}(f \times h)R = R\Delta(f \times h)$$

  $\Delta^{op} \equiv \sigma\Delta$ (i.e. Δ followed by a trivial permutation σ of the two strands).
- $\Rightarrow$ The n-particle states form representations of $D(H) \otimes B(n)$
- Non-abelian statistics if higher dimensional reps of B_n are involved.
- Generalized spin-statistics theorem (suspenders relation) (see also Figure)

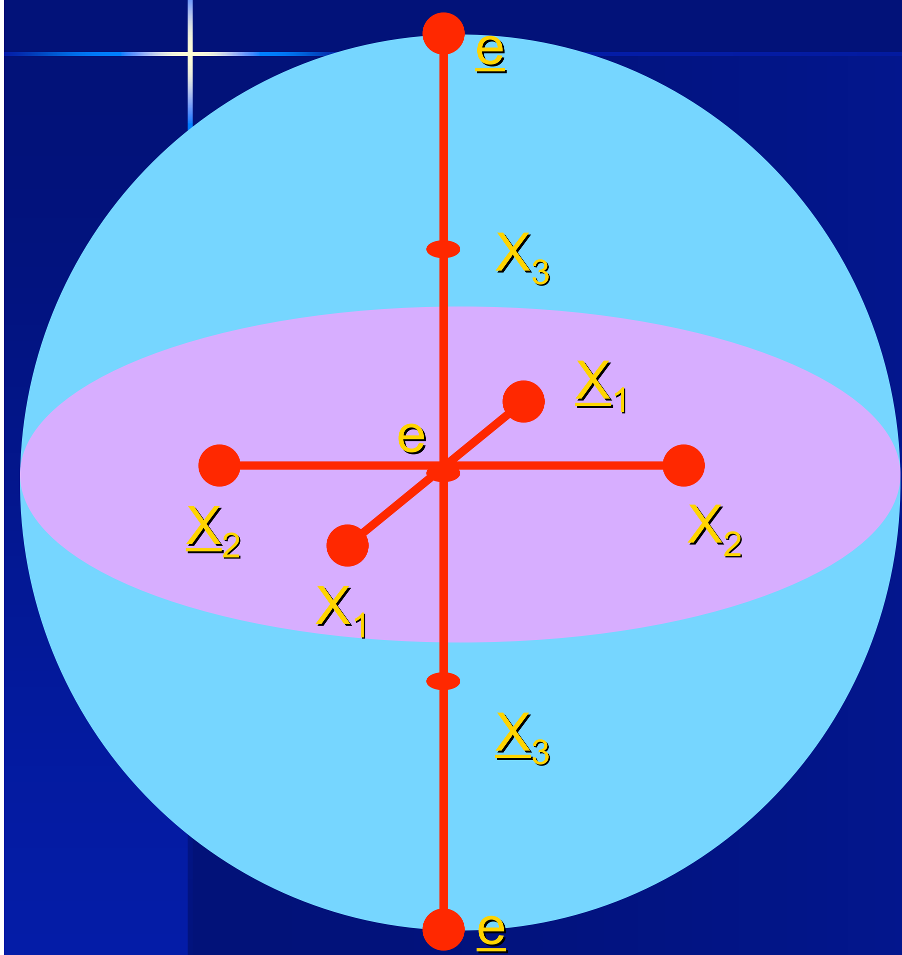
$$(c \otimes c)\Delta c^{-1} = (R_{21}R_{12})\Delta$$

Suspenders diagram



$$(R_{21}\ R_{12})\ \Delta = (c \times c)\Delta\ c^{-1}$$

The dihedral group $\underline{D}_2$



Conjugacy class
$e = \{e\}$
$\bar{e} = \{\bar{e}\}$
$X_1 = \{X_1, \bar{X}_1\}$
$X_2 = \{X_2, \bar{X}_2\}$
$X_3 = \{X_3, \bar{X}_3\}$

The double dihedral group $\underline{D}_2$

Conjugacy class	Centralizer
$e = \{e\}$	$\bar{D}_2$
$\bar{e} = \{\bar{e}\}$	$\bar{D}_2$
$X_1 = \{X_1, \bar{X}_1\}$	$\mathbf{Z}_4 \simeq \{e, X_1, \bar{e}, \bar{X}_1\}$
$X_2 = \{X_2, \bar{X}_2\}$	$\mathbf{Z}_4 \simeq \{e, X_2, \bar{e}, \bar{X}_2\}$
$X_3 = \{X_3, \bar{X}_3\}$	$\mathbf{Z}_4 \simeq \{e, X_3, \bar{e}, \bar{X}_3\}$

Table 3.1: *Conjugacy classes of the double dihedral group $\bar{D}_2$ together with their centralizers.*

$\bar{D}_2$	e	$\bar{e}$	X_1	X_2	X_3
1	1	1	1	1	1
J_1	1	1	1	-1	-1
J_2	1	1	-1	1	-1
J_3	1	1	-1	-1	1
χ	2	-2	0	0	0

$\mathbf{Z}_4$	e	X_a	$\bar{e}$	$\bar{X}_a$
Γ^0	1	1	1	1
Γ^1	1	$\imath$	-1	$-\imath$
Γ^2	1	-1	1	-1
Γ^3	1	$-\imath$	-1	$\imath$

Table 3.2: *Character tables of $\bar{D}_2$ and $\mathbf{Z}_4$.*

Representations of D($\underline{D}_2$) → Spectrum of excitations

Particle types:

electric	$1 := (e, 1)$	$\bar{1} := (\bar{e}, 1)$
	$J_a := (e, J_a)$	$\bar{J}_a := (\bar{e}, J_a)$
	$\chi := (e, \chi)$	$\bar{\chi} := (\bar{e}, \chi)$
magnetic	$\sigma_a^+ := (X_a, \Gamma^0)$	$\sigma_a^- := (X_a, \Gamma^2)$
dyonic	$\tau_a^+ := (X_a, \Gamma^1)$	$\tau_a^- := (X_a, \Gamma^3)$

Spin factor:

particle	$\exp(2\pi\imath s)$
$1, J_a$	1
$\bar{1}, \bar{J}_a$	1
$\chi, \bar{\chi}$	$1, -1$
$\sigma_a^{\pm}$	± 1
$\tau_a^{\pm}$	$\pm \imath.$

Fusion rules for different particle types

$$J_a \times J_a = 1, \qquad J_a \times J_b = J_c, \qquad J_a \times \chi = \chi, \qquad \chi \times \chi = 1 + \sum_a J_a.$$

$$J_a \times \bar{1} = \bar{J}_a, \qquad \chi \times \bar{1} = \bar{\chi}.$$

$$J_a \times \sigma_a^+ = \sigma_a^+, \qquad J_b \times \sigma_a^+ = \sigma_a^-, \qquad \chi \times \sigma_a^+ = \tau_a^+ + \tau_a^-.$$

Fusion rules for different particle types

$$\bar{1} \times \bar{1} = 1, \qquad \bar{1} \times \sigma_a^{\pm} = \sigma_a^{\pm}, \qquad \bar{1} \times \tau_a^{\pm} = \tau_a^{\mp}.$$

$$J_a \times \tau_a^{\pm} = \tau_a^{\pm}, \qquad J_b \times \tau_a^{\pm} = \tau_a^{\mp}, \qquad \chi \times \tau_a^{\pm} = \sigma_a^{+} + \sigma_a^{-},$$

$$\begin{aligned}
\sigma_a^s \times \sigma_a^s &= 1 + J_a + \bar{1} + \bar{J}_a \\
\sigma_a^s \times \sigma_b^s &= \sigma_c^+ + \sigma_c^- \\
\sigma_a^s \times \tau_a^s &= \chi + \bar{\chi} \\
\sigma_a^s \times \tau_b^s &= \tau_c^+ + \tau_c^- \\
\tau_a^s \times \tau_a^s &= 1 + J_a + \bar{J}_b + \bar{J}_c \\
\tau_a^s \times \tau_b^s &= \sigma_c^+ + \sigma_c^- ,
\end{aligned}$$

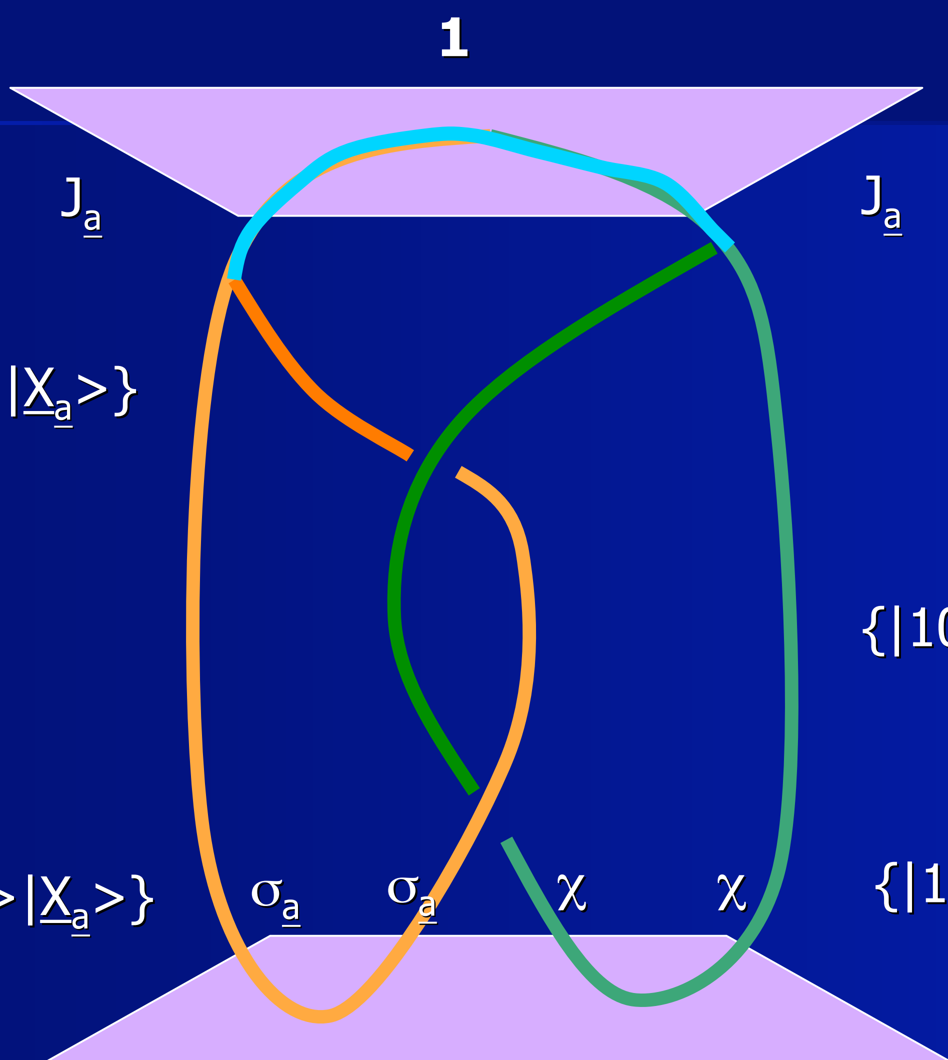
Cheshire charge

Entanglements

$\{|\underline{X}_{\underline{a}}>|X_{\underline{a}}>-|X_{\underline{a}}>|\underline{X}_{\underline{a}}>\}$

$\{|\underline{X}_{\underline{a}}>|X_{\underline{a}}>+|X_{\underline{a}}>|\underline{X}_{\underline{a}}>\}$

1



$\{|10>|01>+|01>|10>\}$

$\{|10>|01>-|01>|10>\}$

1

The truncated braid group B(3,4)

$$\begin{aligned} \tau_1\tau_2\tau_1 &= \tau_2\tau_1\tau_2 \\ \tau_1^4 &= \tau_2^4 = e. \end{aligned}$$

The group B(3,4) has 96 elements and 15 conjugacy classes.

Investigate the following three particle fusion product

$$\tau_1^+ \times \tau_1^+ \times \tau_1^+ = 4\,\tau_1^+.$$

(2 x2 x2 = 4 x 2)

Conjugacy classes of B(3,4)

$$
\begin{aligned}
C_0^1 &= \{e\} \qquad (3.B.\\
C_0^2 &= \{\tau_1\tau_2\tau_1\tau_2\tau_1\tau_2\}\\
C_0^3 &= \{\tau_2^2\tau_1^2\tau_2^2\tau_1^2\}\\
C_0^4 &= \{\tau_2^2\tau_1^3\tau_2^2\tau_1^3\}\\
C_1^1 &= \{\tau_1\,,\ \tau_2\,,\ \tau_2\tau_1\tau_2^3\,,\ \tau_2^2\tau_1\tau_2^2\,,\ \tau_2^3\tau_1\tau_2\,,\ \tau_1^2\tau_2\tau_1^2\}\\
C_1^2 &= \{\tau_1^3\tau_2\tau_1^2\tau_2\,,\ \tau_2^3\tau_1\tau_2^2\tau_1\,,\ \tau_2\tau_1^3\tau_2\tau_1^2\,,\ \tau_2^2\tau_1^2\tau_2^2\tau_1\,,\ \tau_1\tau_2^3\tau_1\tau_2^2\,,\ \tau_1^2\tau_2^2\tau_1^2\tau_2\}\\
C_1^3 &= \{\tau_2\tau_1^3\tau_2\tau_1^3\tau_2\,,\ \tau_1^2\tau_2\tau_1^3\tau_2^2\tau_1\,,\ \tau_2^3\tau_1\tau_2^3\tau_1^2\,,\ \tau_1\tau_2^2\tau_1^3\tau_2^2\tau_1\,,\ \tau_2\tau_1\tau_2^3\tau_1^2\tau_2^2\,,\ \tau_2\tau_1^2\tau_2^3\tau_1^2\tau_2\}\\
C_1^4 &= \{\tau_2^2\tau_1^3\tau_2^2\,,\ \tau_1^2\tau_2^3\tau_1^2\,,\ \tau_2^3\tau_1^3\tau_2\,,\ \tau_1^3\,,\ \tau_2\tau_1^3\tau_2^3\,,\ \tau_2^3\}\\
C_2^1 &= \{\tau_1\tau_2\,,\ \tau_2\tau_1\,,\ \tau_1^2\tau_2\tau_1^3\,,\ \tau_1^3\tau_2\tau_1^2\,,\ \tau_2\tau_1^2\tau_2^2\tau_1\,,\ \tau_2^2\tau_1\tau_2^3\,,\ \tau_2^3\tau_1\tau_2^2\,,\ \tau_1\tau_2^2\tau_1^2\tau_2\}\\
C_2^2 &= \{\tau_1^2\tau_2\tau_1^3\tau_2\tau_1\,,\ \tau_1\tau_2\tau_1^3\tau_2\tau_1^2\,,\ \tau_2\tau_1^2\tau_2^2\tau_1^3\,,\ \tau_1\tau_2\tau_1^2\tau_2^2\tau_1^2\,,\\
&\qquad \tau_2\tau_1^3\tau_2^2\tau_1^2\,,\ \tau_1\tau_2\tau_1^3\tau_2^2\tau_1\,,\ \tau_1^2\tau_2\tau_1^3\tau_2^2\,,\ \tau_1^2\tau_2^2\tau_1^3\tau_2\}\\
C_2^3 &= \{\tau_1^3\tau_2\tau_1^3\tau_2^2\tau_1\,,\ \tau_1\tau_2^2\tau_1^3\tau_2\tau_1^3\,,\ \tau_2\tau_1^3\tau_2^2\,,\ \tau_2^2\tau_1^3\tau_2\,,\ \tau_2^3\tau_1^3\,,\ \tau_1\tau_2^3\tau_1^2\,,\ \tau_1^2\tau_2^3\tau_1\,,\ \tau_1^3\tau_2^3\}\\
C_2^4 &= \{\tau_1^3\tau_2^3\tau_1^2\,,\ \tau_1^2\tau_2^3\tau_1^3\,,\ \tau_2^3\tau_1\,,\ \tau_1\tau_2^3\,,\ \tau_1\tau_2\tau_1\tau_2\,,\ \tau_1^3\tau_2\,,\ \tau_2\tau_1^3\,,\ \tau_2\tau_1\tau_2\tau_1\}\\
C_3^1 &= \{\tau_1^2\,,\ \tau_2^2\,,\ \tau_1\tau_2^2\tau_1^3\,,\ \tau_2^2\tau_1^2\tau_2^2\,,\ \tau_1^2\tau_2^2\tau_1^2\,,\ \tau_1^3\tau_2^2\tau_1\}\\
C_3^2 &= \{\tau_2\tau_1^2\tau_2\,,\ \tau_1\tau_2^2\tau_1\,,\ \tau_1^2\tau_2^2\,,\ \tau_2^3\tau_1^2\tau_2^3\,,\ \tau_1^3\tau_2^2\tau_1^3\,,\ \tau_2^2\tau_1^2\}\\
C_4^1 &= \{\tau_1\tau_2\tau_1\,,\ \tau_1^2\tau_2\,,\ \tau_2^2\tau_1\,,\ \tau_2\tau_1^2\,,\ \tau_1\tau_2^2\,,\ \tau_1^3\tau_2\tau_1^3\,,\ \tau_1^3\tau_2\tau_1^3\tau_2^2\tau_1^2\,,\\
&\qquad \tau_2\tau_1^3\tau_2^2\tau_1\,,\ \tau_1^2\tau_2^2\tau_1^3\,,\ \tau_1\tau_2^2\tau_1^3\tau_2\,,\ \tau_1^3\tau_2^2\tau_1^2\,,\ \tau_1\tau_2\tau_1^3\tau_2^2\}\\
C_4^2 &= \{\tau_1\tau_2\tau_1\tau_2\tau_1\tau_2\tau_1\tau_2\tau_1\,,\ \tau_2\tau_1^2\tau_2^2\,,\ \tau_1\tau_2^2\tau_1^2\,,\ \tau_2^2\tau_1^2\tau_2\,,\ \tau_1^2\tau_2^2\tau_1\,,\ \tau_2\tau_1^3\tau_2\,,\\
&\qquad \tau_2^3\tau_1^3\tau_2^3\,,\ \tau_2^3\tau_1^2\,,\ \tau_1^3\tau_2^2\,,\ \tau_2^2\tau_1^3\,,\ \tau_1^2\tau_2^3\,,\ \tau_1\tau_2^3\tau_1\}.
\end{aligned}
$$

Character table of B(3,4)

	C_0^1	C_0^2	C_0^3	C_0^4	C_1^1	C_1^2	C_1^3	C_1^4	C_2^1	C_2^2	C_2^3	C_2^4	C_3^1	C_3^2	C_4^1	C_4^2
Λ_0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Λ_1	1	-1	1	-1	$\imath$	$-\imath$	$\imath$	$-\imath$	-1	1	-1	1	-1	1	$-\imath$	$\imath$
Λ_2	1	1	1	1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1
Λ_3	1	-1	1	-1	$-\imath$	$\imath$	$-\imath$	$\imath$	-1	1	-1	1	-1	1	$\imath$	$-\imath$
Λ_4	2	2	2	2	0	0	0	0	-1	-1	-1	-1	2	2	0	0
Λ_5	2	-2	2	-2	0	0	0	0	1	-1	1	-1	-2	2	0	0
Λ_6	2	$2\imath$	-2	$-2\imath$	η	$-\eta^*$	$-\eta$	η^*	$\imath$	-1	$-\imath$	1	0	0	0	0
Λ_7	2	$2\imath$	-2	$-2\imath$	$-\eta$	η^*	η	$-\eta^*$	$\imath$	-1	$-\imath$	1	0	0	0	0
Λ_8	2	$-2\imath$	-2	$2\imath$	$-\eta^*$	η	η^*	$-\eta$	$-\imath$	-1	$\imath$	1	0	0	0	0
Λ_9	2	$-2\imath$	-2	$2\imath$	η^*	$-\eta$	$-\eta^*$	η	$-\imath$	-1	$\imath$	1	0	0	0	0
Λ_{10}	3	3	3	3	1	1	1	1	0	0	0	0	-1	-1	-1	-1
Λ_{11}	3	-3	3	-3	$\imath$	$-\imath$	$\imath$	$-\imath$	0	0	0	0	1	-1	$\imath$	$-\imath$
Λ_{12}	3	3	3	3	-1	-1	-1	-1	0	0	0	0	-1	-1	1	1
Λ_{13}	3	-3	3	-3	$-\imath$	$\imath$	$-\imath$	$\imath$	0	0	0	0	1	-1	$-\imath$	$\imath$
Λ_{14}	4	4	-4	-4	0	0	0	0	1	1	-1	-1	0	0	0	0
Λ_{15}	4	-4	-4	4	0	0	0	0	-1	1	1	-1	0	0	0	0

Table 3.4: *Character table of the truncated braid group $B(3,4)$. We used $\eta := \imath + 1$.*

Non abelian braidgroup representations

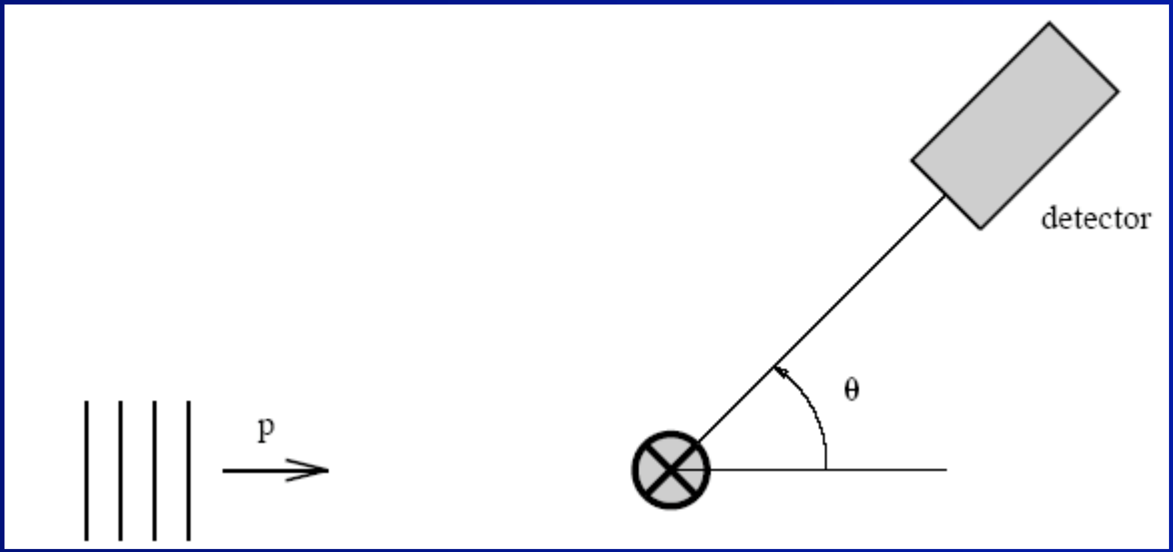
Decompose the tensor product representation under D(D2) x B(3,4)

For D(D2) → $\tau_1^+ \times \tau_1^+ \times \tau_1^+ \;=\; 4\,\tau_1^+.$

Under B(3,4) → $\Lambda_{B(3,4)} \;=\; 4\,\Lambda_1 + 2\,\Lambda_5,$

Combined → $2\,(\tau_1^+, \Lambda_1) \;+\; (\tau_1^+, \Lambda_5),$

Non abelian Ahoronov Bohm scattering (E. Verlinde)



$$\frac{d\sigma}{d\theta}\Big|_{\mathrm{in}\to\mathrm{all}} = \frac{1}{4\pi p \sin^2(\theta/2)} \left[1 - \mathrm{Re}\langle \mathrm{in}|\mathcal{R}^2|\mathrm{in}\rangle\right].$$

Summary/Perspective

- Variety of defects/quasiparticles: anyonic composites
- Determination of a consistent labeling of charge/flux sectors
- Non-abelian fusion and braiding properties
- Quantum groups/ Hopf algebras

  → provide a natural language in which ordinary/electric and topological/magnetic quantum numbers appear on equal footing!
- Multi (quasi)particle reps of braid group (non-abelian statistics)
- Breaking of quantum symmetries by electric/magnetic/dyonic condensates
- Classification of many possible confinement phenomena
- Applications in Hall effect en BEC/Discrete gauge theories/ Gravity/Nematics, Defect mediated melting etc
- (Topological) Quantum computation
- Similar phenomena in d>2
- Statistics – distribution functions etc

Hopf symmetry breaking

- Imagine a condensate to form in a state $|v>$ in Π_α^A
- Look for maximal Hopf algebra that leaves |v> invariant:

$$\pi_a^e(P)\,|v> \;=\; \varepsilon(P)\,|v> \;=\; f(e)\,|v>$$

- Examples:
- Electric condensate:

$$\pi_\alpha^A\,(f\,\underline{x}\,p)\,|v> \;=\; f(e)\;\;\alpha(p)\,|v> \;\;=\; f(e)\,|v>$$

$$\rightarrow\;\; p\;\varepsilon\;N_v \quad \text{and} \quad T= F(H)\,\underline{x}\,CN_v$$

- Magnetic condensate:

$$\pi_o^A\,(f\,\underline{x}\,p)\;\;|v(y)> \;\;=\; f(y\,g_A\,y^{-1})\,|v(p^{-1}y)>$$

$$\rightarrow\; p\;\varepsilon\;N_g \quad \text{and} \quad T= F(H/K)\,\underline{x}\,CN_g$$

Gauge invariant condensates

- Gauge invariant magnetic condensate:
  $|v\rangle = \Sigma\, v(y)$ with $\Sigma\; yg_A y^{-1} = C_A$

  $\Pi_o^A\, (f\, \underline{x}\, p)\;\; |v(y)\rangle \;=\; \Sigma\, |v(p^{-1}y)\rangle = |v\rangle$

  (class sum is per definition invariant under conjugation)

  $\rightarrow\; T = F(H/K)\, \underline{x}\, CH$

- Dyonic condensates

What is the physics of breaking?

- Construct representations Ω_j of $T \leq D(H)$
- Decompose Π^A_α into $\{\Omega_j\}$
- Look at braid relations of |v> and states in othe T reps. (hard)
- If */v>* and */w>* ε Ω_j have nontrivial braiding (R^2) then:

  → */v>* cannot be single valued around */w>*

  → Ω_j particles will have a string (domain wall) attached

  → Ω_j particle will be confined!

- Particles with trivial braiding survive
- Tenssor products Ω_i X Ω_j = $N_{ijk}\,\Omega_k$

Residual Hopf symmetry

- The nonconfined representations of T form a closed set under tensor product of *T*
- This set can be viewed as the representations of yet another Hopf algebra *U*
- There is a surjective map $\Gamma : T \rightarrow U$
- Walls are characterizec by reps of *Ker* Γ

Final picture