Braid Topologies for Quantum Computation

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Inspiration

From “A topological modular functor which is universal for quantum computation”


Universal Quantum Gates

Single Qubit Rotation

\[ |\psi\rangle \rightarrow U_{\phi} |\psi\rangle \]

Controlled Not

\[ |0\rangle \rightarrow |0\rangle \quad |0\rangle \rightarrow |0\rangle \]
\[ |0\rangle \rightarrow |1\rangle \quad |1\rangle \rightarrow |1\rangle \]
\[ |1\rangle \rightarrow |1\rangle \quad |1\rangle \rightarrow |1\rangle \]

Any N qubit operation can be carried out using these two gates.

\[ \left| \Psi_f \right\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} \left| \Psi_i \right\rangle \]
One way to go... \( |0\rangle = \uparrow \quad |1\rangle = \downarrow \)

Loss and DiVincenzo, ‘98

Manipulate electron spins with electric and magnetic fields to carry out quantum gates.

**Problem:** Errors and Decoherence! May be solvable, but it won’t be easy!
Another way to go…

Fault-tolerant quantum computation by anyons

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A Modular Functor Which is Universal for Quantum Computation

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An incompressible quantum liquid can form when the Landau level filling fraction \( \nu = \frac{n_{\text{elec}}(hc/eB)}{eB} \) is a rational fraction.

Quasiparticle excitations can have fractional charge.

Great stuff, but what does this have to do with quantum computing?
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**Fractional Quantum Hall (FQH) States**

An *incompressible quantum liquid* can form when the Landau level filling fraction \( \nu = \frac{n_{\text{elec}}(hc/eB)}{B} \) is a rational fraction.

Quasiparticle excitations can have *fractional charge*.

Great stuff, but what does this have to do with quantum computing?
Topological Degeneracy (X.G. Wen)

A theoretical curiosity: FQH states on topologically nontrivial surfaces have degenerate ground states which can only be distinguished by global measurements.

For the $\nu = 1/3$ state:

\[ \begin{array}{c}
1 \\
3 \\
9 \\
3^N \\
\end{array} \]

Degeneracy
Non-Abelian FQH States  (Moore, Read ’91)

Fractionally charged quasiparticles

Essential features:

A degenerate Hilbert space whose dimensionality is \textit{exponentially large in the number of quasiparticles}.

States in this space \textit{can only be distinguished by global measurements} provided quasiparticles are far apart.

\textcolor{red}{A perfect place to hide quantum information!}
Exchanging Particles in 2+1 Dimensions

Particle “world-lines” form **braids** in 2+1 (=3) dimensions
Exchanging Particles in 2+1 Dimensions

Particle “world-lines” form braids in 2+1 (=3) dimensions
Fractional (Abelian) Statistics

\[ |\psi_f\rangle = e^{i\theta} |\psi_i\rangle \]

\[ |\psi_i\rangle \]

Phase

\[ \theta = 0 \quad \text{Bosons} \]
\[ \theta = \pi \quad \text{Fermions} \]
\[ \theta = \pi/3 \quad \nu = 1/3 \quad \text{quasiparticles} \]

Only possible for particles in 2 space dimensions.
Non-Abelian Statistics (Moore, Read ’91)

\[
|\psi_f\rangle = \tilde{\alpha} |\psi_0\rangle + \tilde{\beta} |\psi_1\rangle
\]

\[
|\psi_i\rangle = \alpha |\psi_0\rangle + \beta |\psi_1\rangle
\]

degenerate states
Non-Abelian Statistics (Moore, Read ‘91)

Matrices form a **non-Abelian** representation of the **braid group**.

(Related to the Jones Polynomial, TQFT (Witten), Conformal Field Theory (Moore, Seiberg), etc.)
Many Non-Abelian Anyons

\[ |\Psi_f\rangle = \begin{pmatrix} a_{11} & \cdots & a_{1M} \\ \vdots & \ddots & \vdots \\ a_{M1} & \cdots & a_{MM} \end{pmatrix} |\Psi_i\rangle \]
Many Non-Abelian Anyons

Matrix depends only on the topology of the braid swept out by anyon world lines!

Robust quantum computation?
Possible Non-Abelian FQH States

\( \nu = \frac{5}{2} \)

Very likely a Moore-Read “Pfaffian” state.

Moore and Read, 1991
Morf, 1998

Charge \( \frac{e}{4} \) quasiparticles with braiding properties described by \( SU(2)_2 \) Chern-Simons Theory.

Nayak and Wilczek, 1996

Not sufficiently “rich” nonabelian statistics to do universal quantum computation.

Possible Non-Abelian FQH States

\( \nu = \frac{12}{5} \)

Possibly a Read-Rezayi \( k = 3 \) “Parafermion” state.

Read and Rezyai, 1999

Charge \( \frac{e}{5} \) quasiparticles with braiding properties described by \( SU(2)_3 \) Chern-Simons Theory.

Slingerland and Bais, 2001

\( SU(2)_3 \) is sufficiently “rich” to do universal quantum computation.

Freedman, Larsen, and Wang, 2001

J.S. Xia et al., PRL (2004).
Fibonacci Anyons (Kuperberg, Preskill)

A Fibonacci Anyon $\rightarrow$ 0,1  Fibonacci $\rightarrow$

The laws of Fibonacci anyons:

1. Fibonacci anyons have a quantum attribute known as $q$-spin:

\[ q\text{-spin} = 1 \]

2. A collection of Fibonacci anyons can have a total $q$-spin of either 0 or 1:

Notation: Ovals are labeled by total $q$-spin of enclosed particles.
Fibonacci Anyons

3. The “fusion” rule for combining q-spin is: \[ 1 \times 1 = 0 + 1 \]

This means that two Fibonacci anyons can have total q-spin 0 or 1, or be in any quantum superposition of the two.

\[ \alpha_0 + \beta_1 \]

Two dimensional Hilbert space

Three Fibonacci anyons \[ \rightarrow \] Three dimensional Hilbert space

\[ \alpha_0 \beta_1 + \gamma_1 \]
Fusion Diagram
States are paths in the fusion diagram.
Here’s another one
Count states by counting paths

- Hilbert space dimensionality grows as the **Fibonacci sequence**!

- Exponentially large in the number of quasiparticles, so big enough for quantum computing.
The F Matrix

Changing fusion bases:

\[ \sum_a F_{ab}^c \]

\[
\begin{pmatrix}
-\tau & \sqrt{\tau} & 0 \\
\sqrt{\tau} & \tau & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[ F_{ab}^c \]

\[ \tau = \frac{\sqrt{5} - 1}{2} \]
The R Matrix

Exchanging particles:

\[
\begin{align*}
\text{0} & \quad \Rightarrow e^{-i\frac{4\pi}{5}} \\
\text{1} & \quad \Rightarrow e^{i\frac{3\pi}{5}}
\end{align*}
\]

\[
R = \begin{pmatrix}
e^{-i\frac{4\pi}{5}} & 0 \\
0 & e^{i\frac{3\pi}{5}}
\end{pmatrix}
\]

*F* and *R* must satisfy certain consistency conditions (the “pentagon” and “hexagon” equations). For Fibonacci anyons these equations uniquely determine *F* and *R*. 
Encoding a Qubit

(Freedman, Larsen, and Wang, 2001)

Qubit States

\[ |0\rangle = \left(\begin{array}{c} 0 \\ 1 \end{array}\right) \]

\[ |1\rangle = \left(\begin{array}{c} 1 \\ 1 \end{array}\right) \]

State of qubit is determined by q-spin of two leftmost particles

Non-Computational State

Transitions to this state are leakage errors
Braiding Matrices for 3 Fibonacci Anyons

\[ \sigma_1 = \begin{pmatrix}
  e^{-i4\pi/5} & 0 & 0 \\
  0 & e^{-i2\pi/5} & 0 \\
  0 & 0 & e^{-i2\pi/5}
\end{pmatrix} \quad c = 1 \]

\[ \sigma_2 = \begin{pmatrix}
  -\tau e^{-i\pi/5} & -i\sqrt{\tau} e^{-i\pi/10} & 0 \\
  -i\sqrt{\tau} e^{-i\pi/10} & -\tau & 0 \\
  0 & 0 & e^{-i2\pi/5}
\end{pmatrix} \quad \tau = \frac{\sqrt{5} - 1}{2} \]

\[ \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \text{ = } M \]

\[ |\Psi_f\rangle = M^{-1} |\Psi_i\rangle \]
Single Qubit Operations

General rule: Braiding inside an oval does not change the total q-spin of the enclosed particles.

Important consequence: As long as we braid within a qubit, there is no leakage error.

Can we do arbitrary single qubit rotations this way?
Single Qubit Operations are Rotations

The set of all single qubit rotations lives in a solid sphere of radius $2\pi$. 

$$|\psi\rangle \xrightarrow{U_{\vec{\alpha}}} U_{\vec{\alpha}}U_{\vec{\alpha}}|\psi\rangle$$

$$U_{\vec{\alpha}} = \exp \frac{i \vec{\alpha} \cdot \vec{\sigma}}{2}$$
$2\pi - 2\pi$  
$\sigma_1^2$  
$\sigma_2^2$  
$-2\pi$  
$\sigma_2^{-2}$  
$\sigma_1^{-2}$
N = 1
$N = 3$
$N = 5$
N = 8
$N = 9$
$N = 10$
$N = 11$
Brute force searching rapidly becomes infeasible as braids get longer.

Fortunately, a clever algorithm due to Solovay and Kitaev allows for systematic improvement of the braid given a sufficiently dense covering of $SU(2)$. 

\[
\sigma_1^{-2} \sigma_2^{-4} \sigma_1^4 \sigma_2^{-2} \sigma_2^2 \sigma_1^{-2} \sigma_2^4 \sigma_1^2 \sigma_2^{-2} \sigma_2^4 \sigma_1^2 \sigma_2^{-2} \sigma_1^2 \sigma_2^{-2} \sigma_1^2 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + O(10^{-3})
\]
Solovay-Kitaev Construction

(Actual calculation)

\[
\begin{bmatrix}
0 & i & 0 \\
0 & 0 & 1 \\
i & 0 & 0
\end{bmatrix} + O(10^{-4})
\]

Braid Length \( \sim |\ln \varepsilon|^c \), \( c \approx 4 \)
What About Two Qubit Gates?

Problems:

1. We are pulling quasiparticles out of qubits: **Leakage error**!

2. **87 dimensional** search space (as opposed to 3 for three-particle braids). Straightforward “brute force” search is problematic.
Two Qubit Controlled Gates

Goal: Find a braid in which some rotation is performed on the target qubit only if the control qubit is in the state 1. \((b=1)\)
Constructing Two Qubit Gates by “Weaving”

*Weave a pair* of anyons from the **control qubit** between anyons in the **target qubit**.

**Important Rule:** Braiding a q-spin 0 object does not induce transitions.

Target qubit is only affected if control qubit is in state $|1\rangle$ 

$$b = 1$$
Constructing Two Qubit Gates by “Weaving”

Only nontrivial case is when the control pair has q-spin 1.

We’ve reduced the problem to weaving one anyon around three others. **Still too hard for brute force approach!**
OK, Try Weaving Through Only Two Particles

We’re back to $SU(2)$, so this is numerically feasible.

**Question:** Can we find a weave which does not lead to leakage errors?
A Trick: Effective Braiding

Actual Weaving

Effective Braiding

\[ \sigma_2^3 \sigma_1^2 \sigma_2^{-4} \sigma_1^2 \sigma_2^2 \sigma_1^{-2} \sigma_2^{-2} \sigma_2^{-2} \sigma_1^2 \sigma_2^2 \sigma_1^{-2} \sigma_2^2 \sigma_1^{-2} \sigma_2^4 \sigma_1^{-2} \sigma_2^2 \sigma_1^2 \sigma_2^4 \sigma_1^{-2} \sigma_2^{-2} \sigma_2 \approx \sigma_1^2 \]

The effect of weaving the blue anyon through the two green anyons has approximately the same effect as braiding the two green anyons twice.
Controlled–“Knot” Gate

Effective braiding is all within the target qubit \( \rightarrow \) **No leakage!**

Not a CNOT, but sufficient for universal quantum computation.
Another Trick: Injection Weaving

Step 1: Inject the control pair into the target qubit.
Step 2: Weave the control pair inside the injected target qubit.
Step 3: Extract the control pair from the target using the inverse of the injection weave.

Putting it all together we have a CNOT gate:
Solovay-Kitaev Improved CNOT
Universal Set of Fault Tolerant Gates

Single qubit rotations: \( |\psi\rangle \xrightarrow{U\phi} U\phi |\psi\rangle \)

Controlled NOT:
Quantum Circuit
Braid