Topological order and quantum entanglement

Michael Levin, Xiao-Gang Wen

MIT
Topological phases
Topological phases

- Gapped
Topological phases

- Gapped
- Degenerate ground state on torus
Topological phases

- Gapped
- Degenerate ground state on torus
- Fractional statistics

\[ e^{i\theta} \]
Topological phases

- Gapped
- Degenerate ground state on torus
- Fractional statistics
- “Topological order”
Real life examples

- FQH liquids.
Real life examples

- FQH liquids.

- Hope: Frustrated magnets
  - Many theoretical models
  - A few candidate materials
    - $\text{Cs}_2\text{CuCl}_4$
    - $\kappa-(\text{BEDT-TTF})_2\text{Cu}_2(\text{CN})_3$
Theory of topological phases
Theory of topological phases

We have:

- Low energy effective theory: TQFT
- Mathematical framework: Tensor categories
- Physical picture: String condensation, etc.
Theory of topological phases

- We have:
  - Low energy effective theory: TQFT
  - Mathematical framework: Tensor categories
  - Physical picture: String condensation, etc.

- We’re missing a lot
Physical characterization is incomplete
Physical characterization is incomplete

- Symmetry breaking order can be detected in a wave function
- But we don’t know how to detect topological order in a wave function
Many wave functions

- Theoretical wave functions
  - Gutzwiller projected states: $\Psi_{\text{spin}} = P \Psi_{\text{ferm}}$
  - Quantum loop gases: $\Psi_d(X) = d^{N_{\text{loops}}}(X)$
Many wave functions

- Theoretical wave functions
  - Gutzwiller projected states: \( \Psi_{\text{spin}} = P \Psi_{\text{ferm}} \)
  - Quantum loop gases: \( \Psi_d(X) = d^{N_{\text{loops}}(X)} \)

- Numerical wave functions
  - \( J_1-J_3 \) spin-1/2 Heisenberg model:
    \[
    H = J_1 \sum_{n.n} S_i \cdot S_j + J_3 \sum_{3rd\ n.n} S_i \cdot S_j
    \]
Many wave functions

- Theoretical wave functions
  - Gutzwiller projected states: $\Psi_{\text{spin}} = P \Psi_{\text{ferm}}$
  - Quantum loop gases: $\Psi_d(X) = d^{N_{\text{loops}}(X)}$

- Numerical wave functions
  - $J_1$-$J_3$ spin-1/2 Heisenberg model:

    $H = J_1 \sum_{\text{n.n}} S_i \cdot S_j + J_3 \sum_{\text{3rd n.n}} S_i \cdot S_j$

- How do we know if they’re topologically ordered?
Topological entropy

Define: $-S_{\text{top}} = (S_1 - S_2) - (S_3 - S_4)$
Main Result

- Then: $S_{\text{top}} = 0$ for normal states, $S_{\text{top}} \neq 0$ for topologically ordered states

- $S_{\text{top}}$ is universal for each topological phase:

  $$S_{\text{top}} = \log(D^2)$$

  where $D^2 = \sum_{\alpha} d_{\alpha}^2$

- cond-mat/0510613 (Kitaev/Preskill posted similar result on hep-th/0510092)
Why does $S_{\text{top}} = 0$ for normal phases, $S_{\text{top}} \neq 0$ for topological phases?
**Physical picture**

- Why does $S_{\text{top}} = 0$ for normal phases, $S_{\text{top}} \neq 0$ for topological phases?

**Nonlocal entanglement!**

- Topologically ordered states have nonlocal entanglement
- $S_{\text{top}}$ measures nonlocal entanglement
Topologically ordered states have nonlocal entanglement

$$H = -V \sum_{l} \sigma_{l1}^{x} \sigma_{l2}^{x} \sigma_{l3}^{x} - t \sum_{p} \sigma_{p1}^{z} \sigma_{p2}^{z} \sigma_{p3}^{z} \sigma_{p4}^{z} \sigma_{p5}^{z} \sigma_{p6}^{z}$$

$Z_2$ topological order
Ground state wave function

- Use string picture:
  \[ \sigma^x_i = -1 \text{ - string on link} \]
  \[ \sigma^x_i = +1 \text{ - no string on link} \]
- \( \Psi \) is uniform superposition of closed string configurations
Ground state wave function

- Use string picture:
  \[ \sigma^x_i = -1 \rightarrow \text{string on link} \]
  \[ \sigma^x_i = +1 \rightarrow \text{no string on link} \]

- \( \Psi \) is uniform superposition of closed string configurations
Ground state wave function

- Use string picture:
  - $\sigma^x_i = -1$ - string on link
  - $\sigma^x_i = +1$ - no string on link
- $\Psi$ is uniform superposition of closed string configurations
- All local correlations $i\sigma^x_i \sigma^x_j$ vanish
Ground state wave function

- Use string picture:
  \[ \sigma^x_i = -1 \] - string on link
  \[ \sigma^x_i = +1 \] - no string on link
- \[ \Psi \] is uniform superposition of closed string configurations
- All local correlations \[ i \sigma^x_i \sigma^x_j \] vanish
- There is a nonlocal correlation: \[ i \prod_{i \in C} \sigma^x_i = 1 \]
String correlations are very general

- Perturb Hamiltonian:

\[ H \rightarrow H + \varepsilon \sum_i \sigma_i^z \]
String correlations are very general

- Perturb Hamiltonian:
  \[ H \rightarrow H + \varepsilon \sum_i \sigma_i^z \]
String correlations are very general

- Perturb Hamiltonian:
  \[ H \to H + \varepsilon \sum_i \sigma_i^z \]

- Thin string operator fails:
  \[ \langle \prod_{i \in C} \sigma_i^x \rangle = 0 \]
String correlations are very general

- Perturb Hamiltonian:
  \[ H \to H + \varepsilon \sum_i \sigma_i^z \]

- Thin string operator fails:
  \[ \langle \prod_{i \in C} \sigma_i^x \rangle = 0 \]

- But “fattened” string operator works
$S_{\text{top}}$ measures string correlations

$-S_{\text{top}} = (S_1 - S_2) - (S_3 - S_4)$
We have argued:

- $S_{\text{top}} = 0$ for normal phases
- $S_{\text{top}} \neq 0$ for topological phases
- $S_{\text{top}}$ is universal
- But why does $S_{\text{top}} = \log(D^2)$?
Exactly soluble example

\[ H = -V \sum_{l} \sigma_{l1}^{x} \sigma_{l2}^{x} \sigma_{l3}^{x} - t \sum_{p} \sigma_{p1}^{z} \sigma_{p2}^{z} \sigma_{p3}^{z} \sigma_{p4}^{z} \sigma_{p5}^{z} \sigma_{p6}^{z} \]
Ground state wave function

\[ \Psi \] is uniform superposition of closed string configurations
Entanglement entropy

For any \( q_1, \ldots, q_n = 0, 1, \sum_m q_m \) even, define \( \Psi_{in}^{q_1 \ldots q_n} \).

Similarly define \( \Psi_{out}^{q_1 \ldots q_n} \).
For any $q_1, \ldots, q_n = 0,1$, $\sum_m q_m$ even, define
\[ \Psi_{q_1, \ldots, q_n}^{\text{in}} \]
Similarly define $\Psi_{q_1, \ldots, q_n}^{\text{out}}$
For any $q_1, \ldots, q_n = 0, 1$, $\sum_m q_m$ even, define $\Psi_{\text{in}}_{q_1, \ldots, q_n}$.

Similarly define $\Psi_{\text{out}}_{q_1, \ldots, q_n}$. 
Entanglement entropy

For any $q_1, \ldots, q_n = 0, 1$, $\sum_m q_m$ even, define $\Psi_{\text{in}}^{q_1, \ldots, q_n}$.

Similarly define $\Psi_{\text{out}}^{q_1, \ldots, q_n}$. 
Entanglement entropy

Then: $\Psi = \sum_q \Psi_{\text{in}}^q \Psi_{\text{out}}^q$
Then: $\Psi = \sum_q \Psi_{\text{in}_q} \Psi_{\text{out}_q}$

Therefore $\rho_R$ is an equal mixture of all $\Psi_{\text{in}_q}$. 
Entanglement entropy

Then: $\Psi = \sum_q \Psi_{\text{in}_q} \Psi_{\text{out}_q}$

Therefore $\rho_R$ is an equal mixture of all $\Psi_{\text{in}_q}$

There are $2^{n-1}$ different $\Psi_{\text{in}_q}$ * $S_R = (n-1) \log 2$
Topological entropy

- \( S_R = (n-k) \log 2 \) where
- \( k = \# \) boundary curves
Topological entropy

- $S_R = (n-k) \log 2$ where $k = \#\text{ boundary curves}$

- $S_1 = (n_1-2) \log 2$;
- $S_2 = (n_2-1) \log 2$;
- $S_3 = (n_3-1) \log 2$;
- $S_4 = (n_4-2) \log 2$;
Topological entropy

- $S_R = (n-k) \log 2$ where $k = \# \text{ boundary curves}$

- $S_1 = (n_1-2) \log 2$; $S_2 = (n_2-1) \log 2$; $S_3 = (n_3-1) \log 2$; $S_4 = (n_4-2) \log 2$;

- $-S_{top} = (n_1-n_2-n_3+n_4-2) \log 2 = -2 \log 2 = -\log(2^2)$
Topological entropy

- $S_R = (n-k) \log 2$ where $k = \# \text{boundary curves}$

- $S_1 = (n_1-2) \log 2$;
  $S_2 = (n_2-1) \log 2$;
  $S_3 = (n_3-1) \log 2$;
  $S_4 = (n_4-2) \log 2$;

- $-S_{\text{top}} = (n_1-n_2-n_3+n_4-2) \log 2 = -2 \log 2 = -\log(2^2)$

Right result: $D=2$ for $Z_2$ topological order!
Topological entropy in the continuum

\[ S = c \cdot L + \ldots \]
Topological entropy in the continuum

\[ S = c \ L - S_{\text{top}} \]
Topological entropy in the continuum

\[ S = c \, L - S_{\text{top}} \]

Universal finite size correction!
Topological entropy for disk

\[ S = c L - S_{\text{top}}/2 \]

Universal finite size correction!
Conclusions/New directions

- Compute $S_{\text{top}}$ for $J_1$-$J_3$ model, quantum loop gas, etc.

- $S_{\text{top}}$ and critical theories

- Can we get more information from $\Psi$ – e.g. statistics of quasiparticles?