q-Q.H.E. and Topology

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From the Hall Effect to integrability

1. Hall effect.
2. Transfer Matrix.
3. Annular algebras.
4. deformed Hall effect wave function as a toy model for TQFT.
5. Conclusion.
Hall effect

- Lowest Landau Level wave functions

\[ \psi_n(z) = z^n e^{-\frac{zz}{l^2}} \]

\[ r = l\sqrt{n} \]

\[ n=1, 2, \ldots, \frac{A}{2\pi l^2} \]
\[ \frac{A}{2\pi l^2} = n_0 \]  
Number of available cells also the **maximal degree** in each variable

\[ Z_1^{\lambda_1} \ldots Z_n^{\lambda_n} \]  
Is a basis of **states** for the system
Interactions

\[(Z_i - Z_j)^m\]

\(m\) measures the strength of the interactions.

Competition between interactions which spread electrons apart and high Compression which minimizes the degree \(n\).
With adiabatic time $\text{QHE}=\text{TQFT}$

- Bulk and edge.

Compute Feynman path integrals
Two layer system.

- Spin singlet projected system of 2 layers

\[ \prod (x_i - x_j)^m (y_i - y_j)^m \]

When 3 electrons are put together, the wave function vanishes as:

\[ \mathcal{E}^m \]
Projection onto the singlet state

Crossings forbidden to avoid double counting
RVB basis:

Projection onto the **singlet state**

\[
\begin{align*}
\tilde{e}_{5\ 3\ 1} & = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{array}{c}
\text{Diagram 1}
\end{array} \\
\tilde{e}_{5\ 2\ 3\ 1} & = \begin{pmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{pmatrix} = \begin{array}{c}
\text{Diagram 2}
\end{array} \\
\tilde{e}_{4\ 3\ 2\ 1} & = \begin{pmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 6 \end{pmatrix} = \begin{array}{c}
\text{Diagram 3}
\end{array} \\
\tilde{e}_{4\ 5\ 2\ 3\ 1} & = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 6 \end{pmatrix} = \begin{array}{c}
\text{Diagram 4}
\end{array} \\
\tilde{e}_{3\ 4\ 5\ 2\ 3\ 1} & = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} = \begin{array}{c}
\text{Diagram 5}
\end{array}
\end{align*}
\]
RVB Basis
Also with fluxes
Exemples of wave functions

• Haldane-Rezayi: singlet state for 2 layer system.

\[
\text{Perm} \left[ \frac{1}{x_i - y_j} \right] \prod (x_i - x_j)(y_i - y_j)(x_i - y_j)
\]

When 3 electrons are put together, the wave function vanishes as:

\[ e^{2} \]
Moore Read

No spin

When 3 electrons are put together, the wave function vanishes as: $\mathcal{E}^2$

$$\text{Pfaff}\left[\frac{1}{z_i - z_j}\right]\prod (z_i - z_j)$$
Razumov Stroganov Conjectures

I.K. Partition function:

Also eigenvector of:

Stochastic matrix

\[ H = \sum_{i=1}^{6} e_i \]
$e_1 =$

\[
H = \begin{pmatrix}
3 & 2 & 2 & 0 & 2 \\
1 & 2 & 0 & 1 & 0 \\
1 & 0 & 2 & 1 & 0 \\
0 & 2 & 2 & 3 & 2 \\
1 & 0 & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
2 \\
1 \\
1 \\
2 \\
1
\end{pmatrix}
\]

Stochastic matrix
If $d=1$
Not hermitian
Transfer Matrix
Consider inhomogeneous transfer matrix:

\[ T \left( z, z_i \right) = tr \left( L \left( \frac{Z^1}{z} \right) \ldots \ L \left( \frac{Z^n}{z} \right) \right) \]

\[ L = \underbrace{\quad + \quad \quad} \]

\[ qz - q^{-1}z^{-1} \quad \quad \quad \quad z - z^{-1} \]
Transfer Matrix

\[ T(z, z_i) \Psi(z_1 \ldots z_n) = \Lambda(z_1 \ldots z_n) \Psi \]

\[ [T(z), T(w)] = 0 \]
I.K. Partition function

Total partition function can be expressed as a Gaudin Determinant

When $d=q+1/q=1$ symmetrical and given by a Schur function.

Stroganov
Hecke and Yang generators

- Consider Hecke algebra generators:
  \[ t_1 t_2 t_1 = t_2 t_1 t_2 \]  
  Braid group relations

- And Yang operators:
  \[ (t_1 - q)(t_1 + q^{-1}) = 0 \]

- Permutation relations:
  \[ Y_1 = \frac{z_1 t_1 - z_2 t_1^{-1}}{q z_2 - q^{-1} z_1} \]
  Or Yang-Baxter algebra

\[ Y_1 Y_2 Y_1 = Y_2 Y_1 Y_2 \]

\[ Y_1 Y_1 = 1 \]
T.L. (Jones)

\[ x = 9 \]

Diagram: Two disconnected loops connected by a line.
\[ Y_4 \]

\[ [Y_i, Y_j] = 0 \]
\[ y_1 T^{-1} = T y_2 \]
Content of Representation

Read eigenvalues of $y_i$

Initial height 1 2 3 4 .......

Final height
Bosonic Ground State

$$\Psi = 1 = \sum \pi \otimes \pi$$

$$t_i \Psi = \Psi \Psi t_i,$$

$$y_i \Psi = \Psi y_i,$$

Look for dual representation of AHA on polynomials
AHA

- Spin representation

\[ y_1 \]
\[ y_2 = t_1 y_1 t_1 \]
\[ y_3 = t_2 y_2 t_2 \cdots \]

- Polynomial representation

\[ y_1 = t_1 \cdots t_n \sigma \]

With: \[ z_i \sigma = z_{i+1} \]

And: \[ z_{i+n} = sz_i \]

Triangular matrices
Two \( q \)-layer system.

- Spin singlet projected system of 2 layers

\[
\prod (q^{-1} x_i x_j)(q^{-1} y_i y_j)
\]

(\( P \)) If \( i < j < k \) cyclically ordered, then

\[
\Psi(z_i = z, z_j = q^2 z, z_k = q^4 z) = 0
\]

Imposes \( s = q^6 \) for no new condition to occur
(e + \tau) \Psi = (\Psi(z_1, z_2) - \Psi(z_2, z_1)) \frac{z_1 q - z_2 q^{-1}}{z_1 - z_2}

e + \tau \text{ projects onto polynomials divisible by: }

z_1 q - z_2 q^{-1}

e \text{ Measures the Amplitude for 2 electrons to be in the same layer}
k q-layer system

- Spin singlet projected system of k layers

\[ \prod_{a=1} \left( q \ x_i^a - q^{-1} x_j^a \right) \]

If \( i < j < k \) cyclically ordered, then

(\( P \)) \[ \Psi(z_i = z, z_j = q^2 z, z_k = q^{2k} z) = 0 \]

Imposes \( S = q^{2(k+1)} \) for no new condition to occur
Other generalizations

• q-Haldane-Rezayi

\[
\text{Det} \left[ \frac{1}{(x_i - y_j)(qx_i - q^{-1}y_j)} \right]
\]

Generalized Wheel condition, Gaudin Determinant
Related in some way to the Izergin-Korepin partition function?

Fractional hall effect Flux \( \frac{1}{2} \) electron
Kasatani wheel conditions

\[ t^{k+1} q^{r-1} = 1 \]

\[ \frac{Z_{i_{a+1}}}{Z_{i_a}} = tq^{b_{aa+1}} \]

\[ b_{aa+1} = 0 \Rightarrow i_{a+1} > i_a \]

\[ \sum b_{aa+1} \leq r - 2 \]

With \( r \), Flux=1/r particle.
Moore-Read

- Property (P) with $s$ arbitrary.
- Affine Hecke replaced by Birman-Wenzl-Murakami.
- R.S replaced by Nienhuis De Gier in the symmetric case.

\[
sq^{2(k+1)} = 1
\]

\[
Pfaff \left[ \frac{1}{qx_i - q^{-1}x_j} \right]
\]
The case $q$ root of unity

- When $q+1/q=1$, Hecke representation is no more semisimple and degenerates into a trivial representation.
- Stroganov Partition function (Schur function) can be recovered as the unique symmetrical polynomial of the minimal degree obeying (P).

- Other roots of unity?
Conclusions

• T.Q.F.T. realized on q-deformed wave functions of the Hall effect.

• All connected to Razumov-Stroganov type conjectures. Proof of conjecture still missing.

• Relations with works of Feigin, Jimbo, Miwa, Mukhin and Kasatani on polynomials obeying wheel condition.

• Excited states of the Hall effect.

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