Quasi-holes for non-abelian quantum Hall states

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outline

1. quasi-hole heuristics
2. non-abelian qH states
3. quasi-hole specifics
4. qH-CFT connection
5. quasi-hole counting
6. quasi-hole wavefunctions
7. quasi-hole braiding
 quasi-hole heuristics

• following Laughlin (1983): fundamental excitation over 
\[ \nu = \frac{1}{m} \]
fqH state associated to insertion of a flux quantum \( \Phi_0 \)
• the resulting quasi-hole carries an electric charge equal to
\[ q = \sigma_{\text{Hall}} \Phi_0 = \nu e = e/m \]
• the excitations are anyonic: braiding is represented by a phase factor
\[ \exp(i\pi/m) \]
quasi-hole heuristics

- in multi-layer and spin-1/2 state, flux insertion `per layer' or `per spin'
- example --

Halperin \((m+1,m+1,m)\) spin-singlet state, filling \(\nu=2/(2m+1)\),

\[
\tilde{\Psi}_H(z_1^\uparrow,...,z_N^\uparrow,z_1^\downarrow,...,z_N^\downarrow) = \prod_{i<j}(z_i^\uparrow - z_j^\uparrow)^{m+1} \prod_{i<j}(z_i^\downarrow - z_j^\downarrow)^{m+1} \prod_{i,j}(z_i^\uparrow - z_j^\downarrow)^m
\]

has quasi-holes with quantum numbers

\[q = e/(2m+1), \quad S=1/2\]
quasi-hole heuristics

- non-abelian quantum Hall states generally characterised by pairing (or order-$k$ clustering)
- as in a superconductor, pairing (clustering) of particles leads to a reduction of the minimal allowed flux-insertion, which is now $\Phi_0/k$
- this leads to further fractionalization of quasi-holes
- the resulting quasi-holes are the ones exhibiting non-abelian braid statistics
quasi-hole heuristics

for paired qH states with pfaffian factor the quasi-particle break-up can take different forms:

• \( \nu = 1/2 \) Moore-Read state:
  
  \( q = e/2 \) splits into twice \( q = e/4 \)
  
  [charge fractionalisation]

• \( \nu = 2/3 \) spin-singlet state [Ardonne et al, 2002]
  
  \( [q = e/3, S = 1/2] \) splits into \( [q = e/3, S = 0], [q = 0, S = 1/2] \)
  
  [spin-charge separation]

• pfaffian \( \nu = 1 \) state for rotating spin-1 bosons [Reijnders et al, 2002]
  
  \( [q = 1, S = 1] \) splits into twice \( [q = 1/2, S = 1/2] \)
  
  [spin fractionalisation]
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Read-Rezayi *k*-clustered states

- qH states for spin-polarized electrons
- filling factor $\nu=k/(kM+2)$
- expected in 2nd LL [$\nu=5/2, 12/5$ (?, ..)]
- expected in rapidly rotating BEC [$\nu=1/2, 1, 3/2, ..$]
- *k*-clustering: $M=0$ (bosonic) state obtained as maximal density zero energy eigenstate of Hamiltonian

$$H = V \sum_{i_1 < .. < i_{k+1}} \delta^2(z_{i_1} - z_{i_2}) \delta^2(z_{i_2} - z_{i_3}) \ldots \delta^2(z_{i_k} - z_{i_{k+1}})$$

- quasi-holes:
  - $q = 1/(kM+2)$
  - counting, wavefunctions, braiding
double layer states

Many possibilities for non-abelian states in multi-layer samples.

Example --

double layer pfaffian state

\[
\tilde{\Psi}_{pf}(z_1^{\uparrow}, ..., z_N^{\uparrow}, z_1^{\downarrow}, ..., z_N^{\downarrow}) = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i < j} (z_i^{\uparrow} - z_j^{\uparrow})^m \prod_{i < j} (z_i^{\downarrow} - z_j^{\downarrow})^m \prod_{i < j} (z_i^{\uparrow} - z_j^{\downarrow})^n
\]

Excitations include non-abelian excitons.

For \(m=2, n=1\) this is a \(\nu=2/3\) spin-singlet state exhibiting spin-charge separation, with non-abelian spinon (spin 1/2 and zero charge) and holon (charge e/3 and zero spin) excitations.

Ardonne-v. Lankvelt-Ludwig-S, 2002
spin-1/2 spin-singlet states

- spin-singlet qH states for spin-1/2 electrons
- filling factor $\nu = \frac{2k}{2kM+3}$
- $k$-clustering: $M=0$ (bosonic) state obtained as maximal density zero energy eigenstate of hamiltonian

$$H = V \sum_{i_1 < \ldots < i_{k+1}} \delta^2(z_{i_1} - z_{i_2}) \delta^2(z_{i_2} - z_{i_3}) \ldots \delta^2(z_{i_k} - z_{i_{k+1}})$$

- quasi-holes:
  - $[ q = \frac{1}{2kM+3}, S = 1/2 ]$ or $[ q = \frac{2}{2kM+3}, S = 0 ]$
  - counting, wavefunctions, braiding
spin-1/2 spin-singlet states: $k=2$

- paired states; filling factor $\nu=4/3$ [M=0], $\nu=4/7$ [M=1], etc.
- explicit [M=0, $N_\uparrow=2$, $N_\downarrow=2$]
  \[
  \tilde{\Psi}_{AS}(z_1^\uparrow,z_2^\uparrow,z_1^\downarrow,z_2^\downarrow) = (z_1^\uparrow - z_1^\downarrow)(z_2^\uparrow - z_2^\downarrow) + (z_1^\uparrow - z_2^\downarrow)(z_2^\uparrow - z_1^\downarrow)
  \]

- the pairing is governed by same Hamiltonian as for Moore-Read state; the AS state smoothly connects to the MR state upon varying $N_\uparrow$, $N_\downarrow$ from spin-singlet to fully polarized with (for $M=0$)
  \[
  \nu = \frac{N_\uparrow^2 + 2N_\uparrow N_\downarrow + N_\downarrow^2}{N_\uparrow^2 + N_\uparrow N_\downarrow + N_\downarrow^2}
  \]

- the quasi-hole braiding is in the universality class of Fibonacci anyons; hence the prospect of universal QC with a paired qH state!

- experimental access to spin-singlet states: hydrostatic pressure and tilted field [Kang et al. 1997, Cho et al. 1998]
qH states for rotating (spin-1) bosons

Reijnders-v. Lankvelt-Read-S, 2002

• if qH states for rotating bosons can be achieved, then spin-1 bosons are an interesting and viable option

• two independent channels \((S=0, 2)\) in contact interactions give non-trivial parameter \(g=g_2/g_0\)

• spin interactions can be ferromagnetic \([g<0, ^{85}\text{Rb}]\) or antiferromagnetic \([g>0, ^{23}\text{Na}]\)

• qH states compete with various lattices of spin-textures (below)

• qH states include spin-1 version of RR series \([SU(4)_k]\) and a series \([SO(5)_k]\) exhibiting spin fractionalisation
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Quasi-holes generated by inserting excess flux in qH ground states come with an ‘internal register’ associated to fusion channel assignments. Issues

• counting the dimensionality of ‘internal register’ associated with given number of quasi-holes

• explicit wavefunctions, setting the amplitudes of various fusion channel states as a function of the positions of the quasi-holes

• braiding properties: enabling operations on the internal register by weaving braids of quasi-hole positions
quasi-holes: the strategies

- **direct analysis**: (numerically) determining zero-energy eigenstates for excess flux, and braiding e.g. with a Berry phase approach
  [Read-Rezayi, Tserkovnyak-Simon, …]

- `coordinate CFT`: concrete evaluation of quasi-hole wavefunctions, leading to braiding properties and beyond [Nayak-Wilczek, Ardonne-S]

- `algebraic CFT`: relying on quantum group structure associated to RCFT, for example to match fusion and braiding properties
  [Slingerland-Bais,…]
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qH wavefunctions from CFT

ground state wave function

$$\Psi_{GS}(z_1,\ldots,z_N) \equiv \langle \psi_e(z_1) \cdots \psi_e(z_N) \psi_{\text{background}}(z_{\infty}) \rangle_{\text{CFT}}$$

electron (boson) condensate operator
neutralizing background charge

quasi-hole excitations: fixed by

$$\phi_{\text{qh}}(w)\psi_e(z_1) = (z-w)^{\text{integer}} \left[ \phi_2(w) + \ldots \right]$$

excited state wave function:

$$\Psi_{\text{qh}}(w_1,w_2,\ldots;z_1,z_2,\ldots) \equiv \langle \phi_{\text{qh}}(w_1)\phi_{\text{qh}}(w_2) \cdots \psi_e(z_1)\psi_e(z_2) \cdots \rangle_{\text{CFT}}$$
parafermions and vertex operators

Generic form of electron and quasi-hole operators

\[ \psi_e = \psi_{PF} \times V.O.(\text{spin}) \times V.O.(\text{charge}) \]

\[ \phi_{qh} = \sigma_{PF} \times V.O.(\text{spin}) \times V.O.(\text{charge}) \]

Parafermionic fields taken from CFT of the form

\[ \text{PF}(G, k) \cong \hat{G}_k / [U(1)^{\text{rank}(G)}] \]  

Gepner, 1987

Vertex Operators for spin and charge have the form

\[ V.O.(\text{charge}) \cong \exp(i \alpha_c \varphi_c), \quad V.O.(\text{spin}) \cong \exp(i \alpha_s \varphi_s) \]
The parafermion theory $PF(G,k)$, together with the vertex operators for spin and charge (depending on the Laughlin exponent $M$), build up a deformation of the chiral WZW model $WZW(G,k)$

CFT data: $G_{k,M}$

$G_k=SU(2)_k$ for the RR series and $G_k=SU(3)_k$ for the non-abelian spin-singlet states.

**The properties of the $SU(2)_k$ and $SU(3)_k$ parafermions guarantee that the PF correlators give wave functions that satisfy the order-$k$ clustering property.**
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quasi-hole counting

Agreement between the `direct’ and `parafermion’ approaches for the counting problem in spherical geometry

• **direct approach:**
  numerical evaluation of zero energy eigenstates of clustering hamiltonian in presence of excess flux, meaning a fixed number of quasi-holes

• **parafermion approach:**
  analytic counting formulas based on parafermion combinatorics and on a physical picture relating parafermions to broken clusters in the condensate

[Read-Rezayi, Rezayi-Gurarie, Ardonne-Read-Rezayi-S, Ardonne]
**direct approach**

$k=2$ spin-singlet state: 

*N*=8 electrons  

*n*=8 quasi-holes

\[ \begin{array}{|l|c|c|c|c|}
\hline
& S=0 & S=1 & S=2 & S=3 \\
\hline L=0 & 4 & 1 & 3 & 0 \\
\hline L=1 & 1 & 7 & 2 & 1 \\
\hline L=2 & 7 & 7 & 6 & 1 \\
\hline L=3 & 3 & 9 & 3 & 1 \\
\hline L=4 & 6 & 6 & 4 & 0 \\
\hline L=5 & 2 & 5 & 1 & 0 \\
\hline L=6 & 3 & 2 & 1 & 0 \\
\hline L=7 & 0 & 1 & 0 & 0 \\
\hline L=8 & 1 & 0 & 0 & 0 \\
\hline \end{array} \]

(tot al # states: 
\[ \#_{N=8,n=8} = 1719 \])
parafermion approach

Analytical expression for general $N_{\uparrow}, N_{\downarrow}, n_{\uparrow}, n_{\downarrow}$:

$$
\#_{N,n} = \sum_{F_1, F_2} \left\{ \begin{array}{c} n_{\uparrow} \\ F_1 \\ \end{array} \begin{array}{c} n_{\downarrow} \\ F_2 \\ \end{array} \right\}^{(k)} \left( \frac{(N_{\uparrow} - F_1)}{k + n_{\uparrow}} \right) \left( \frac{(N_{\downarrow} - F_2)}{k + n_{\downarrow}} \right)
$$

(with explicit expressions for the symbols $\{ \}$\(^{(k)} \) …]

Ardonne-Read-Rezayi-S 2000, Ardonne 2001
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We determined explicit wavefunctions for the excitations with 4 quasi-holes (of various kinds) over the $k=3$ Read-Rezayi state and the $k=2$ non-abelian spin-singlet state, generalizing results of Nayak and Wilczek, 1996, for the MR state. From these, braiding properties follow by inspection.

Here we present the case of 4 spin-less quasi-holes $\phi_3$, of charge $2/3$, over the paired spin-singlet state. For $M>0$ these are the lowest dimension quasi-holes, and thereby the most natural agents in a QC protocol.
The qH-CFT correspondence gives

\[
\Psi_{3333}(w_1, w_2, \ldots; z_1^\uparrow, z_2^\uparrow, \ldots, z_1^\downarrow, z_2^\downarrow, \ldots) = \\
\left\langle \sigma_3(w_1)\sigma_3(w_2)\ldots \psi_1(z_1^\uparrow)\psi_1(z_2^\uparrow)\ldots \psi_2(z_1^\downarrow)\psi_2(z_2^\downarrow)\ldots \right\rangle_{\text{CFT}} \\
\times \left[ \Psi_{221}^{221}\left(\{z_i^\uparrow, z_j^\downarrow\}\right) \right]^{1/2} \prod_{i,j} (z_i^\uparrow - w_j)^{1/2} \prod_{i,j} (z_i^\downarrow - w_j)^{1/2} \prod_{i<j} (w_i - w_j)^{1/3}
\]
quasi-hole wavefunctions

Going into the correlator

\[
\langle \sigma_3(w_1)\sigma_3(w_2) \ldots \psi_1(z_1^\uparrow)\psi_1(z_2^\uparrow) \ldots \psi_2(z_1'^\downarrow)\psi_2(z_2'^\downarrow) \ldots \rangle_{\text{CFT}}
\]

\( S U(3)_2 \) parafermion algebra

\[
\psi_1(z)\psi_1(w) = (z - w)^{-1}I + \ldots ,
\]

\[
\psi_2(z)\psi_2(w) = (z - w)^{-1}I + \ldots
\]

\[
\psi_1(z)\psi_2(w) = (z - w)^{-1/2}\psi_{12} + \ldots
\]

and the spin-field OPE, with two independent fusion channels

\[
\sigma_3(z)\sigma_3(w) = (z - w)^{-1/5}I + (z - w)^{2/5}\rho_3(w) + \ldots
\]
For the MR state, Nayak and Wilczek applied \textit{bosonization} to obtain the analogous correlator, giving a final expression of the type

$$\Psi^{(0,1)}(w_1, w_2, w_3, w_4; z_1, z_2, \ldots) = \quad \text{basis for two-fold degenerate internal register; polynomial in } w_i, z_j$$

pre-factors depending on fusion channel \((0, 1)\) and on quasi-hole locations \(w_i\)
To obtain a similar expression we proceed in a number of steps

**Step1.** In absence of quasi-holes, we have the following expression for wavefunction [Cappelli et al, Ardonne et al]

\[
\Psi_{GS} = \frac{1}{N} \sum_{\{S_1, S_2\}} \Psi^{221}_{S_1}(z^\uparrow_i, z^\downarrow_j) \Psi^{221}_{S_2}(z^\uparrow_i, z^\downarrow_j)
\]

with particles in disjoint subsets \( S_1, S_2 \) each forming a Halperin 221 state

\[
\Psi^{221}(z^\uparrow_1, \ldots, z^\uparrow_N, z'^\downarrow_1, \ldots, z'^\downarrow_N) = \prod_{i<j} (z^\uparrow_i - z^\uparrow_j)^2 \prod_{i<j} (z'^\downarrow_i - z'^\downarrow_j)^2 \prod_{i,j'} (z^\uparrow_i - z'^\downarrow_{j'})
\]
quasi-hole wavefunctions

**Step 2.** Basis for 4 quasi-hole state obtained by distributing the quasiholes over the sets $S_1, S_2$; two independent choices for this are $\Psi_{12,34}$ and $\Psi_{13,24}$ with

\[
\Psi_{12,34} = \frac{1}{N} \sum_{\{S_1, S_2\}} \left[ \prod_{i,j \in S_1} (z_i^\uparrow - w_1)(z_j^\uparrow - w_2)(z_i^\downarrow - w_1)(z_j^\downarrow - w_2) \right] \Psi_{S_1}^{221}(z_i^\uparrow, z_j^\uparrow) \\
\times \left[ \prod_{i,j \in S_2} (z_i^\uparrow - w_3)(z_j^\uparrow - w_4)(z_i^\downarrow - w_3)(z_j^\downarrow - w_4) \right] \Psi_{S_2}^{221}(z_i^\uparrow, z_j^\uparrow)
\]

\[
\Psi_{13,24} = \ldots
\]
quasi-hole wavefunctions

**Step 3.** Decompose wavefunction over $\Psi_{12,34}$ and $\Psi_{13,24}$ and impose consistency upon fusing some of the parafermions $\psi_{1,2}$ with the $\sigma_3$.

This requires detailed knowledge of short distance Operator Products Expansions (OPE), and of 4-point functions in the $SU(3)_2$ WZW model [Knizhnik-Zamolodchikov, 1984]

Building blocks are hypergeometric functions

\[
F_1^{(0)} = x^{-8/15} (1 - x)^{1/15} F\left(\frac{1}{5}, -\frac{1}{5}, \frac{2}{5}, x\right)
\]

\[
F_2^{(0)} = \frac{1}{2} x^{7/15} (1 - x)^{1/15} F\left(\frac{6}{5}, \frac{4}{5}, \frac{7}{5}, x\right)
\]

\[
F_1^{(1)} = x^{1/15} (1 - x)^{1/15} F\left(\frac{2}{5}, \frac{4}{5}, \frac{8}{5}, x\right)
\]

\[
F_2^{(1)} = -3x^{1/15} (1 - x)^{1/15} F\left(\frac{2}{5}, \frac{4}{5}, \frac{3}{5}, x\right)
\]

\[
x = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_4)(w_3 - w_2)}
\]
quasi-hole wavefunctions

Final result.

\[
\Psi^{(0,1)}_{3333}(w_1, w_2, w_3, w_4; z_1^{\uparrow}, z_2^{\uparrow}, \ldots, z_1^{\downarrow}, z_2^{\downarrow}, \ldots) = \\
A^{(0,1)}_{3333}(\{w_i\})\Psi_{[12,34]}(\{w_i; z_i, z_j^{\uparrow}\}) + B^{(0,1)}_{3333}(\{w_i\})\Psi_{[13,24]}(\{w_i; z_i, z_j^{\downarrow}\})
\]

\[
A^{(0)}_{3333} = \left[w_{12}w_{34}\right]^{4/5} x^{-2/15} (1 - x)^{2/3} F_2^{(0)}(x)
\]

\[
B^{(0)}_{3333} = \left[w_{12}w_{34}\right]^{4/5} x^{-2/15} (1 - x)^{2/3} F_1^{(0)}(x)
\]

\[
A^{(1)}_{3333} = \left[w_{12}w_{34}\right]^{4/5} (-1)^{8/5} C x^{-2/15} (1 - x)^{2/3} F_2^{(1)}(x)
\]

\[
B^{(1)}_{3333} = \left[w_{12}w_{34}\right]^{4/5} (-1)^{8/5} C x^{-2/15} (1 - x)^{2/3} F_1^{(1)}(x)
\]

\[
C^2 = \frac{1}{9} \frac{\Gamma^3\left(\frac{2}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\Gamma^3\left(\frac{3}{5}\right)\Gamma\left(\frac{1}{5}\right)}
\]
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quasi-hole braiding

The full parafermion theory has eight sectors

\[ \{I, \psi_1, \psi_2, \psi_{12}, \sigma_1, \sigma_2, \sigma_3, \rho\} \]

with Fibonacci type fusion if we set

\[ 0 \equiv \{I, \psi_1, \psi_2, \psi_{12}\} \quad 1 \equiv \{\sigma_1, \sigma_2, \sigma_3, \rho\} \]

One thus expects Fibonacci braiding properties, suitable for universal QC.
quasi-hole braiding

Evaluation of braiding by direct inspection of:

\[
\Psi^{(0,1)}_{3333}(w_1, w_2, w_3, w_4; z^+_1, z^+_2, \ldots, z^+_1, z^+_2, \ldots) = A^{(0,1)}_{3333}(\{w_i\})\Psi_{[12,34]}(\{w_i; z_i, z_j'\}) + B^{(0,1)}_{3333}(\{w_i\})\Psi_{[13,24]}(\{w_i; z_i, z_j'\})
\]

\[
A^{(0)}_{3333} = [w_1 w_3]^{4/5} x^{-2/15} (1 - x)^{2/3} F^{(0)}_2(x)
\]

\[
B^{(0)}_{3333} = [w_1 w_3]^{4/5} x^{-2/15} (1 - x)^{2/3} F^{(0)}_1(x)
\]

\[
A^{(1)}_{3333} = [w_1 w_3]^{4/5} (-1)^{8/5} C x^{-2/15} (1 - x)^{2/3} F^{(1)}_2(x)
\]

\[
B^{(1)}_{3333} = [w_1 w_3]^{4/5} (-1)^{8/5} C x^{-2/15} (1 - x)^{2/3} F^{(1)}_1(x)
\]

Example: \(w_2 \leftrightarrow w_3\).

this swaps \(\Psi_{12,34}\) and \(\Psi_{13,24}\); furthermore

\[
F^{(0)}_2(1 - x) = C_0^0 F^{(0)}_1(x) + C_1^0 F^{(1)}_1(x), \quad \text{etc.}
\]

\[
C_0^0 = \frac{1}{2}(\sqrt{5} - 1) = \tau, \quad C_1^0 / C = -\sqrt{\tau}
\]

\[
U_{2\leftrightarrow3} = (-1)^{4/5} \begin{pmatrix} \tau & \sqrt{\tau} \\ \sqrt{\tau} & -\tau \end{pmatrix}
\]
quasi-hole braiding

Full set of braiding relations on the 4 quasi-hole wavefunctions at $M=0$

\[
U_{1\leftrightarrow 2} = (-1)^{-2/3} \begin{pmatrix}
(-1)^{4/5} & 0 \\
0 & (-1)^{-3/5}
\end{pmatrix}
\]

\[
U_{2\leftrightarrow 3} = (-1)^{4/5} \begin{pmatrix}
\tau & \sqrt{\tau} \\
\sqrt{\tau} & -\tau
\end{pmatrix}
\]

\[
U_{1\leftrightarrow 3} = (-1)^{8/15} \begin{pmatrix}
\tau & (-1)^{-3/5} \sqrt{\tau} \\
(-1)^{-3/5} \sqrt{\tau} & (-1)^{-1/5} \tau
\end{pmatrix}
\]

Similar relations are obtained for spin-full quasi-holes over the $k=2$ spin-singlet state and for quasi-holes over the $k=3$ RR state.

Ardonne-S, in preparation
summary

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