Edge versus Bulk in Non-Abelian Topological Phases

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With thanks to Eduardo Fradkin, Paul Goldbart, Tony Leggett and Smitha Vishveshwara.

Outline of Talk

- Pfaffian wave functions
- $p_x + ip_y$ superconductor: vortices and Majorana edge states
- Hall droplet edge states, group theory, and bosonization
- generalized Pfaffians: k-clustering
- Bose gas representations of $\mathrm{su}(2)_k$
- $\mathrm{su}(n)_k \Rightarrow q$-refinement of fusion rules

Pfaffian wave functions

Moore-Read Pfaffian States

$$\mathrm{Pf}\,\mathbf{A} = \frac{1}{2^N N!}\epsilon_{i_1 \ldots i_{2N}} A_{i_1 i_2} \cdots A_{i_{2N-1} i_{2N}}$$

$$\left(\mathrm{Pf}\,\mathbf{A}\right)^2 = \det \mathbf{A}$$

Moore-Read Pfaffian States

$$\mathrm{Pf}\,\mathbf{A} = \frac{1}{2^N N!}\epsilon_{i_1 \ldots i_{2N}} A_{i_1 i_2} \cdots A_{i_{2N-1} i_{2N}}$$

$$\Psi(z_1, \ldots, z_{2N}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^n$$

Moore-Read Pfaffian States

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$\nu = 1/2$ quantum Hall state

Moore-Read Pfaffian States

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$$\Psi(z_1, \ldots, z_{2N}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^{1}$$

rotating Bose gas QH-like state

Moore-Read Pfaffian States

$$\mathrm{Pf}\,\mathbf{A} = \frac{1}{2^N N!}\epsilon_{i_1 \ldots i_{2N}} A_{i_1 i_2} \cdots A_{i_{2N-1} i_{2N}}$$

$$\Psi(z_1, \ldots, z_{2N}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)$$

planar $p_x + ip_y$ superconductor

$p_x + ip_y$ superconductor: vortices and Majorana edge states

$p_x + ip_y$ superconductors

$$\langle \psi_\alpha \psi_\beta \rangle = \text{spin} \times \text{orbit}$$

$$\text{spin} = \{(-i\sigma_2)(\mathbf{d}\cdot\sigma)\}_{\alpha\beta} \qquad \text{triplet}$$

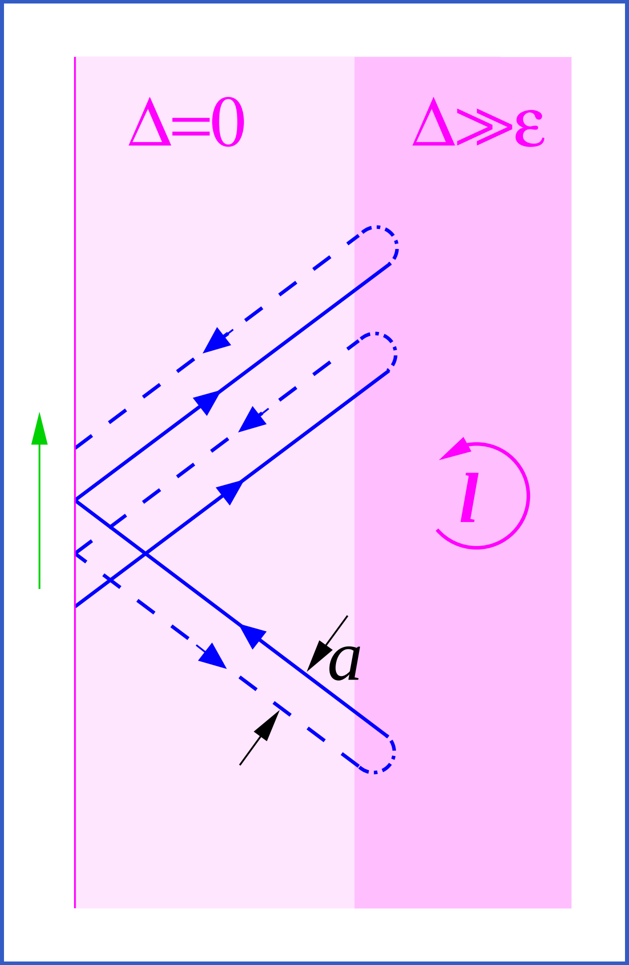
$$\text{orbit} \propto (\hat{p}_x + i\hat{p}_y) \qquad p\text{—wave}$$

If $\mathbf{d} = \mathbf{e}_y$ then

$$\langle \psi_\alpha \psi_\beta \rangle \propto \delta_{\alpha\beta}\,(\hat{p}_x + i\hat{p}_y)$$

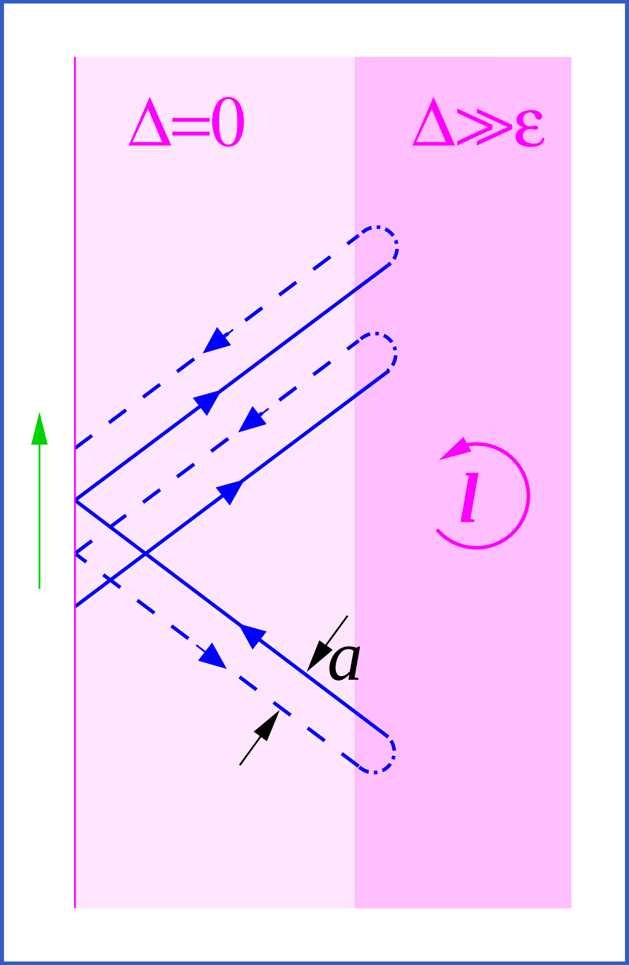
$\Rightarrow$ up/down spins decoupled

Chiral Majorana edge mode



Spin-triplet $p_x + ip_y$ SC has chiral Majorana edge-mode.

Chiral Majorana edge mode



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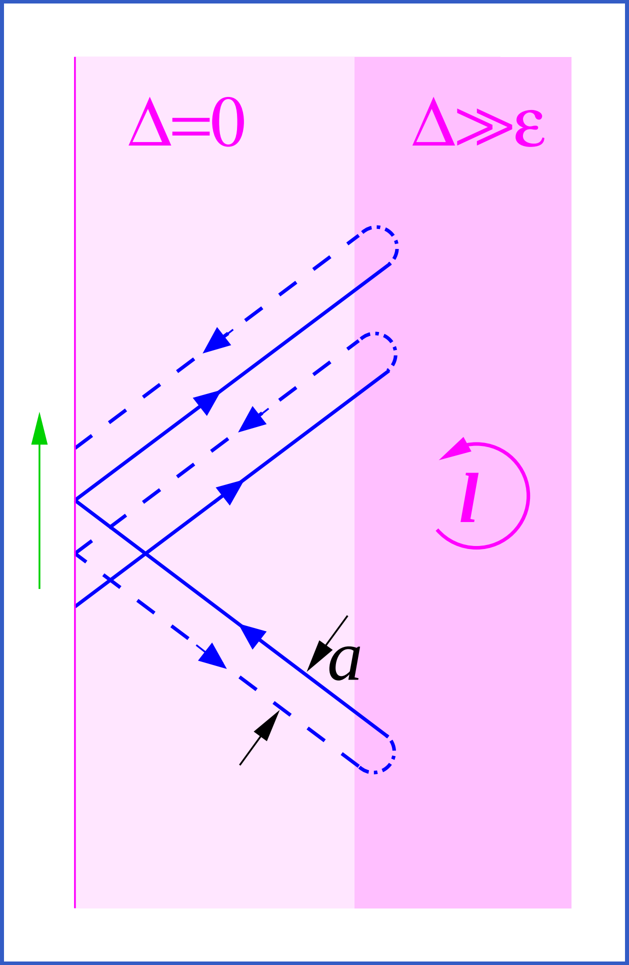
- Why *chiral*?

Chiral Majorana edge mode



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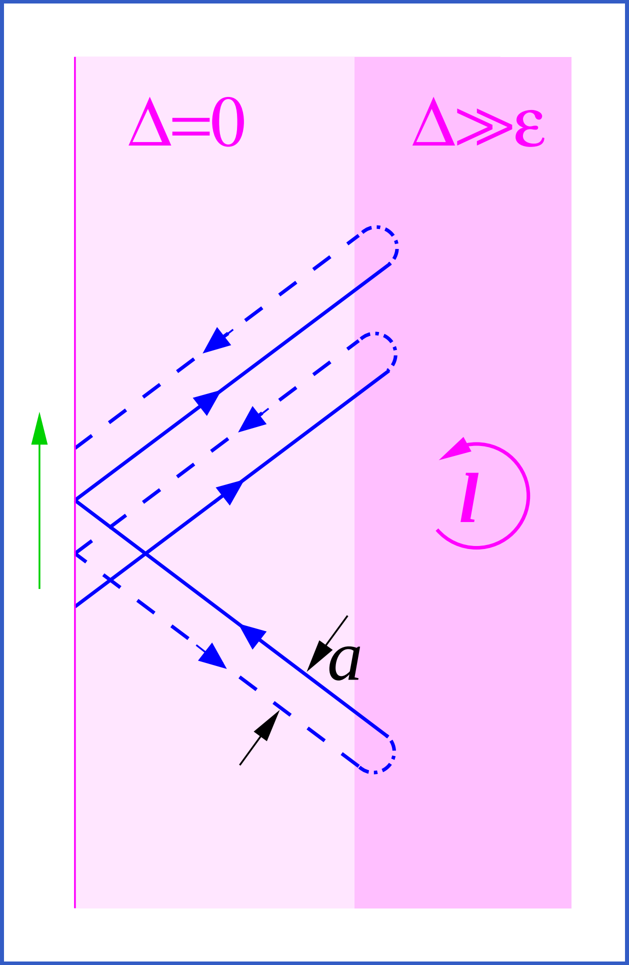
- Why *chiral?*
- Cooper pair has $l = \hbar$

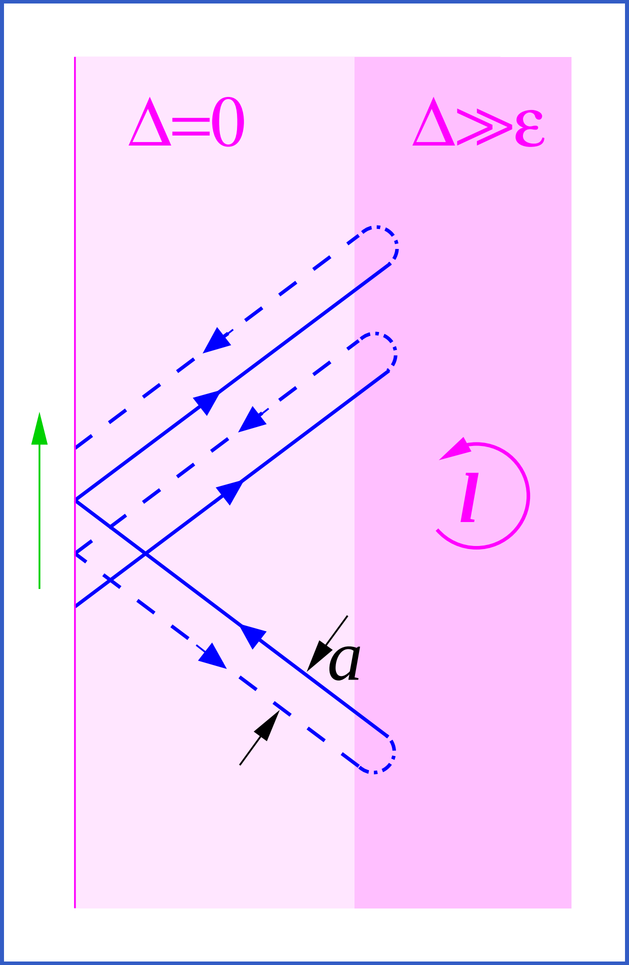
Chiral Majorana edge mode



Spin-triplet $p_x + ip_y$ SC has chiral Majorana edge-mode.

- Why *chiral?*
- Cooper pair has $l = \hbar$
- $\Rightarrow$ Andreev reflection offset $k_{\mathrm{Fermi}} a = \hbar$.

Chiral Majorana edge mode



Spin-triplet $p_x + ip_y$ SC has chiral Majorana edge-mode.

- Why *chiral?*
- Cooper pair has $l = \hbar$
- $\Rightarrow$ Andreev reflection offset $k_{\text{Fermi}} a = \hbar$.
- $\Rightarrow$ one-way edge creep

$$\varepsilon_k = ck$$

Why Majorana?

S-wave, S=0, superconductor

$$b_{\uparrow,k} = a_{\uparrow,k} + a^{\dagger}_{\downarrow,-k}$$

$$\boxed{b_{\uparrow,k} = b^{\dagger}_{\downarrow,-k}}$$

distinct anti-particle $\Rightarrow$ not Majorana

Why Majorana?

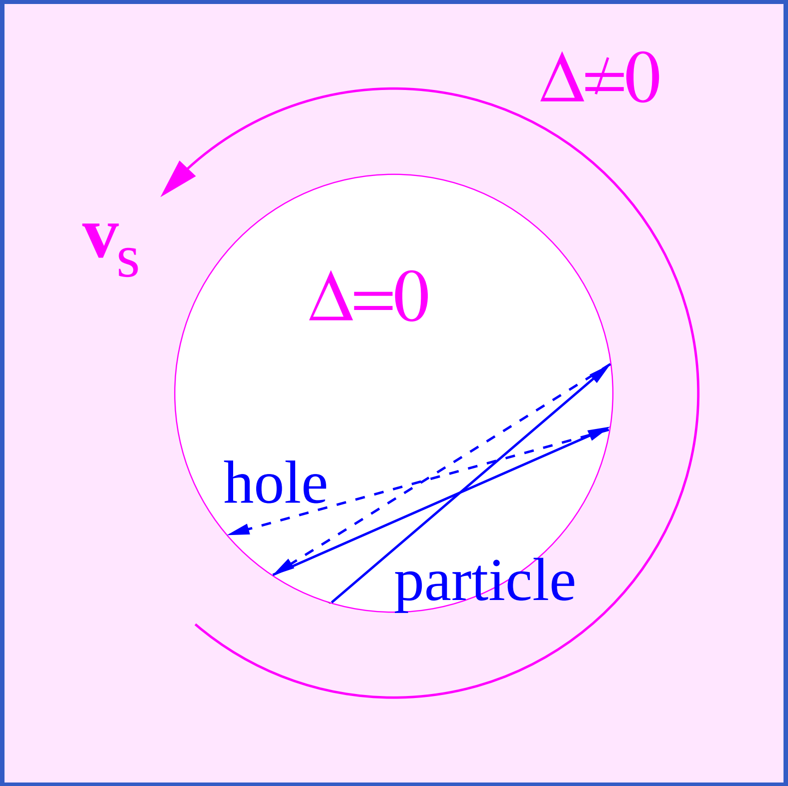
P-wave, S=1, superconductor

$$b_{\uparrow,k} = a_{\uparrow,k} + a^{\dagger}_{\uparrow,-k}$$

$$\boxed{b_{\uparrow,k} = b^{\dagger}_{\uparrow,-k}}$$

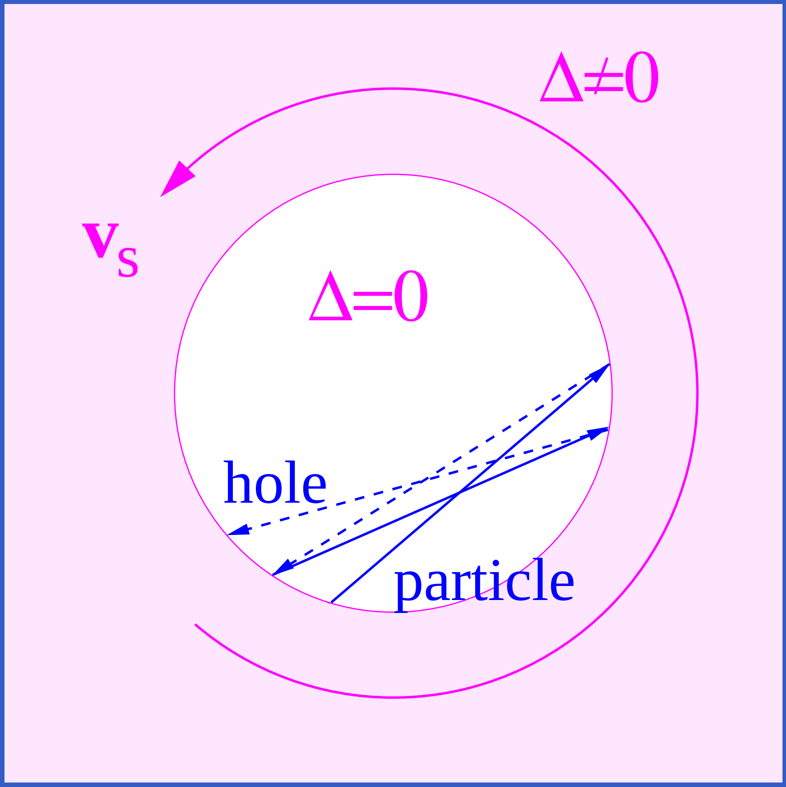
own anti-particle $\Rightarrow$ Majorana

vortex core states



Andreev bound state

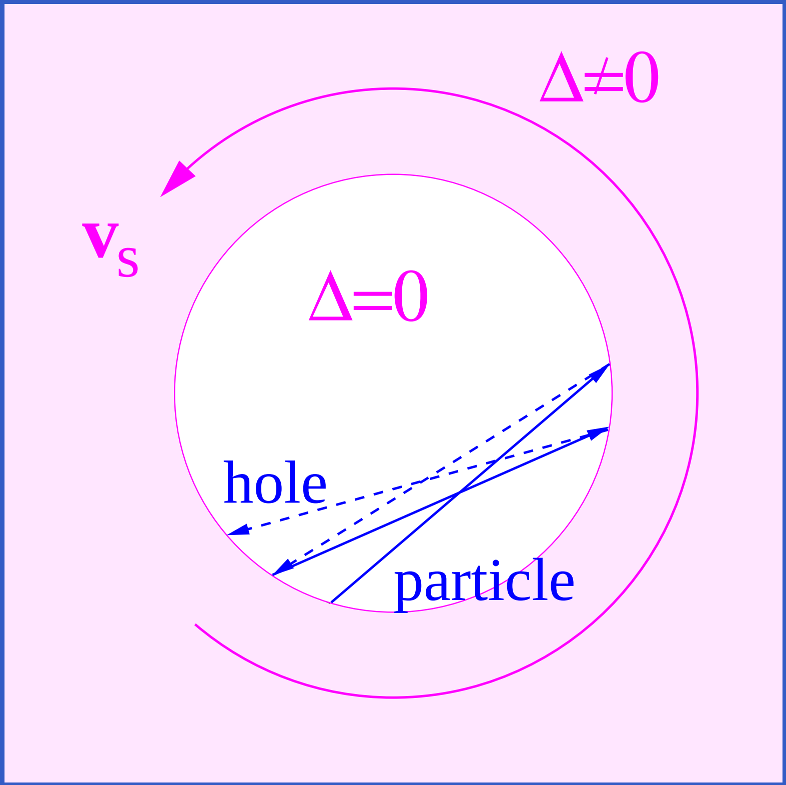
vortex core states



Andreev bound state

- Andreev reflection not *quite* retro-reflective

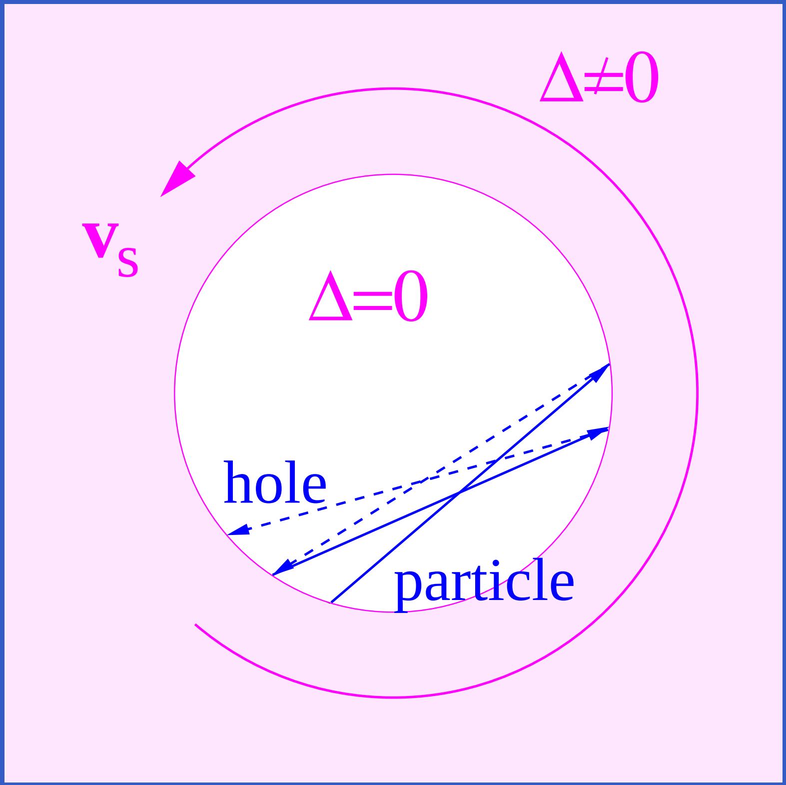
vortex core states



Andreev bound state

- Andreev reflection not *quite* retro-reflective
- ⇒ backward creep

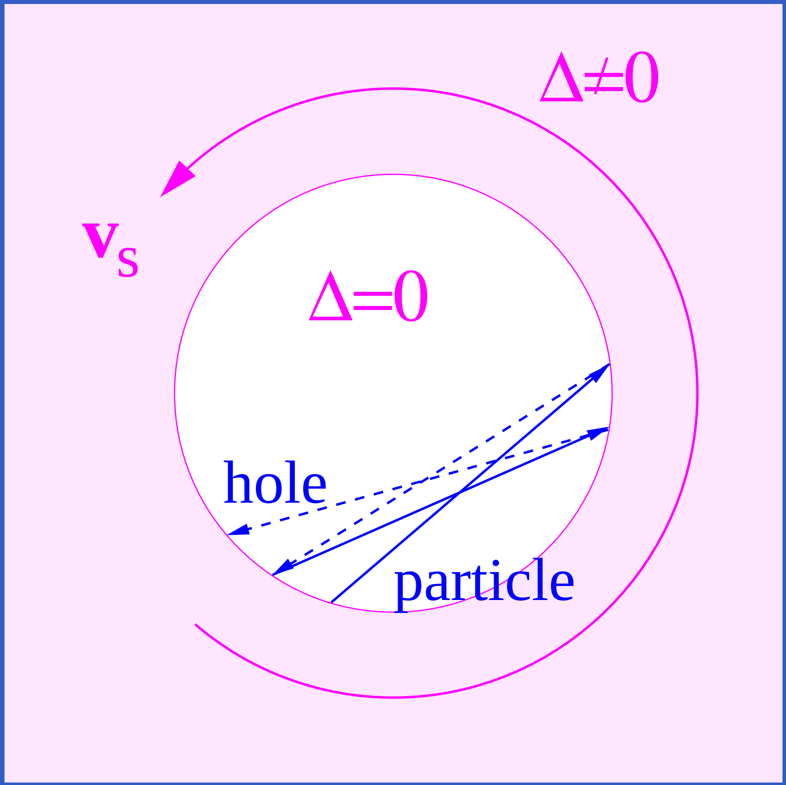
vortex core states



Andreev bound state

- Andreev reflection not *quite* retro-reflective
- $\Rightarrow$ backward creep
- $\Rightarrow \varepsilon_l = -\omega_0(l+\alpha)$

vortex core states



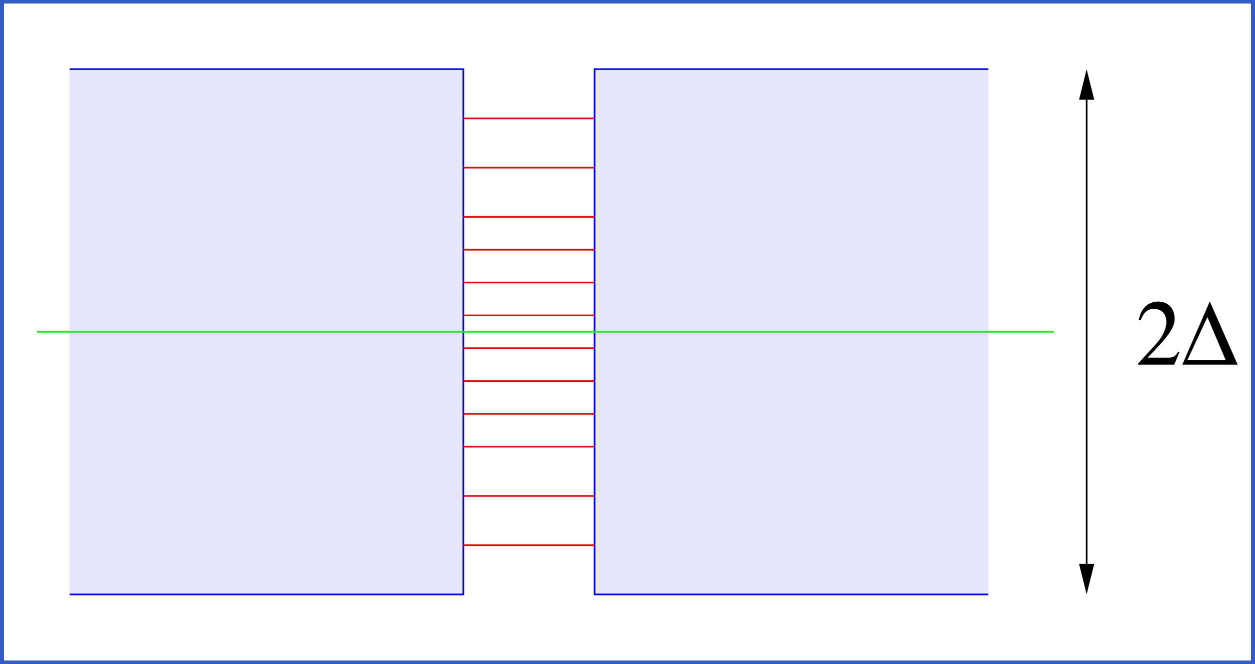
Andreev bound state

- Andreev reflection not *quite* retro-reflective
- $\Rightarrow$ backward creep
- $\Rightarrow \varepsilon_l = -\omega_0(l + \alpha)$
- α?

Core spectrum

Vortex-core bound-state spectrum always has

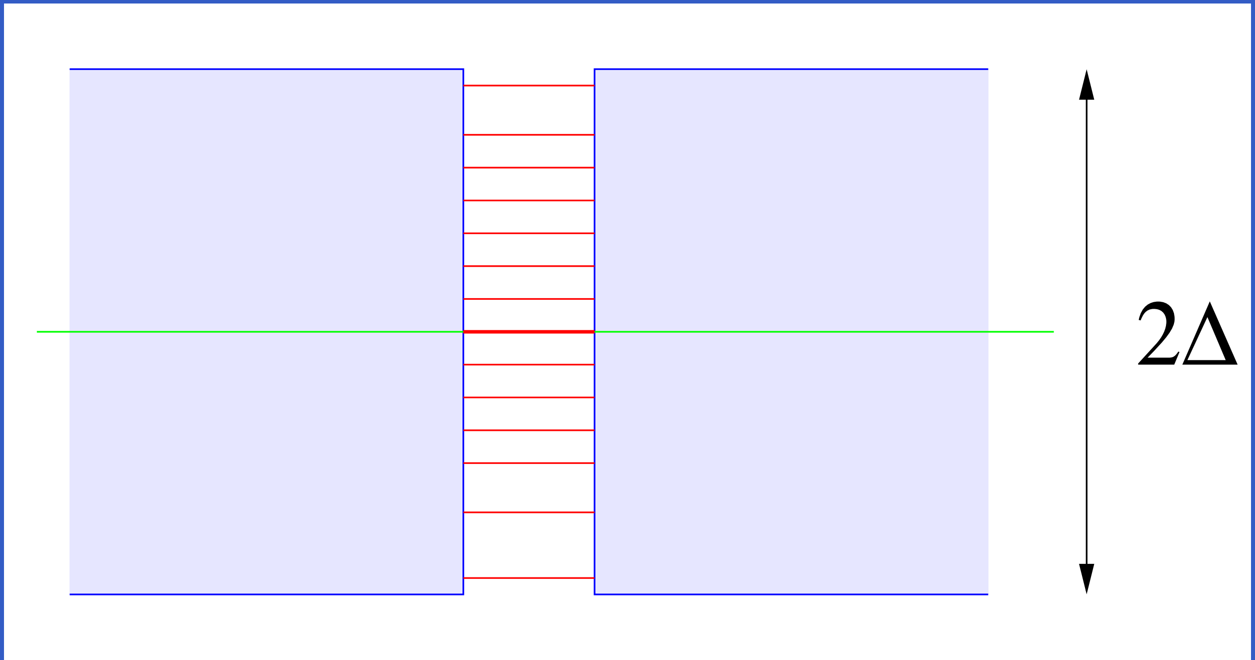
$\boxed{\varepsilon \to -\varepsilon}$ BdG symmetry $\Rightarrow \alpha = 0, \frac{1}{2}$.



S-wave bound states $\alpha = \frac{1}{2} \Rightarrow$ no zero mode

Core spectrum

Vortex-core bound-state spectrum always has
$\boxed{\varepsilon \to -\varepsilon}$ BdG symmetry $\Rightarrow \alpha = 0, \frac{1}{2}$.



P-wave bound states $\alpha = 0 \Rightarrow$ exact zero mode

statistics and fusion algebra

The exact zero-mode core states must come in pairs, and are responsible for:

statistics and fusion algebra

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- Changing the BC's of the edge-mode from antiperiodic (no edge zero mode) to periodic (edge zero mode)

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statistics and fusion algebra

The exact zero-mode core states must come in pairs, and are responsible for:

- Changing the BC's of the edge-mode from antiperiodic (no edge zero mode) to periodic (edge zero mode)
- non-Abelian statistics (Ivanov, Stern *et al.*)
- The Ising-like fusion rules:

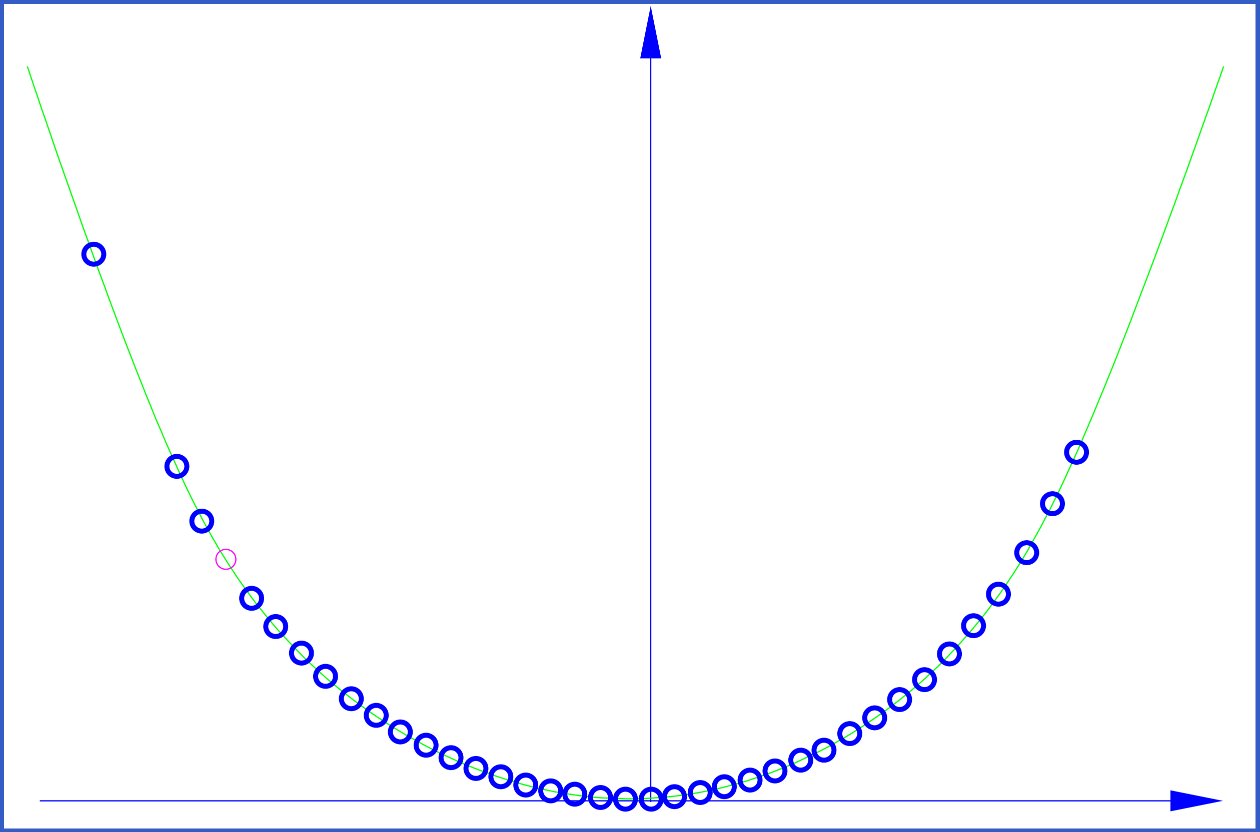
$$
\begin{aligned}
\psi \times \psi &= \mathbb{I}, \\
\sigma \times \sigma &= \mathbb{I} + \psi, \quad \psi \times \sigma = \sigma.
\end{aligned}
$$

statistics and fusion algebra

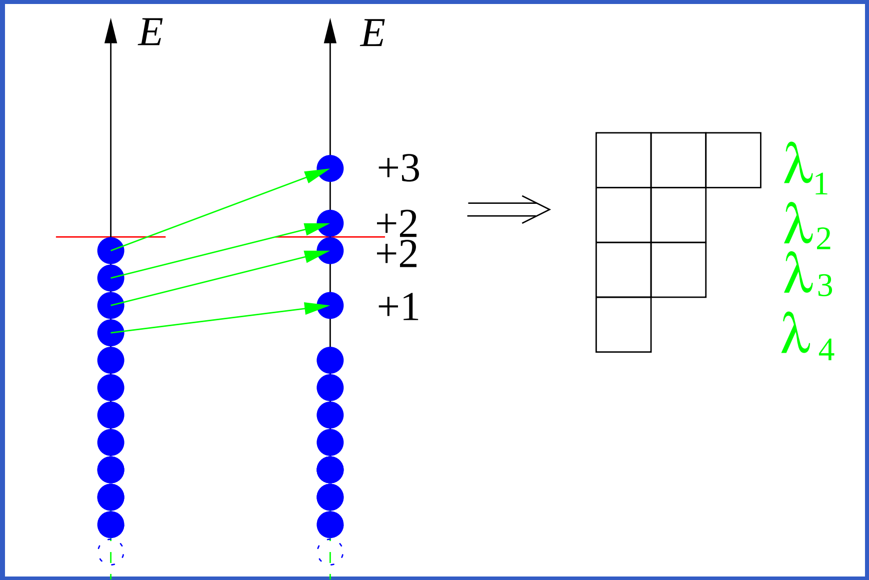
Analogous features will occur in all wave functions in the Pfaffian family, and to their natural k-clustered generalization.

Hall droplet edge states, group theory, and bosonization

integer Hall edge



edge states and Schur functions

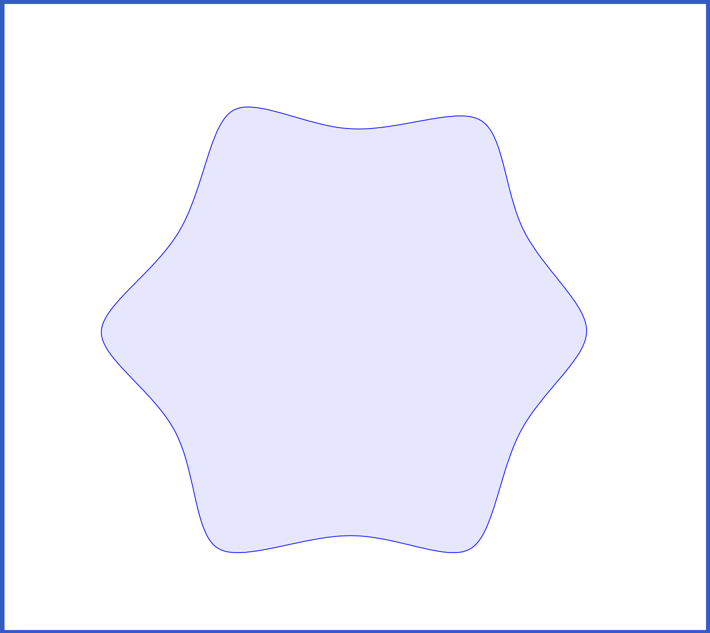


$$\psi_\lambda(z) = \begin{vmatrix} z_1^{\lambda_1+N-1} & z_1^{\lambda_2+N-2} & \dots & 1 \\ z_2^{\lambda_1+N-1} & z_2^{\lambda_2+N-2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ z_N^{\lambda_1+N-1} & z_N^{\lambda_2+N-2} & \dots & 1 \end{vmatrix}$$

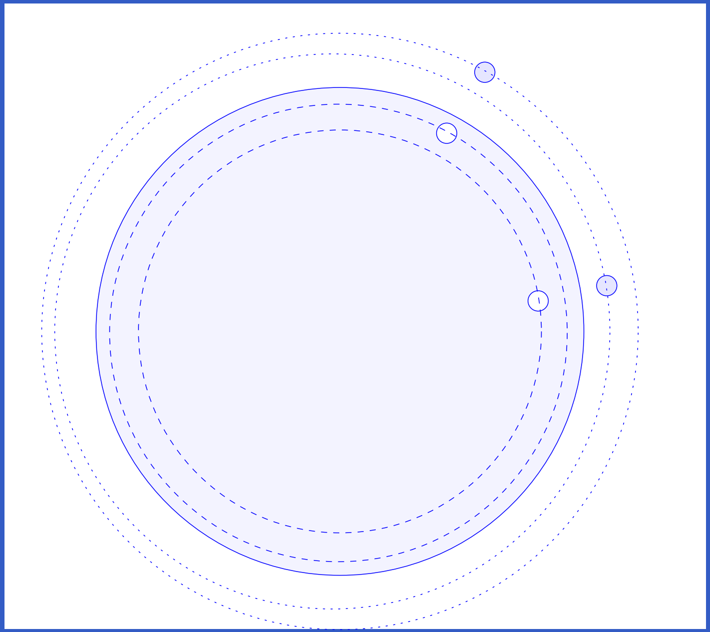
$S_n(z) = z_1^n + z_2^n + \cdots + z_N^n$ (Girard 1629, Isaac Newton 1666)

$\Psi_\lambda(z) = \psi_\lambda(z)/\psi_0(z)$ (Cauchy 1815, Issai Schur 1901)

bosonization identity



$\Leftrightarrow$



$$\boxed{S_1^{l_1} S_2^{l_2} \ldots S_N^{l_N} = \sum_\lambda \chi_\lambda^{(l)} \Psi_\lambda(z)}$$

G. Frobenius 1903

generalized Pfaffians: k-clustering

local to global *via* k-clustering

Read and Rezayi introduced a family of generalized Pfaffian states

$$\psi(z_1, z_2, \ldots, z_N)$$

with the property that $\psi(z) = 0$ if any *k+1* z_i's coincide. We will show that (for bosons at least) these states possess an $\mathrm{su}(2)$ current-algebra symmetry. Therefore the local k-clustering property lead to global topological order and to quasiparticles with non-abelian statistics.

Bose gas representations of $\mathrm{su}(2)_k$

We will study k-clustered symmetric polynomials, and show that they can be identified with states in representations of the affine Lie algebra $\mathrm{su}(2)_k$.

affine Lie algebra

Finite $\mathrm{su}(2)$ algebra

e, f and h such that:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

This is a mathematician's Chevalley basis.
Physicists usually set

$$e \to J_+, \quad f \to J_-, \quad h \to 2J_3$$

affine Lie algebra

Affine $\mathrm{su}(2)_k$ algebra

e_n, f_n, h_n, $\hat{k}$, $\hat{d}$ such that:

$$[e_m, e_n] = [f_m, f_n] = 0$$
$$[e_m, \hat{k}] = [f_m, \hat{k}] = [h_m, \hat{k}] = [\hat{d}, \hat{k}] = 0$$
$$[h_m, e_n] = 2e_{m+n}, \quad [h_n, f_m] = -2f_{m+n},$$
$$[e_m, f_n] = h_{m+n} + m\hat{k}\delta_{n+m,0}, \quad [h_m, h_n] = 2m\hat{k}\delta_{m+n}$$
$$[\hat{d}, e_n] = ne_n, \quad [\hat{d}, f_n] = nf_n, \quad [\hat{d}, h_n] = nh_n.$$

representations and weights

Can have simultaneous eigenstates

$$\begin{aligned}
\hat{k}|m,\lambda,i\rangle &= k|m,\lambda,i\rangle, \\
\hat{d}|m,\lambda,i\rangle &= m|m,\lambda,i\rangle, \\
h_0|m,\lambda,i\rangle &= \lambda|m,\lambda,i\rangle.
\end{aligned}$$

- k, m and λ are integers known as weights.
- positive integer k is the level of representation
- label "i" distinguishes between states with same m and λ.

highest weight

A highest weight state is a $|\mathbf{v}_0\rangle$ such that

$$\begin{aligned} f_n|\mathbf{v}_0\rangle &= h_n|\mathbf{v}_0\rangle = 0 \quad n > 0, \\ e_n|\mathbf{v}_0\rangle &= 0, \quad n \geq 0. \end{aligned}$$

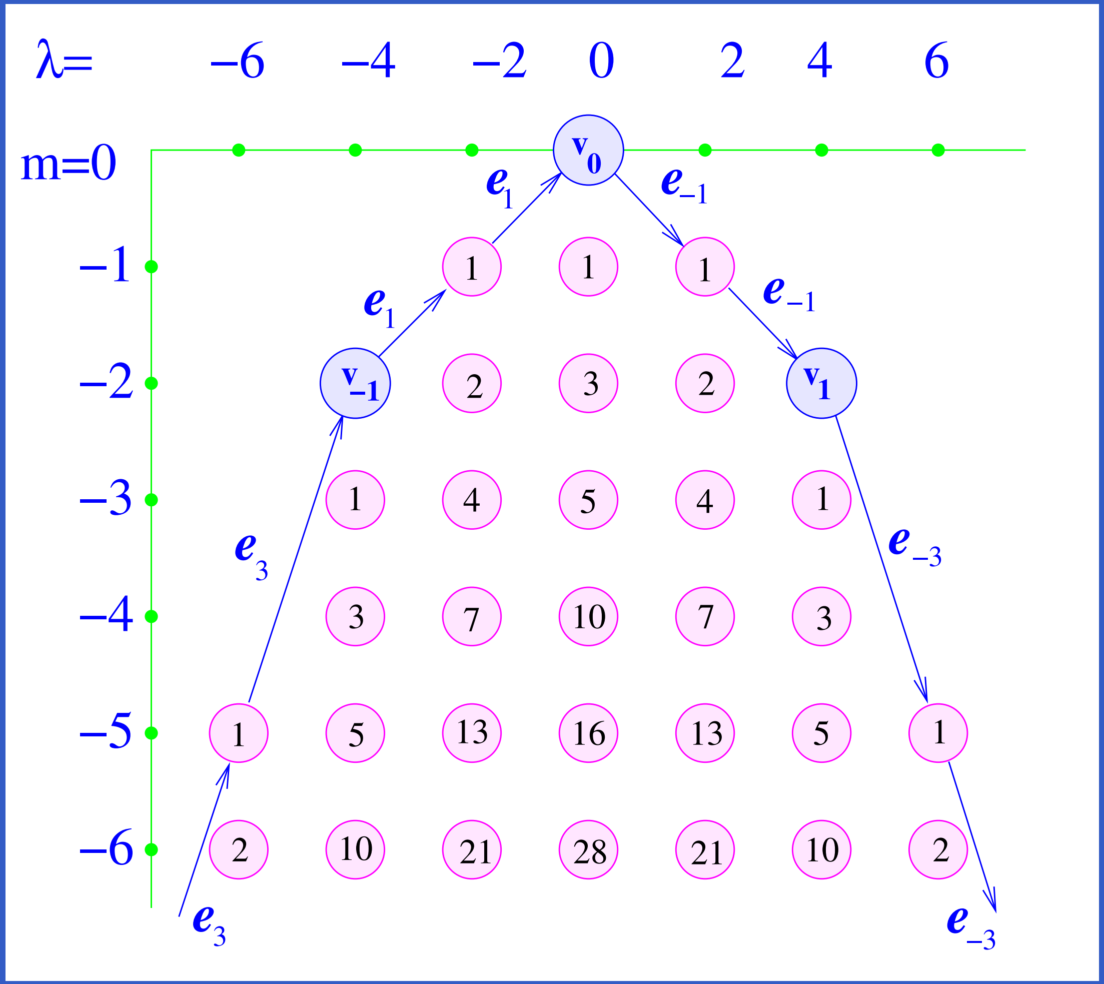
One highest weight with

$$h_0|\mathbf{v}_0\rangle = l|\mathbf{v}_0\rangle, \quad \hat{d}|\mathbf{v}_0\rangle = 0$$

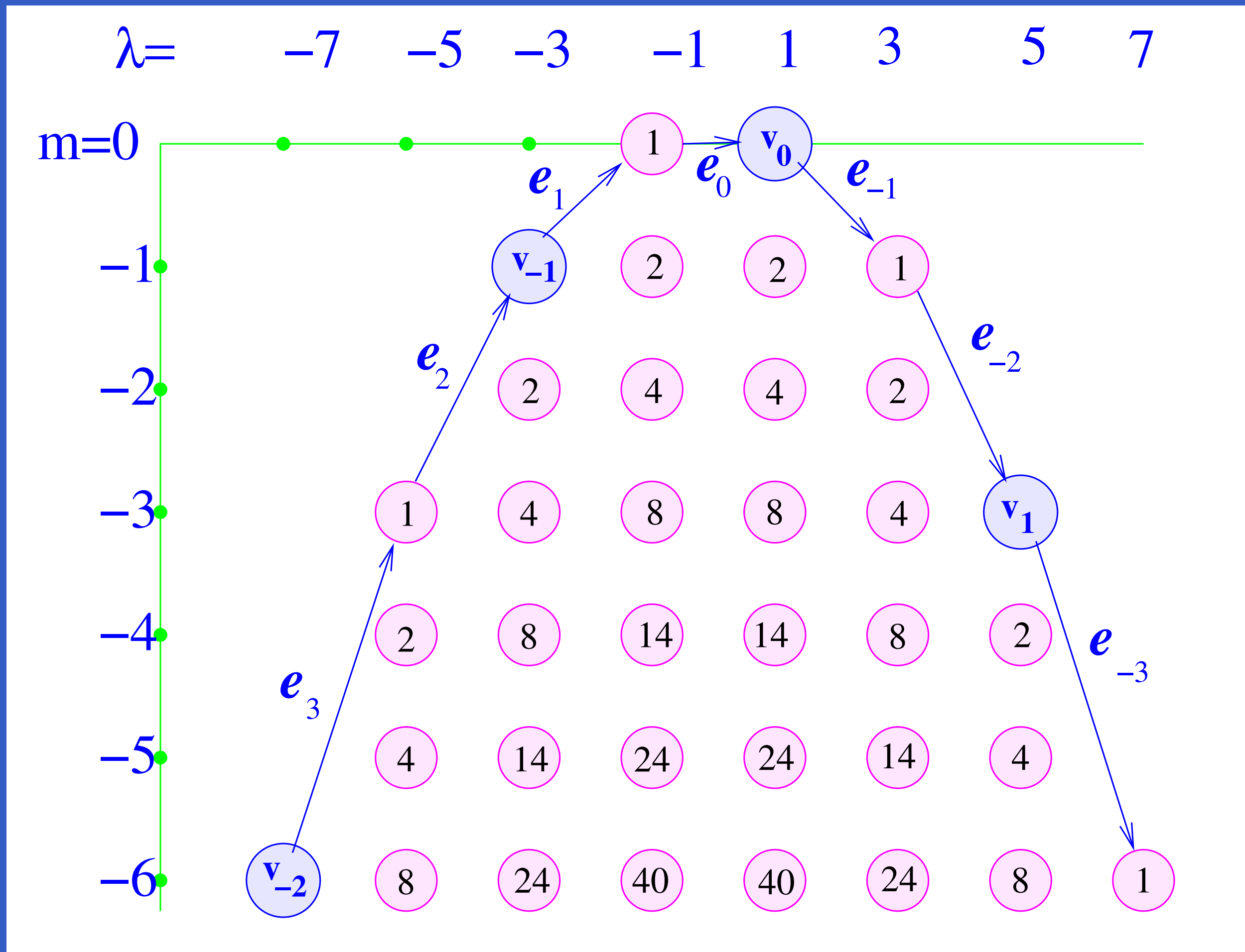
in each representation.

Use k and l to label representations

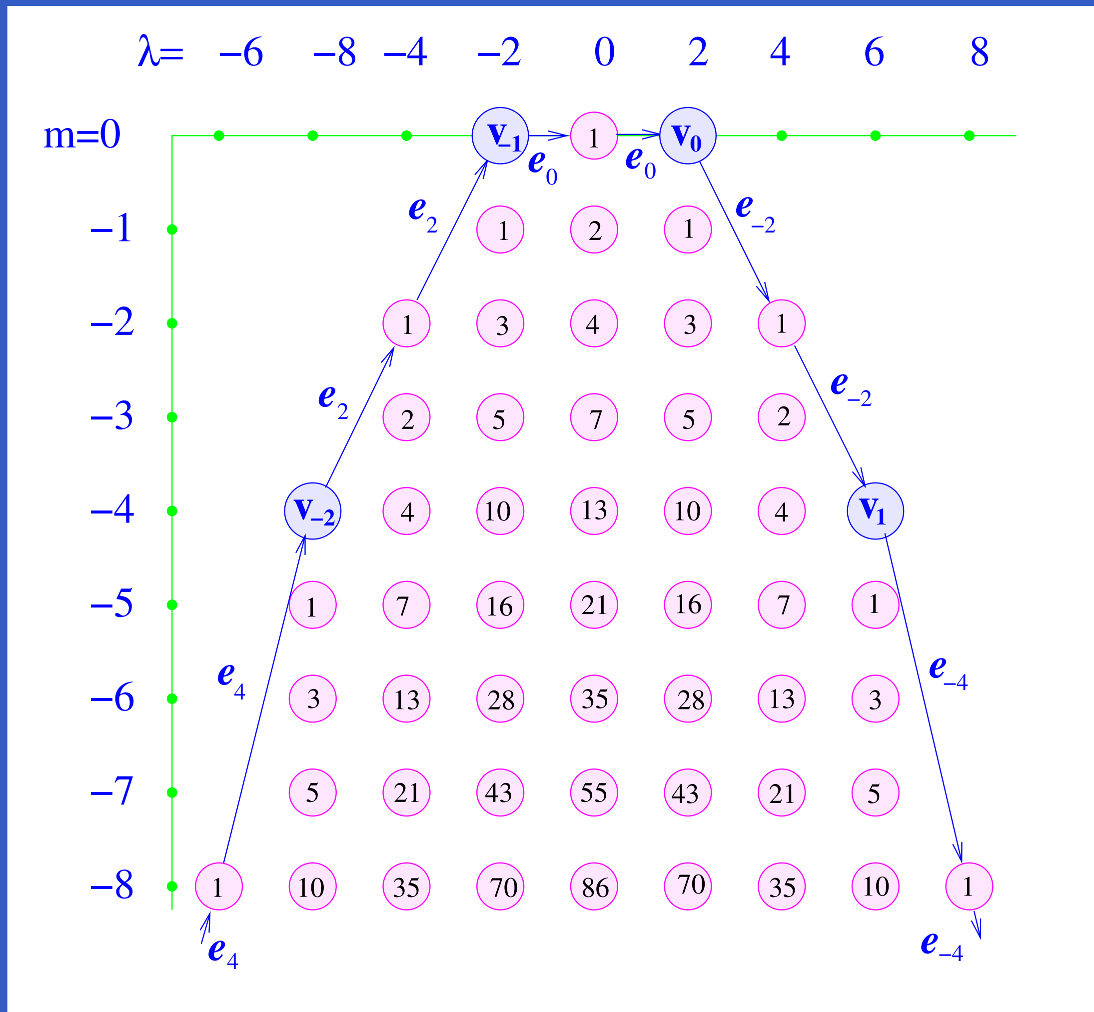
$k = 2, l = 0$


λ=
−6
−4
−2
0
2
4
6
m=0
−1
−2
−3
−4
−5
−6
v0
v−1
v1
e1
e−1
e3
e−3
1 1 1
2 3 2
1 4 5 4 1
3 7 10 7 3
1 5 13 16 13 5 1
2 10 21 28 21 10 2

$k = 2, l = 1$


λ= −7 −5 −3 −1 1 3 5 7
m=0
−1
−2
−3
−4
−5
−6
1 v0
v−1 2 2 1
2 4 4 2
1 4 8 8 4 v1
2 8 14 14 8 2
4 14 24 24 14 4
v−2 8 24 40 40 24 8 1
e0
e1
e2
e3
e−1
e−2
e−3

$k = 2, l = 2$



λ= −6 −8 −4 −2 0 2 4 6 8
m=0
−1
−2
−3
−4
−5
−6
−7
−8
V−1 1 V0
1 2 1
1 3 4 3 1
2 5 7 5 2
V−2 4 10 13 10 4 V1
1 7 16 21 16 7 1
3 13 28 35 28 13 3
5 21 43 55 43 21 5
1 10 35 70 86 70 35 10 1
e0
e0
e2
e2
e−2
e−2
e4
e4
e−4
e−4

making k-clustered wavefunctions

$$F_{|\mathbf{v}\rangle}(z) \equiv \langle \mathbf{v}|e(z_1)e(z_2)\dots e(z_p)|\mathbf{v}_0\rangle$$

$$e(z) = \sum_{n=-\infty}^{\infty} e_n z^{-n-1}$$

making k-clustered wavefunctions

$$F_{|\mathbf{v}\rangle}(z) \equiv \langle \mathbf{v}|e(z_1)e(z_2)\ldots e(z_p)|\mathbf{v}_0\rangle$$

$$e(z) = \sum_{n=-\infty}^{\infty} e_n z^{-n-1}$$

Properties of $e(z)$:

- $[e(z), e(z')] = 0$
- $[e(z)]^{k+1} = 0$
- $(e_{-1})^{k+1-l}|\mathbf{v}_0\rangle = 0$

making k-clustered wavefunctions

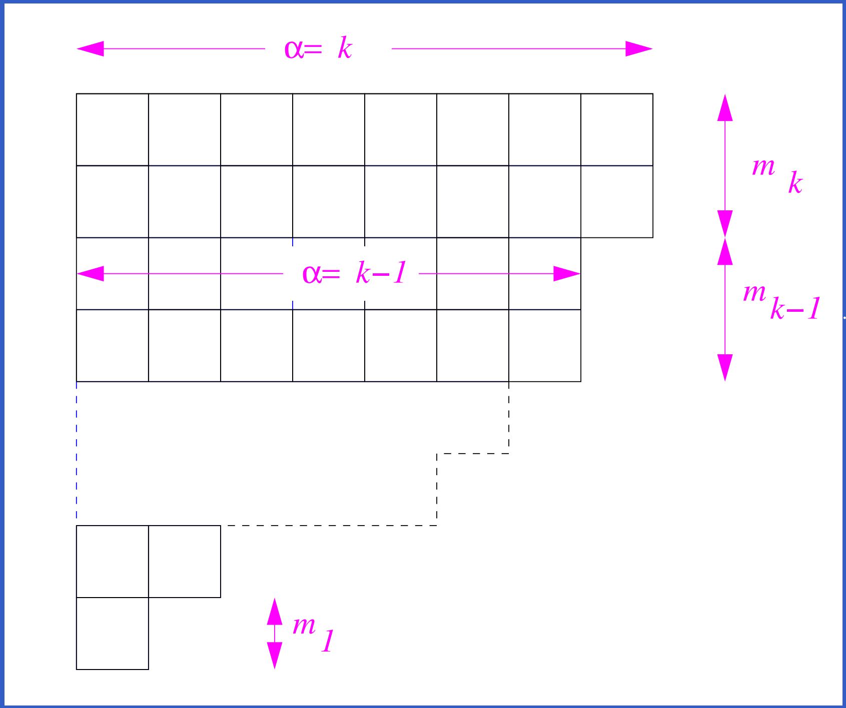
$$\boxed{F_{|\mathbf{v}\rangle}(z) \equiv \langle \mathbf{v}|e(z_1)e(z_2)\ldots e(z_p)|\mathbf{v}_0\rangle}$$

$$e(z) = \sum_{n=-\infty}^{\infty} e_n z^{-n-1}$$

Properties of $F_{|\mathbf{v}\rangle}(z)$:

- F is a symmetric polynomial
- F is zero if any $k+1$ z's coincide
- F is zero if any $k+1-l$ z's become zero

counting polynomials

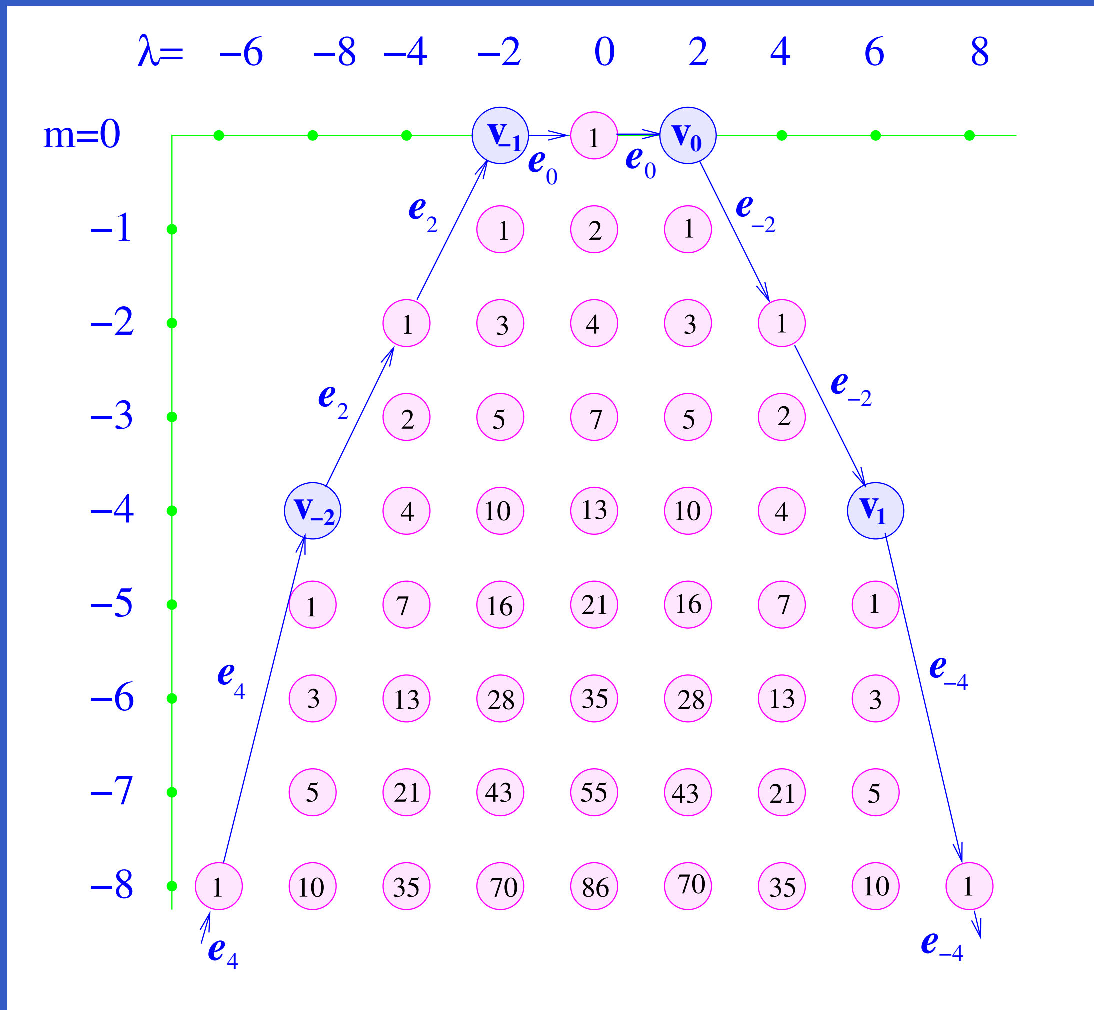


$\mathrm{mult}(p,d)$ = number of F's of degree d in p variables

$$\begin{aligned} M_{ij} &= \min(i,j) \\ \mathbf{m}^t &= (m_1, m_2, \ldots, m_k) \\ (q)_m &= (1-q)\cdots(1-q^m) \\ \mathbf{d}^t &= (0,\ldots,0,1,2,\ldots,l) \end{aligned}$$

$$\sum_d \mathrm{mult}(p,d)\, q^d = q^{-p} \sum_{\text{partitions}} \frac{q^{\mathbf{m}^t \mathbf{M} \mathbf{m} + \mathbf{d}^t \mathbf{m}}}{(q)_{m_1} (q)_{m_2} \ldots (q)_{m_k}}$$

$k = 2, l = 2$, reprise



Filling the Bose sea

Connection with weights:

$$\begin{aligned} -m &= \deg(F_{\mathbf{v}}) + p, \\ \lambda &= l + 2p \end{aligned}$$

Action of affine Weyl group leads to k-clustered ground state

$$|\mathbf{v}_0\rangle = (e_0^l e_1^{k-l})(e_2^l e_3^{k-l}) \ldots (e_{2N-2}^l e_{2N-1}^{k-l})|\mathbf{v}_{-N}\rangle$$

and takes

characters: counting states

$$\mathrm{ch}_{W_{k,l}}(q,x) = \sum_{p=0}^{\infty} x^{2p+l} \left\{ \sum_{N_1+\cdots+N_k=p} \frac{q^{N_1^2+N_2^2+\cdots N_k^2+N_{k-l+1}+N_{k-l+2}+\cdots N_k}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\cdots(q)_{N_k}} \right\}$$

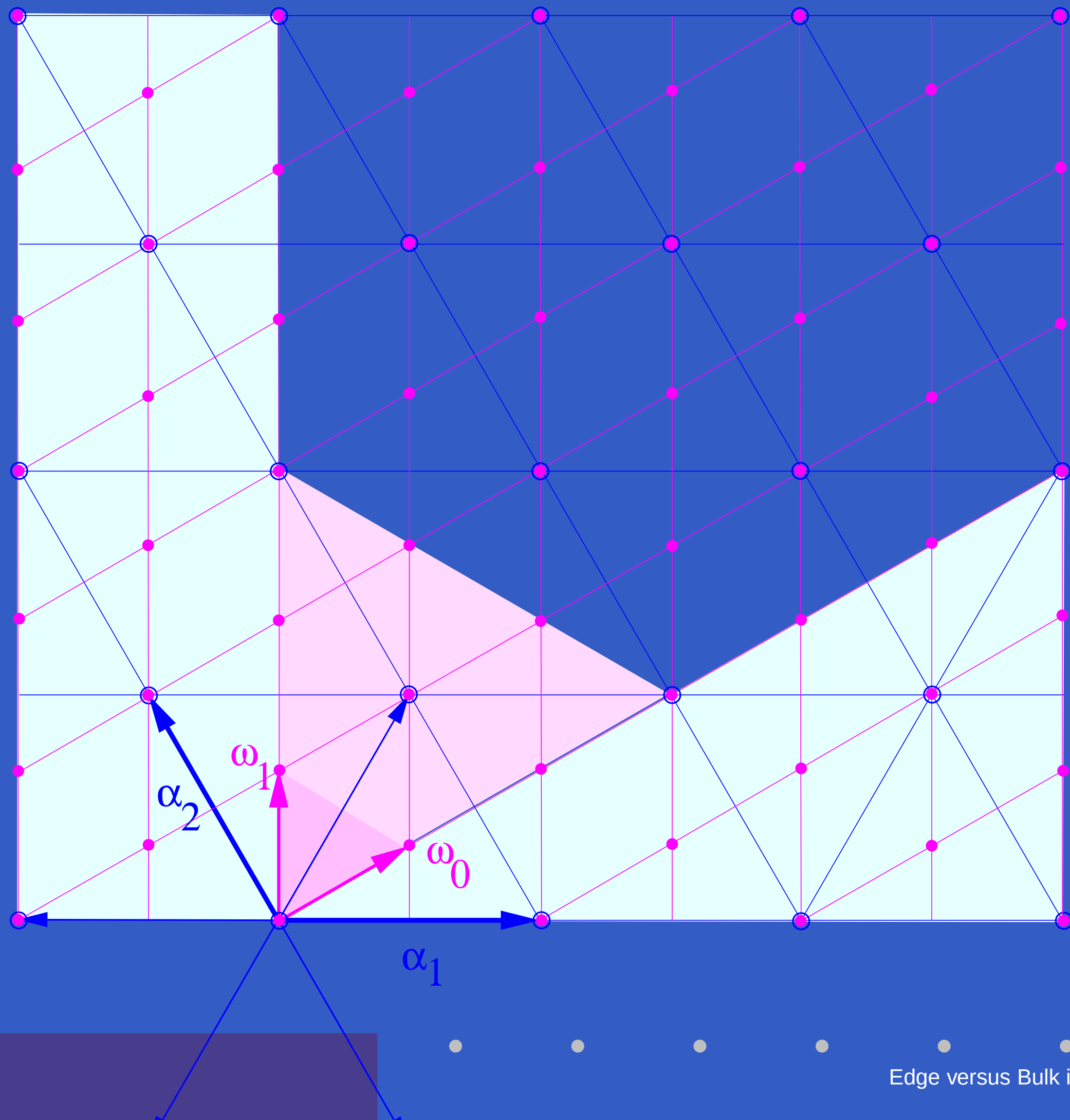
to

$$\frac{1}{(q)_\infty} \sum_{N_1\geq N_2\geq\ldots\geq N_k} \frac{x^{2N_1+2N_2\cdots+2N_k+l} q^{N_1^2+N_2^2+\cdots N_k^2+N_{k-l+1}+N_{k-l+2}+\cdots N_k}}{(q)_{N_1-N_2}(q)_{N_2-N_3}\cdots(q)_{N_{k-1}-N_k}}$$

where

$$\begin{aligned} N_1 &= m_1+m_2+\cdots+m_k, \\ N_2 &= m_2+\cdots+m_k, \\ &\vdots \\ N_k &= m_k \end{aligned}$$

$\mathrm{su}(n)_k \Rightarrow q$-refinement of fusion rules


ω
1
α
2
ω
0
α
1

$\mathrm{su}(n)_k$ and Kostka polynomials

$\mathrm{su}(n)_k$ and Kostka polynomials

function characters

$$\begin{aligned}\mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,1;4} \quad &= \quad \mathrm{ch}\,V_{1,1,1;4} + \frac{1}{q}\mathrm{ch}\,V_{0,0,2;4} + \frac{1}{q}\mathrm{ch}\,V_{2,0,0;4} \\ &\qquad + \left(\frac{1}{q} + \frac{1}{q^2}\right)\mathrm{ch}\,V_{0,1,0;4}\end{aligned}$$

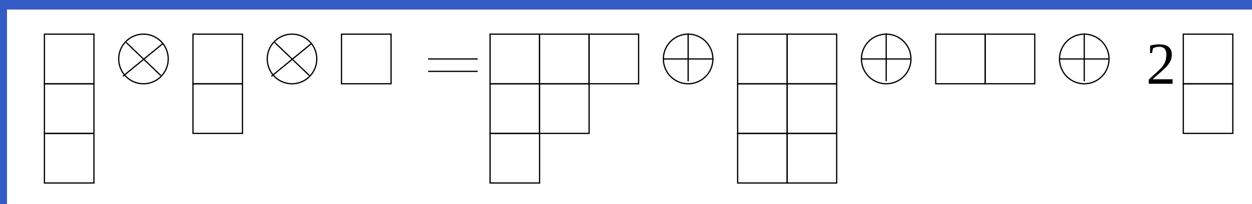
$\mathrm{su}(n)_k$ and Kostka polynomials

function characters

$$\mathrm{ch}\,\mathcal{F}^{\infty}_{1,1,1;4} = \mathrm{ch}\,V_{1,1,1;4} + \frac{1}{q}\mathrm{ch}\,V_{0,0,2;4} + \frac{1}{q}\mathrm{ch}\,V_{2,0,0;4} + \left(\frac{1}{q} + \frac{1}{q^2}\right)\mathrm{ch}\,V_{0,1,0;4}$$

$\mathrm{su}(4)_4$

Littlewood-Richardson



conclusions

- have better understanding of local $\longrightarrow$ global
- have a better understanding of bulk $\longrightarrow$ edge
- obtained q-refinement of fusion rules
- obtained new character formulæ

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still to do

- fermions: orbifolds?
- other groups: Kirrilov-Reshitikin modules?
- general lessons?

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Our work

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