Quantum Control of Qubits

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Quantum Circuit model

$$\text{input } |6\rangle = |1\rangle$$

$$|0\rangle$$

$$(1)$$

$$(0)$$

$$\sum_{k=0}^{7} e^{3\pi i k / 2} |k\rangle$$

output

e.g., Quantum Fourier Transform on 3 qubits

wire = carrier of quantum information

(qubit \{\ket{0}, \ket{1}\}, qudit \{\ket{0}, \ket{1}, \ldots, \ket{d}\}

gate = time evolution of quantum information

$$U = T\left[\exp\left(-iH(t)t_0 / \hbar\right)\right]$$
Universal sets of quantum gates

Theorem: every n-qubit unitary can be decomposed into combinations of 1-qubit and 2-qubit operations (Barenco et al, 1995)

1. single qubit gates

\[ U(\theta, \hat{n}, \phi) = e^{i\phi} (\cos \theta I + i \sin \theta \hat{n} \cdot \sigma) \]

2. Two qubit gates

\[ |00\rangle |01\rangle |10\rangle |11\rangle \]

\[ CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
Quantum simulators

• What can we implement, given a physical system?

  e.g., CPHASE → SWAP

  atoms in OL → interacting quantum spins
  controlled collisions → spin-spin exchange

• What control fields are required? What cost?

• Can we ‘simply’ generate arbitrary quantum operations?
QIP requires ultra-high level of quantum control

- high fidelity quantum operations required for fault-tolerant quantum computation in standard model
- admissible error threshold
  - generic threshold value $\sim 0.0001$ [Aharonov, Gottesman '02]
  - scaling: $\text{#levels recursion} \quad \text{#qubits} \quad \text{#operations}$
    
    \[
    \begin{array}{ccc}
    2 & 50 & 20000 \\
    3 & 350 & 4000000 \\
    \end{array}
    \]

- experimental fidelities $\sim 0.01$
  need $\text{#qubits}$, $\text{#operations}$ 1-3 orders of magnitude larger

- [ Roadmap Goal: recursion level 2 by 2012 ]
quantum control and robustness

• How generate gates and arbitrary quantum operations from Hamiltonians?
• Efficiency – various criteria for optimality
  – Time
  – On/off switching of interactions and external fields
  – Energy input from external fields
  – Minimal decoherence
  – All of the above together, with accurate gates....?
Algebraic approach

- Tunable interactions: 2-qubit gates by Weyl chamber steering
- Non-tunable interactions: algebraic decoupling for 1-qubit gates
- allows some gate optimization

add optimal control

- optimize with respect to cost function
  - time
  - energy
  - decoherence
all 1- and 2-qubit gates: SU(4)

schematic partition of SU(4)

Local gates
SU(2) ⊗ SU(2)

Non-local gates
SU(4) \ SU(2) ⊗ SU(2)

• CNOT
• SWAP
• SWAP^{1/2}

Perfect Entanglers

Cartan decomposition
U = k_1 A k_2

local

non-local

algebra su(4): \sigma_1^i, \sigma_2^i, \sigma_1^i \sigma_2^j, \ldots i, j = x, y, z

Abelian subalgebra: \sigma_1^x \sigma_2^x, \sigma_1^y \sigma_2^y, \sigma_1^z \sigma_2^z

\boxed{\text{Cartan decomposition}}
su(4) algebra = \( \mathfrak{k} \oplus \mathfrak{p} \)

\[ \mathfrak{k} = \text{span} \frac{i}{2} \{ \sigma_x^1, \sigma_y^1, \sigma_z^1, \sigma_x^2, \sigma_y^2, \sigma_z^2 \} \quad \text{local} \]

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ \mathfrak{p} = \text{span} \frac{i}{2} \{ \sigma_x^1 \sigma_x^2, \sigma_x^1 \sigma_y^2, \sigma_x^1 \sigma_z^2, \sigma_y^1 \sigma_x^2, \sigma_y^1 \sigma_y^2, \sigma_y^1 \sigma_z^2, \sigma_z^1 \sigma_x^2, \sigma_z^1 \sigma_y^2, \sigma_z^1 \sigma_z^2 \} \quad \text{non-local} \]

\[ \sigma_x^1 \sigma_x^2 = \begin{pmatrix} 0 & \sigma_x^2 \\ \sigma_x^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \]

Commutation relation, e.g.: \( [\sigma_y^2, \sigma_y^1 \sigma_z^2] \sim -\sigma_y^1 \sigma_x^2 \)
Cartan decomposition: \( \text{su}(4) \) algebra = \( k \oplus p \)

\[
k = \text{span} \frac{i}{2} \{ \sigma_x^1, \sigma_y^1, \sigma_z^1, \sigma_x^2, \sigma_y^2, \sigma_z^2 \}
\]

\[
p = \text{span} \frac{i}{2} \{ \sigma_x^1 \sigma_x^2, \sigma_x^1 \sigma_y^2, \sigma_x^1 \sigma_z^2, \sigma_y^1 \sigma_x^2, \sigma_y^1 \sigma_y^2, \sigma_y^1 \sigma_z^2, \sigma_z^1 \sigma_x^2, \sigma_z^1 \sigma_y^2, \sigma_z^1 \sigma_z^2 \}
\]

Maximal Abelian subalgebra

\[
a = \text{span} \frac{i}{2} \{ \sigma_x^1 \sigma_x^2, \sigma_y^1 \sigma_y^2, \sigma_z^1 \sigma_z^2 \}
\]

Decomposition of a unitary transformation \( U \) in \( \text{SU}(4) \)

\[
U = k_1 A k_2 = k_1 \exp[ \left( c_1 \sigma_x^1 \sigma_x^2 + c_2 \sigma_y^1 \sigma_y^2 + c_3 \sigma_z^1 \sigma_z^2 \right)] k_2
\]

local \quad local \quad non-local
Local gates and local equivalence (~)

\[
\begin{align*}
\text{SWAP} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\text{CNOT} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
\text{C-z} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\end{align*}
\]

\[U_1 \sim U_2 \text{ if } U_1 = k_1 U_2 k_2, \text{ where } k_1 \text{ and } k_2 \text{ are local gates,}\]

\[\text{e.g., CNOT} \sim \text{C-z}\]

\[
\text{CNOT} = \frac{1}{\sqrt{2}} (I \otimes H) \cdot \text{C-z} \cdot \frac{1}{\sqrt{2}} (I \otimes H)
\]

\[\text{SWAP} \sim \text{CNOT}\]

local equivalence can be determined by evaluating 3 invariants

(Makhlin quant-ph/0002045)
Makhlin’s local invariants

Given a two-qubit operation $U$

$$m = (Q^\dagger U Q)^T (Q^\dagger U Q), \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & i & 1 & 0 \\ 0 & i & -1 & 0 \\ 1 & 0 & 0 & -i \end{pmatrix}$$

$$G_1(U) = \frac{\text{tr}^2 m}{16 \det U}$$

$$G_2(U) = \frac{\text{tr}^2 m - \text{tr} m^2}{4 \det U}$$

$G_1$: complex number

$G_2$: real number

3 invariants

<table>
<thead>
<tr>
<th>Local gates</th>
<th>$G_1$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CNOT</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>SWAP</td>
<td>-1</td>
<td>-3</td>
</tr>
<tr>
<td>$\sqrt{\text{SWAP}}$</td>
<td>$i/4$</td>
<td>0</td>
</tr>
</tbody>
</table>

Claim: If $G_1(U_1) = G_1(U_2)$ and $G_2(U_1) = G_2(U_2)$, then $U_1 \sim U_2$

Makhlin, QIP 1, 243 (2002)
Cartan decomposition on $su(4)$

any $U \in SU(4)$ can be decomposed as:

$$U = e^{i\alpha} \cdot k_1 \cdot \exp\left\{ \frac{i}{2} (c_1 \sigma_1 \sigma_2 + c_2 \sigma_y \sigma_y + c_3 \sigma_z \sigma_z) \right\} \cdot k_2$$

$G_1 = \cos^2 c_1 \cos^2 c_2 \cos^2 c_3 - \sin^2 c_1 \sin^2 c_2 \sin^2 c_3 + \frac{i}{4} \sin 2c_1 \sin 2c_2 \sin 2c_3$

$G_2 = 4 \cos^2 c_1 \cos^2 c_2 \cos^2 c_3 - 4 \sin^2 c_1 \sin^2 c_2 \sin^2 c_3 - \cos 2c_1 \cos 2c_2 \cos 2c_3$

$\rightarrow$ invariants are periodic in $c_1, c_2, c_3$
Geometric Theory of Non-local Gates

Cartan decomposition

\[ U = k_1 A k_2 = k_1 \exp\left[ \frac{i}{2} (c_1 \sigma_x^1 \sigma_x^2 + c_2 \sigma_y^1 \sigma_y^2 + c_3 \sigma_z^1 \sigma_z^2) \right] k_2 \]

\( c_1, c_2, c_3 \) periodic \( \rightarrow \) 3-Torus

Local invariants

\[ G_1 = \cos c_1 \cos c_2 \cos c_3 \]
\[ G_2 = \sin c_1 \sin c_2 \sin c_3 \]
\[ G_3 = 2 (\cos^2 c_1 + \cos^2 c_2 + \cos^2 c_3) - 3 \]

\( G_1, G_2 \) and \( G_3 \) are invariant on permuting \( c_1, c_2, \) and \( c_3 \) with/without sign flips

J. Zhang et al., PRA 67, 042313 (2003)

Weyl tetrahedron

\[ \text{one-to-one correspondence} \]

between the points inside the tetrahedron and local equivalence classes (except on base)
Implications of geometric analysis

Tetrahedral representation of local equivalence classes

Applications:
- physical generation of non-local gates, arbitrary 2-qubit operations
- optimally efficient quantum circuits
- characterization of perfect entanglers
Generation of non-local gates as a steering problem in the Weyl tetrahedron

15 dimensional control problem on U(4)

\[ \dot{U} = -iH(v)U, \quad U(0) = I \]

3 dimensional steering problem in Weyl tetrahedron
Weyl tetrahedron trajectory

System dynamics: \( \dot{U} = -iH(v)U \)

For any \( t \), \( U(t) \) determines a point in the tetrahedron via the Makhlin invariants for the non-local equivalence classes, i.e.,

\[ U(t) \rightarrow G_i(t) \rightarrow c_i(t) \]
Weyl tetrahedron trajectory

\[ \dot{U} = -iH(v)U \]

\[ t = 0, \quad U(0) = I \]
Weyl tetrahedron trajectory

\[ \dot{U} = -iH(v)U \]

\[
t = 0, \quad U(0) = I
\]
\[
t = t_1, \quad U(t_1) = U_1
\]
Weyl tetrahedron trajectory

\[ \dot{U} = -iH(v)U \]

\[ t = 0, \quad U(0) = I \]
\[ t = t_1, \quad U(t_1) = U_1 \]
\[ t = t_2, \quad U(t_2) = U_2 \]
Weyl tetrahedron trajectory

\[ \dot{U} = -iH(v)U \]

\[ t = 0, \quad U(0) = I \]
\[ t = t_1, \quad U(t_1) = U_1 \]
\[ t = t_2, \quad U(t_2) = U_2 \]

As time evolves, we can obtain a continuous trajectory in the Weyl tetrahedron.
Consider $H = -\frac{1}{2}(c_x\sigma_x^1\sigma_x^2 + c_y\sigma_y^1\sigma_y^2 + c_z\sigma_z^1\sigma_z^2)$

$U(t) = \exp(-iHt) = \exp\left(\frac{i}{2}(c_xt\sigma_x^1\sigma_x^2 + c_yt\sigma_y^1\sigma_y^2 + c_zt\sigma_z^1\sigma_z^2)\right)$

$[c_x, c_y, c_z]t$
Pure nonlocal Hamiltonian

Consider $H = -\frac{1}{2}(c_x \sigma_x \sigma_x^2 + c_y \sigma_y \sigma_y^2 + c_z \sigma_z \sigma_z^2)$

$U(t) = \exp(-iHt) = \exp\frac{i}{2}(c_x t \sigma_x \sigma_x^2 + c_y t \sigma_y \sigma_y^2 + c_z t \sigma_z \sigma_z^2)$

$k \cdot \exp(-iHt) \cdot k^\dagger$, where $k \subset$ Weyl group

Reflections of $[c_x, c_y, c_z]$ w.r.t. diagonal planes

New directions: $[c_x, -c_y, -c_z], [-c_x, c_z, -c_y], [c_x, c_z, c_y], [-c_x, -c_z, c_y], \ldots$
Steering a Weyl chamber trajectory

Piece two segments together:

\[ k \cdot \exp(-iHt_2) \cdot k^\dagger \cdot \exp(-iHt_1) \]

we can reach anywhere in the plane spanned by \([c_x, c_y, c_z]\) and \([c_x, -c_y, -c_z]\).

\[ [c_x, -c_y, -c_z]t_2 + [c_x, c_y, c_z]t_1 \]

Changing direction twice suffices:

the trajectory defines a quantum circuit

→ **Theorem:** the following circuit can implement any two-qubit gate

\[ k_0 \exp(-iHt_1) k_1 \exp(-iHt_2) k_2 \exp(-iHt_3) k_3 \]
Minimum bound for Controlled-Unitary

For a Controlled-U gate $e^{\gamma \sigma_z \sigma_z}$, minimum applications needed to implement any arbitrary two-qubit gate is $\left\lceil \frac{3\pi}{2\gamma} \right\rceil$.

Quantum Circuits for arbitrary 2-qubit operations

CNOT
3 applications suffice

\[
e^{i \frac{\pi}{2} \sigma_z \sigma_z} e^{i \frac{\pi}{2} \frac{1}{2} \sigma_y} e^{i \frac{\pi}{2} \frac{1}{2} \sigma_x} e^{i \frac{\pi}{2} \sigma_z \sigma_z}
\]

Double-CNOT
3 applications suffice

\[
e^{i \frac{\pi}{2} \sigma_z \sigma_z + \frac{\pi}{2} \frac{1}{2} \sigma_y \sigma_y} e^{i \frac{\pi}{2} \frac{1}{2} \sigma_x \sigma_x} e^{i \frac{\pi}{2} \frac{1}{2} \sigma_z \sigma_z + \frac{\pi}{2} \frac{1}{2} \sigma_y \sigma_y}
\]

J. Zhang, J. Vala, S. Sastry, K.B. Whaley
PRA 69, 042309 (2004)
Two applications of the B gate suffice to implement any arbitrary two-qubit gate: explicit solutions for $\beta_1$ and $\beta_2$ as functions of $c_2$ and $c_3$

on the computational basis B acts as:

$$|m\rangle \otimes |n\rangle \rightarrow e^{\pi i \sigma x (m \oplus n)} |m\rangle \otimes |m \oplus n\rangle$$
Example: SWAP gate and quantum wire

qubit state transfer via a sequence of SWAP gates:

\[ |\Psi\rangle = (|0\rangle_k + c_1 |1\rangle_k)|0\rangle_{k+1} \]

\[
\text{SWAP} \left[ (c_0 |0\rangle_k + c_1 |1\rangle_k)|0\rangle_{k+1} \right] = |0\rangle_k (c_0 |0\rangle_{k+1} + c_1 |1\rangle_{k+1})
\]

only two B gates needed compared to three CNOTs for each step of the wire

Oskin et al., Proc. ISCA 2002
Example: QFT on two qubits

Computer science implementation of two-qubit QFT:

\[
\begin{array}{c}
\text{FIVE CNOT gates in total} \\
H \quad R_2 \quad H
\end{array}
\]

In matrix representation:

\[
\text{QFT}_2 = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & i^2 & i^3 \\
1 & i^2 & i^4 & i^6 \\
1 & i^3 & i^6 & i^9
\end{pmatrix} \sim \left[ \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4} \right]
\]

Three CNOT gates can implement this*:

\[
\begin{array}{c}
\text{B gate needs only two applications:} \\
H \quad \begin{array}{c}
e^{\frac{\pi}{2} i \sigma_y} \\
e^{\frac{\pi}{2} i \sigma_y} \\
e^{\frac{\pi}{4} i (\sigma_x + \sigma_y)} \\
e^{\frac{\pi}{4} i (\sigma_x + \sigma_y)} \\
e^{\frac{\pi}{4} i \sigma_x} \\
e^{\frac{\pi}{4} i \sigma_x}
\end{array} \\
H \quad \begin{array}{c}
e^{\frac{\pi}{2} i \sigma_y} \\
e^{\frac{3\pi}{8} i \sigma_y}
\end{array}
\end{array}
\]

In n-qubit case, the B gate is slightly better than the CNOT gate

* J. Zhang et al., quant-ph/0308167
I. Josephson junction charge-coupled qubits

Y. Makhlin et al., RMP 73, 357 (2001)

charge qubit with tunable coupling: for 2 qubits

\[ H = \sum_{i=1,2} \frac{1}{2} B_z \sigma_z^i - \sum_{i=1,2} \frac{1}{2} E_J^i \sigma_x^i - \frac{E_J^1 E_J^2}{E_L} \sigma_y^1 \sigma_y^2 \]

switch \( E_J^i \) independently \( \rightarrow \) 1 qubit operations

switch \( E_J^1, E_J^2 \) together \( \rightarrow \) 2 qubit operations

\( E_J \sim 100 \, mK, \, E_L \sim 1-100 \, mK \)
interaction between Josephson junction qubits:

\[ H_{\text{int}} = -(\alpha E_L/2) (\sigma_x^1 + \sigma_x^2) + \alpha^2 E_L \sigma_y^1 \sigma_y^2 \quad (\alpha=\frac{E_J}{E_L}) \]

Weyl chamber trajectory

\[ c_1(t) = \alpha^2 E_L t - \omega(\alpha, t), \]
\[ c_2(t) = \alpha^2 E_L t + \omega(\alpha, t), \]
\[ c_3(t) = 0. \]

time optimal parameters for CNOT, \( \alpha = 1.1991, t=2.73 \)

J. Zhang et al., PRA 67, 042313 (2003)
scaled parameters $E_j = \alpha E_L$, $E_L = 1$:
tune $\alpha$ to implement various gates in minimum time

- time optimal solution for CNOT has $\alpha > 1$
- no CNOT solution for $\alpha < 1$
- B gate has solution for all $\alpha$ regimes

- realistic SC circuit, $\alpha \leq 1$
  - no CNOT from single application of $H_{\text{int}}$
  - but B can be implemented directly

Zhang et al. PRL 93, 020502 (2004)
II. inductively coupled SC flux qubits

\[ H_i = \frac{1}{2} \left[ \epsilon_i(t) \sigma_z^{(i)} + \Delta_i \sigma_x^{(i)} \right] \]

inductive coupling:

natural interaction via flux:
e.g., screening flux of qubit 1 changes flux bias $\varepsilon$ of qubit 2

$\rightarrow \sigma_z^{(1)}\sigma_z^{(2)}$ interaction

$$H = \frac{1}{2} \sum_{i=1,2} \left( \varepsilon_i(t)\sigma_z^{(i)} + \Delta_i\sigma_x^{(i)} \right) + K\sigma_z^{(1)}\sigma_z^{(2)}$$

- coupling via mutual inductance: $K$ fixed
- new - magnetic flux $J$ in the outer loop couples the two qubits: $K$ tunable and can be switched off

Entangling operation with variable inductance

Switch magnetic flux in outer loop on/off with bias current, couples two qubits

\[
H = \frac{1}{2} \sum_{i=1,2} \left( \epsilon_i(t) \sigma_z^{(i)} + \Delta_i \sigma_x^{(i)} \right) + K(t) \sigma_z^{(1)} \sigma_z^{(2)}
\]

\[
\epsilon_i(t) = \epsilon_i^{(0)} + A_i \cos(\omega_i t + \phi_i) + \delta\epsilon_i^{xtalk}(t)
\]

2-qubit operations: Weyl trajectory

1-qubit operations: external control fields $\omega_i$ (off resonant)
Implementation of CNOT

\[
\text{CNOT} = \begin{pmatrix}
  k_{22} & U & k_{12} \\
  k_{21} & & k_{11}
\end{pmatrix}
\]

Non-tunable interactions: how generate 1-qubit gates?

**Direct approach:** exact algebraic decoupling of two-qubit Hamiltonian

\[
H = \frac{\omega_1}{2} (\cos \phi_1 \sigma_x^1 + \sin \phi_1 \sigma_y^1) + \frac{\omega_2}{2} (\cos \phi_2 \sigma_x^2 + \sin \phi_2 \sigma_y^2) + \frac{J}{2} \sigma_z^1 \sigma_z^2
\]

\(J\) is the always-on and untunable coupling strength, \(\omega_j\) and \(\phi_j\) the amplitudes and phases of the external control fields.

**Target:** generate any arbitrary single-qubit operation in each qubit

Simplified problem

It is easy to prove that to implement any arbitrary one-qubit operation, we only need to generate an arbitrary local unitary operation:

\[ e^{-i\gamma_1 \sigma_x/2} \otimes e^{-i\gamma_2 \sigma_x/2} \]

from the Hamiltonian:

\[ H_1 = \frac{\omega_1}{2} \sigma_x^1 + \frac{\omega_2}{2} \sigma_x^2 + \frac{J}{2} \sigma_z^1 \sigma_z^2 \]

Now observe that \( i\sigma_x^{1/2}, i\sigma_x^{2/2}, \) and \( i\sigma_z^{1/2}\sigma_z^{2/2} \) generate the following Lie algebra:

\[ \mathfrak{t}_1 = \frac{i}{2} \{ \sigma_x^1, \sigma_x^2, \sigma_z^1 \sigma_y^2, \sigma_y^1 \sigma_z^2, \sigma_y^1 \sigma_y^2, \sigma_z^1 \sigma_z^2 \} \]

It is straightforward to show that \( \mathfrak{t}_1 \) satisfies the same commutation relations as \( \text{so}(4) \), where \( \text{so}(4) \) denotes the Lie algebra formed by all the 4x4 real skew symmetric matrices.
Lie algebra isomorphism

Let

\[ \epsilon_x^1 = \frac{\sigma_x^1 - \sigma_x^2}{4}, \quad \epsilon_y^1 = \frac{\sigma_y^1 \sigma_x^2 + \sigma_z^1 \sigma_z^2}{4}, \quad \epsilon_z^1 = \frac{\sigma_x^1 \sigma_y^2 - \sigma_y^1 \sigma_z^2}{4}, \]
\[ \epsilon_x^2 = \frac{\sigma_x^1 + \sigma_x^2}{4}, \quad \epsilon_y^2 = \frac{\sigma_y^1 \sigma_x^2 - \sigma_z^1 \sigma_z^2}{4}, \quad \epsilon_z^2 = \frac{\sigma_x^1 \sigma_y^2 + \sigma_y^1 \sigma_z^2}{4}, \]

we have the following commutation relations:

\[
\begin{array}{cccccc}
[\cdot, \cdot] & i\epsilon_x^1 & i\epsilon_y^1 & i\epsilon_z^1 & i\epsilon_x^2 & i\epsilon_y^2 & i\epsilon_z^2 \\
\hline
i\epsilon_x^1 & 0 & -i\epsilon_z^1 & i\epsilon_y^1 & 0 & 0 & 0 \\
i\epsilon_y^1 & i\epsilon_z^1 & 0 & -i\epsilon_x^1 & 0 & 0 & 0 \\
i\epsilon_z^1 & -i\epsilon_y^1 & i\epsilon_x^1 & 0 & 0 & 0 & 0 \\
i\epsilon_x^2 & 0 & 0 & 0 & -i\epsilon_z^1 & i\epsilon_y^2 & 0 \\
i\epsilon_y^2 & 0 & 0 & 0 & i\epsilon_z^2 & 0 & -i\epsilon_x^2 \\
i\epsilon_z^2 & 0 & 0 & 0 & -i\epsilon_y^2 & i\epsilon_x^2 & 0 \\
\end{array}
\]

Therefore, \( \mathfrak{g}_1 \) is isomorphic to \( \text{su}(2) \otimes \text{su}(2) \). This isomorphism allows simplification for the generation of single-qubit operation, because it provides an algebraic way to decouple the entangling Hamiltonian into two unentangled single-qubit Hamiltonians.
Two sub-problems

We can now rewrite the Hamiltonian as

\[ H_1 = (\omega_1 - \omega_2)\epsilon_x^1 + J\epsilon_y^1 + (\omega_1 + \omega_2)\epsilon_x^2 - J\epsilon_y^2 \]

and the target operation as

\[ k_2 = e^{-i\gamma_1\sigma_x/2} \otimes e^{-i\gamma_2\sigma_x/2} = e^{-i((\gamma_1-\gamma_2)\epsilon_x^1 + (\gamma_1+\gamma_2)\epsilon_x^2)} \]

Now the original problem of generating \( e^{-i\gamma_1\sigma_x/2} \otimes e^{-i\gamma_2\sigma_x/2} \) from

\[ H_1 = \frac{\omega_1}{2}\sigma_x^1 + \frac{\omega_2}{2}\sigma_x^2 + \frac{J}{2}\sigma_z^1\sigma_z^2 \]

becomes generating \( e^{-i((\gamma_1-\gamma_2)\sigma_x^1/2 + (\gamma_1+\gamma_2)\sigma_x^2/2)} \) from

\[ \frac{\omega_1 - \omega_2}{2}\sigma_x^1 + \frac{J}{2}\sigma_y^1 + \frac{\omega_1 + \omega_2}{2}\sigma_x^2 - \frac{J}{2}\sigma_y^2 \]

We only need to implement the following two one-qubit operations:

1. Generate \( e^{-i(\gamma_1-\gamma_2)\sigma_x^1/2} \) from the Hamiltonian \( (\omega_1 - \omega_2)\sigma_x^1/2 + J\sigma_y^1/2 \); and
2. Generate \( e^{-i(\gamma_1+\gamma_2)\sigma_x^2/2} \) from the Hamiltonian \( (\omega_1 + \omega_2)\sigma_x^2/2 - J\sigma_y^2/2 \).
One-qubit sub-operations: optimal control

Consider a general one-qubit system:

\[ i\dot{U} = \left( \frac{\omega(t)}{2} \sigma_x + \frac{J}{2} \sigma_y \right) U, \quad U(0) = I \]

Solution of \(\omega(t)\) to exactly implement a target one-qubit operation \(U_T = e^{-i\gamma/2\sigma_x}\) is possible with simultaneous minimization of a cost function

\[ J = \int_0^T L(\omega(t)) \, dt \]

Time optimal

\[ J = \int_0^T 1 \, dt \]

Energy optimal

\[ J = \frac{1}{2} \int_0^T \omega^2(t) \, dt \]

Also have analytic approximate implementation with

\[ \omega = \frac{\gamma J}{n\pi} \cos(Jt), \quad T = \frac{2n\pi}{J} \]
Simple example:

Let $J=200$ Hz in the Hamiltonian and $e^{-i\pi/4\sigma_x^1}$ be the desired target 1-qubit operation. Choosing $\phi_1 = \phi_2 = 0$ and $n=1$, we obtain an approximate solution $\omega_1 = 100 \cos 200t, \quad \omega_2 = 0$. The corresponding pulse time is $T=31.4$ ms.

Numerical optimization via the maximization of the fidelity leads to the improved solution parameters $\omega_1 = 98.062 \cos 196.900t, \quad \omega_2 = 0$,

with corresponding pulse time $T=31.911$ ms and fidelity error $4.104 \times 10^{-11}$.

Control functions that generate $e^{-i\pi/4\sigma_x^1}$. (A) Dashed line: approximate control; solid line: fidelity optimized control; (B) the difference between fidelity optimized control and minimum energy control.
Optimal control of 1-qubit operations subject to random telegraph noise

\[ H = a(t)\sigma_x + \eta(t)\sigma_z \]

Bounded control: \( a \leq a_{\text{max}} \)

Random telegraph noise

Solution for given noise correlation time via numerical optimization of unitary quantum trajectory formulation of fidelity

Summary of Part I

• Geometric approach to non-local gates
  - steering approach to generation of 2-qubit gates
  - analytic construction of quantum circuits

• How implement an arbitrary 2-qubit operation?
  - starting from given Hamiltonian
    → steering in Weyl chamber (tetrahedron)
  - starting from given gate, e.g., CNOT
    → gate B is optimal, only 2 applications

• Constraints
  - physical feasibility of H
  - minimal switchings, time optimization

• Algebraic decoupling for 1-qubit operations when non-tunable interactions present
  - optimization with respect to general cost function
    - time, energy, decoherence …

→ broad route to optimal feasible control of coupled qubits …
Part II

• Geometric approach to non-local gates
  - steering approach to generation of 2-qubit gates
  - analytic construction of quantum circuits
• How implement an arbitrary 2-qubit operation?
  - starting from given Hamiltonian
    → steering in Weyl chamber (tetrahedron)
  - starting from given gate, e.g., CNOT
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• Constraints
  - physical feasibility of H
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Optimal control of 1-qubit operations subject to random telegraph noise

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\[ a \leq a_{\text{max}} \]

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Random telegraph noise

Average sojourn $\tau_c$

$\eta(t)$ vs. time

Lorentzian spectrum

$\Lambda_{\text{cut-off}} = 1/\tau_c$
Random telegraph noise (RTN)

- Described by the correlation time $\tau_c$ and the noise strength $\Delta$
- The noise amplitude jumps between values $\Delta$ and $-\Delta$
- Probability of no jumps in time $t$ is $p_0(t) = e^{-t/\tau_c}$
- Jump time instants and the noise amplitude are given by

\[
t_i = \sum_{j=1}^{i} -\tau_c \ln(p_j)
\]

\[
\eta(t) = (-1) \sum_i \Theta(t-t_i) \eta(0)
\]
Trapping center noise on charge qubits

- Effective only if trap energy level is close to Fermi level;
- High temperature, $k_B T \gg \gamma$: Lorentzian spectrum, semiclassical RTN;
- Low temperature, $k_B T \ll \gamma$: QUANTUM REGIME f-noise, Ohmic

System dynamics

- Let $k$ index the sample paths of RTN
- The dynamics of the system density matrix is given by an average over all different noise samples as

$$\rho(t) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} U_k \rho_0 U_k^\dagger$$

$$U_k = \mathcal{T} e^{-i \int_0^t d\tau [a(\tau)\sigma_x + \eta_k(\tau)\sigma_z]}/\hbar$$
Example operations

**Bit flip** $|0\rangle \rightarrow |1\rangle$

$\rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \rho_t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

**Fidelity**

$\phi(\rho_t, \rho_0) = \text{tr}\{\rho_t^\dagger \rho(T)\}$

**NOT gate** = target gate $U_t$

**Fidelity**

$\Phi(U_t) = \frac{1}{4\pi} \int_{c_x^2 + c_y^2 + c_z^2 = 1} d\Omega \, \phi(U_t \rho_0 U_t^\dagger, \rho_0)$

$\rho_0 = (I + c_x \sigma_x + c_y \sigma_y + c_z \sigma_z)/2$
Gradient ascent pulse engineering (GRAPE)

- Optimizes the fidelity with respect to the control pulse by a gradient method
- Solution not unique
- We used a constant control pulse as an initial condition
- Convergence of fidelity much faster than convergence of the pulse shape
- Compare with standard pulse sequences for correction of systematic (static) error, CORPSE and SCORPSE
Composite pulse sequences

- **π-pulse:** \[ a_{\pi}(t) = a_{\text{max}}, \quad \text{for} \quad t \in [0, \pi\hbar/a_{\text{max}}] \]

- Compensation of off-resonance with a pulse sequence

\[
\text{CORPSE: } a_C(t) = \begin{cases} 
  a_{\text{max}}, & \text{for } 0 < t' < \pi/3 \\
  -a_{\text{max}}, & \text{for } \pi/3 \leq t' \leq 2\pi \\
  a_{\text{max}}, & \text{for } 2\pi < t' < 13\pi/3 
\end{cases}
\]

\[ t' = a_{\text{max}} t / \hbar \]

- **short CORPSE:** \[ a_{SC}(t) = \begin{cases} 
  -a_{\text{max}}, & \text{for } 0 < t' < \pi/3 \\
  a_{\text{max}}, & \text{for } \pi/3 \leq t' \leq 2\pi \\
  -a_{\text{max}}, & \text{for } 2\pi < t' < 7\pi/3 
\end{cases} \]

Best short pulse sequences correcting for systematic static error
CORPSE and short CORPSE
Composite pulses and numerical optimization


\[ U_\pi = e^{i\left(\frac{\pi}{2}\sigma_x + \eta\sigma_z\right)} \Rightarrow \text{Fidelity} \approx 1 - \left(\frac{\eta}{a_{\text{max}}}\right)^2 \]

\[ U_{\text{CORPSE}} = e^{i\left(\frac{7\pi}{6}\sigma_x + \eta\sigma_z\right)} e^{i\left(-\frac{5\pi}{6}\sigma_x + \eta\sigma_z\right)} e^{i\left(\frac{\pi}{6}\sigma_x + \eta\sigma_z\right)} \Rightarrow \text{Fidelity} \approx 1 - 0.065\left(\frac{\eta}{a_{\text{max}}}\right)^4 \]

\[ U_{\text{Short CORPSE}} = e^{i\left(-\frac{\pi}{6}\sigma_x + \eta\sigma_z\right)} e^{i\left(\frac{5\pi}{6}\sigma_x + \eta\sigma_z\right)} e^{i\left(-\frac{\pi}{6}\sigma_x + \eta\sigma_z\right)} \Rightarrow \text{Fidelity} \approx 1 - 2.7\left(\frac{\eta}{a_{\text{max}}}\right)^4 \]

Large number of composite pulses \(\Rightarrow\) Numerical optimization (Gradient Ascent Pulse Engineering)

Fidelity vs noise correlation time for the state transformation $|0\rangle \to |1\rangle$
Fidelity vs noise correlation time for NOT gate

![Graph showing the relationship between Fidelity and noise correlation time for different quantum gate methods. The graph compares GRAPE, short CORPSE, and \( \pi \)-pulse techniques.](image-url)
Optimized operation times

Bit flip $|0\rangle \rightarrow |1\rangle$

\[ \Delta = 0.25 \ a_{\text{max}} \quad \tau_c/(\hbar/a_{\text{max}}) = 5 \]

NOT gate

\[ \Delta = 0.125 \ a_{\text{max}} \quad \tau_c/(\hbar/a_{\text{max}}) = 30 \]
OVERALL SUMMARY:

• High fidelity quantum gate operations from Hamiltonians
• Weyl chamber steering for 2-qubit (non-local) gates
• Algebraic decoupling for 1-qubit (local) gates in presence of untunable interactions
• Efficiency issues – implementation specific
• Decoherence suppression using bounded controls for broad range of noise correlation times
• current/future: combined methodologies for optimal feasible quantum control tailored to specific qubit systems