Pre-thermalization and thermalization in models with weak integrability breaking

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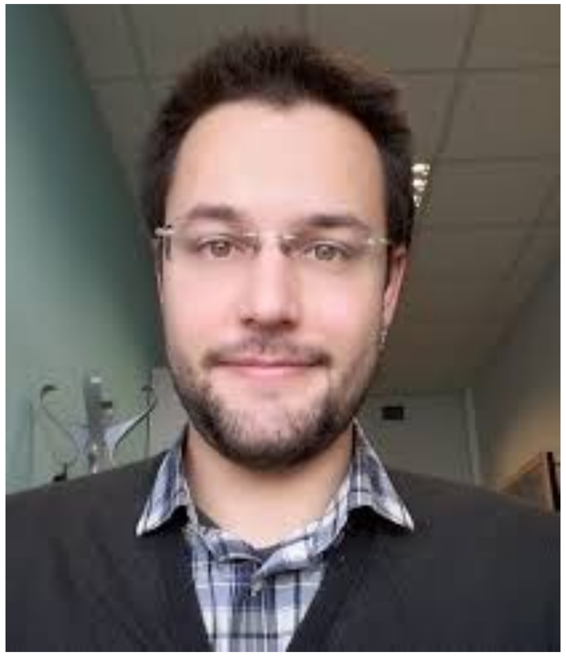
Joint works with



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Outline

A. Quantum Quenches in isolated systems.

B. Local relaxation in integrable/non-integrable models.

C. “Pre-thermalization plateaux”.

D. Beyond Pre-thermaliztion.

Quantum Quenches in isolated many-particle systems

A. Consider a quantum many-particle system with Hamiltonian H (no randomness, translationally invariant, short ranged)

B. Prepare the system in density matrix $\rho(0)$ that does **not** correspond to superposition of small # of eigenstates of H, fulfils cluster decomposition & is translationally invariant.

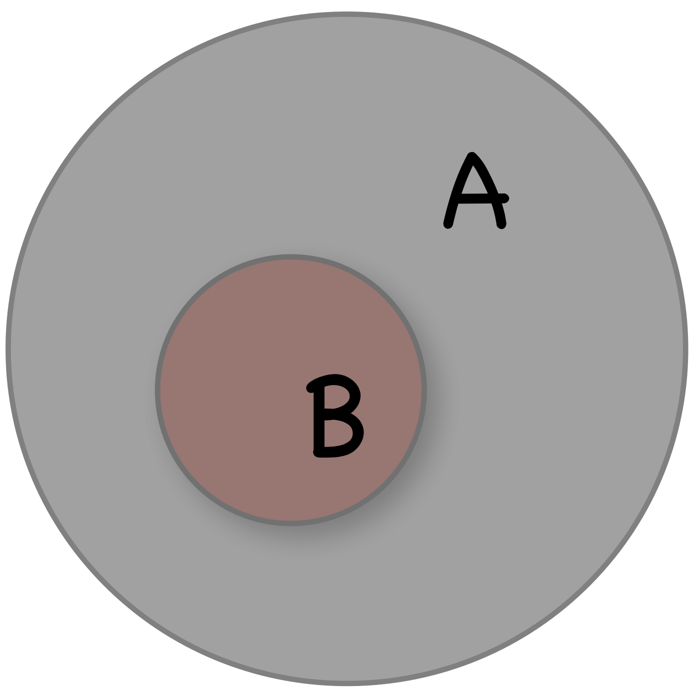
C. Time evolution $\rho(t) = \exp(-iHt)\ \rho(0)\ \exp(iHt)$

D. Study time evolution of **local observables** $\mathrm{Tr}[\rho(t)\ \mathcal{O}(x)]$ (in the **thermodynamic limit).**

Local Relaxation

Given that we are considering an **isolated** system, in what sense does the system relax at late times ?

It relaxes locally (in space):



- Entire System: $A \cup B$
- Take A **infinite**, B finite
- Ask questions only about B:

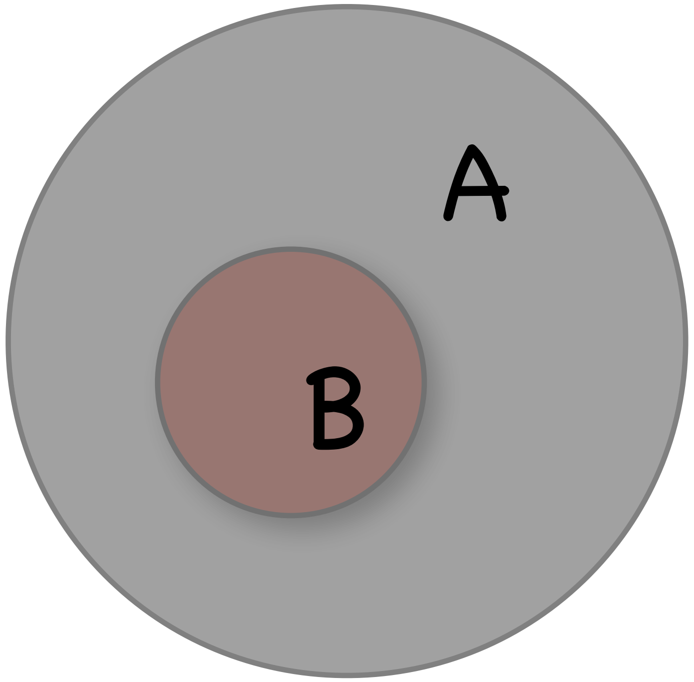
Expectation values of **local** ops:

$$\langle \Psi(t) | O_B(x) | \Psi(t) \rangle$$

Physical Picture: A acts like a bath for B.

Subsystems are described by reduced density matrices:

Reduced density matrix: $\rho_B(t)=\mathrm{tr}_A\, \rho(t)$



The system relaxes locally if $\lim_{t\to\infty} \rho_B(t)= \rho_B(\infty)$ exists for any finite subsystem B in the thermodynamic limit.

Nonequilibrium Steady State

A density matrix ρ^{SS} describes the steady state of a system $A \cup B$ that relaxes locally, if $Tr_A[\rho^{SS}] = Tr_A[\rho(\infty)]$ for any finite subsystem B in the thermodynamic limit $|A| \rightarrow \infty$.

N.B. ρ^{SS} is not unique.

What is ρ^{SS} ?

Conservation laws

Isolated system $\rightarrow$ energy conserved $\quad [H, e^{-iHt}] = 0$

No other conserved quantities $\rightarrow$ system **thermalizes**

Deutsch '91, Srednicki '94,....

Define a Gibbs Ensemble: $$\rho_{\mathrm{GE}} = \frac{1}{Z_{\mathrm{GE}}} e^{-\beta_{\mathrm{eff}} H}$$

fix effective temperature: $$e = \lim_{L\to\infty} \frac{1}{L} \mathrm{Tr}\left[\rho(0) H\right] = \lim_{L\to\infty} \frac{1}{L} \mathrm{Tr}\left[\rho_{\mathrm{GE}} H\right]$$

Thermalization: $\rho^{SS} = \rho_{GE}$

Further conserved quantities: system does not **thermalize**

$$[I_\alpha, H] = 0 \Rightarrow \mathrm{Tr}\left[\rho(t) I_\alpha\right] = \mathrm{const}$$

Define a Generalized Gibbs Ensemble:

Rigol et. al. '07

$$\rho_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} e^{-\sum_\alpha \lambda_\alpha I_\alpha}$$

fix Lagrange multipliers:

$$e_\alpha = \lim_{L\to\infty} \frac{1}{L} \mathrm{Tr}\left[\rho(0) I_\alpha\right] = \lim_{L\to\infty} \frac{1}{L} \mathrm{Tr}\left[\rho_{\mathrm{GGE}} I_\alpha\right]$$

Non-thermal Steady State $\rho^{SS} = \rho_{GGE}$

Barthel&Schollwöck '08
Cramer et al '08
...

Quantum Integrable Models

These have extensive numbers of **local** (in space) integrals of motion $[I_m, I_n]=[I_m, H(h)]=0$.

Example: transverse-field Ising chain

Grady '82, Prosen '98

define operators $$S^{\alpha\beta}_{j,j+\ell} = \sigma^\alpha_j \left[\sigma^z_{j+1} \dots \sigma^z_{j+\ell-1}\right] \sigma^\beta_{j+\ell}$$

$$I_0^+ = H(h) = -J\sum_j S^{xx}_{j,j+1} + h\sum_j \sigma^z_j$$

$$I_1^+ = -J\sum_j \left(S^{xx}_{j,j+2} - \sigma^z_j\right) - h\sum_j \left(S^{xx}_{j,j+1} + S^{yy}_{j,j+1}\right)$$

$$I^+_{n\geq 2} = -J\sum_j \left(S^{xx}_{j,j+n+1} + S^{yy}_{j,j+n-1}\right) - h\sum_j \left(S^{xx}_{j,j+n} + S^{yy}_{j,j+n}\right)$$

$$I_n^- = -J\sum_j \left(S^{xy}_{j,j+n+1} - S^{yx}_{j,j+n+1}\right)$$

I_n involve spins on n+2 neighbouring sites

Integrable vs non-integrable models

Non-equilibrium evolution of quantum integrable models
is **markedly different** from that of non-integrable models:

- Integrable models relax locally to GGEs
- Non-integrable models thermalize.

What happens if we add a small perturbation to a quantum integrable model?

Adding small perturbations to integrable models

Steady state will be thermal, but there could be a "proximity effect" at intermediate times:

"remnants of integrability"?

thermalization?

0

T

time

Manmana et al '07
Moeckel&Kehrein '08
Kollar et al '11
Marcuzzi et al '13
Brandino et al '13
Essler et al '14
Nessi et al '14

Kollath et al '07
Rigol& Santos '09, '10

What are these "remnants of integrability"?

Essler, Kehrein, Manmana &Robinson '14

(1) integrable model = free theory for simplicity

(2) Two tuneable parameters:

1 for quench in integrable model, 1 to break integrability;

$$H(\delta, U) = -J\sum_{l=1}^{L}\left[1+\delta(-1)^l\right]\left(c_l^\dagger c_{l+1} + \text{h.c.}\right) + U\sum_{l=1}^{L} n_l n_{l+1}$$

Non-interacting (integrable) theory:

$$H(\delta, 0) = \sum_{0<k<\pi}\sum_{\alpha=\pm}\epsilon(k,\delta)a_\alpha^\dagger(k)a_\alpha(k)$$

2 bands of free fermions

$$\{a_\alpha^\dagger(k), a_\beta(p)\} = \delta_{\alpha,\beta}\delta_{p,k}$$

$$c_l = \frac{1}{\sqrt{L}}\sum_{k>0}\sum_{\alpha=\pm}\gamma_\alpha(l,k|\delta)a_\alpha(k)\ .$$

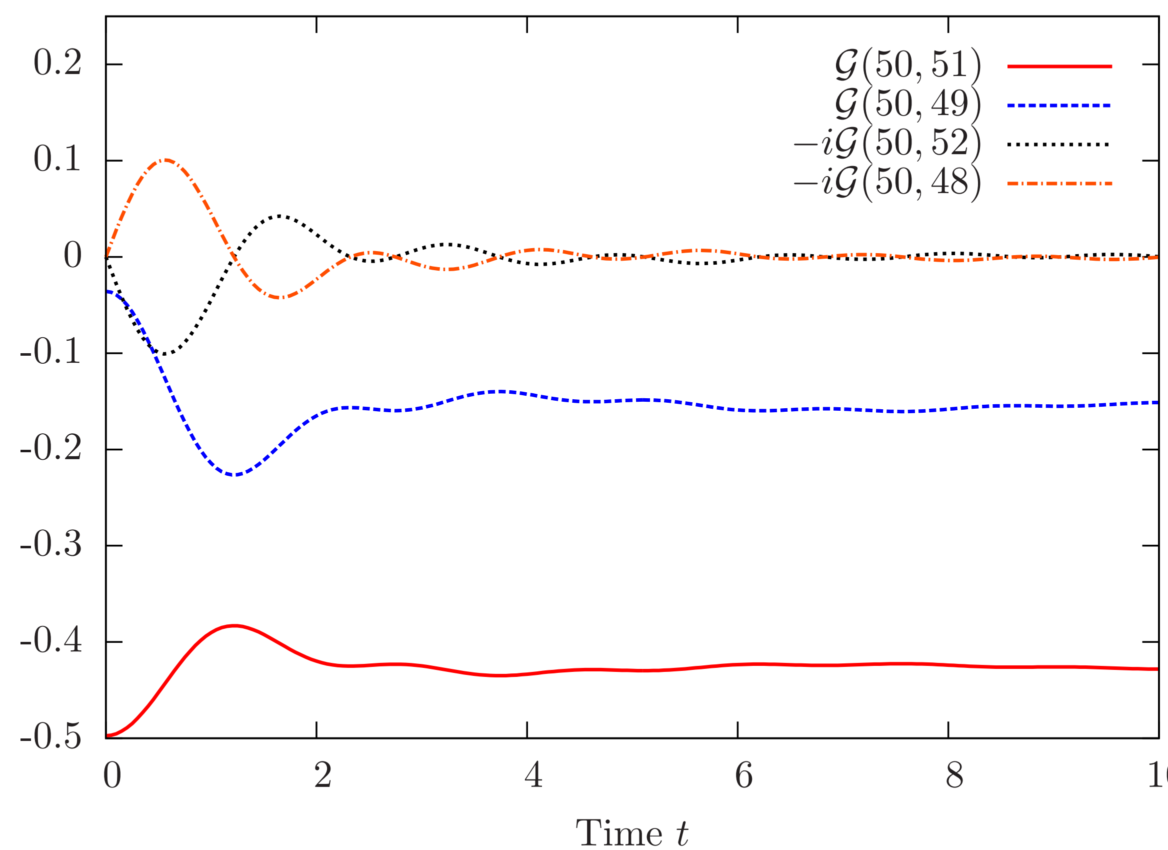
Quenches in the free (integrable) theory

Prepare the system in the ground state $|\Psi_0\rangle$ of $H(\delta_i, 0)$

At t=0 quench $\delta_i \longrightarrow \delta_f$

Single particle Green's function $G(j, \ell, t) = \langle \Psi_0(t)| c_j^\dagger c_\ell |\Psi_0(t)\rangle$

$\delta_i = 0.75$
$\delta_f = 0.25$



$$\lim_{t\to\infty} G_0(j, \ell, t) \sim g_1(j, \ell) + g_2(j, \ell) t^{-3/2} + \dots$$

given by a GGE

Stationary State : GGE

Free theories: local conservation laws ⇔ mode occupation ops

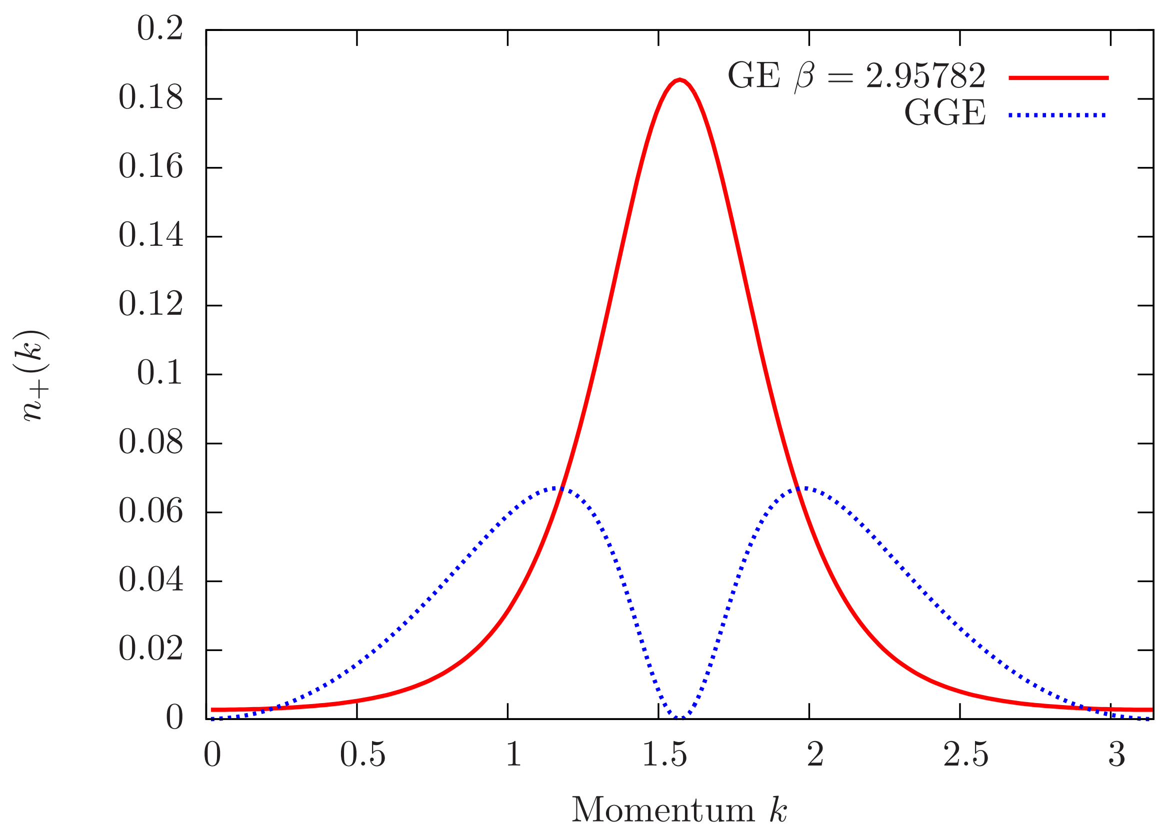
$$n_\alpha(k) = a^\dagger_\alpha(k) a_\alpha(k)$$

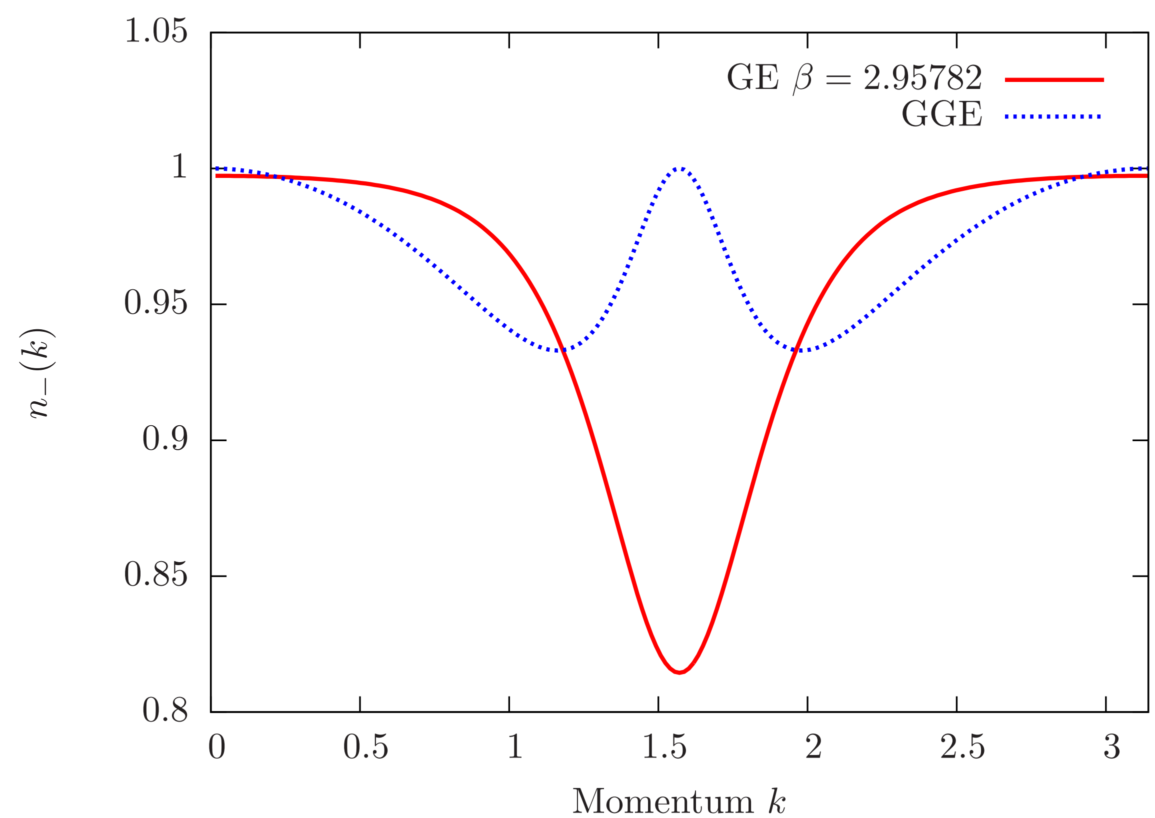
$$[H(\delta_f, 0), n_\alpha(k)] = 0 = [n_\alpha(k), n_\beta(q)]$$

$$\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp\left[\sum_k \sum_{\alpha=\pm} \mu_{k,\alpha}\, n_\alpha(k)\right]$$

Momentum occupation numbers for the two bands:

$$\delta_i = 0.75$$
$$\delta_f = 0.25$$





Very non-thermal!

Break integrability through interactions

(1) Prepare the system in the ground state $|\Psi_0\rangle$ of $H(\delta_i, 0)$

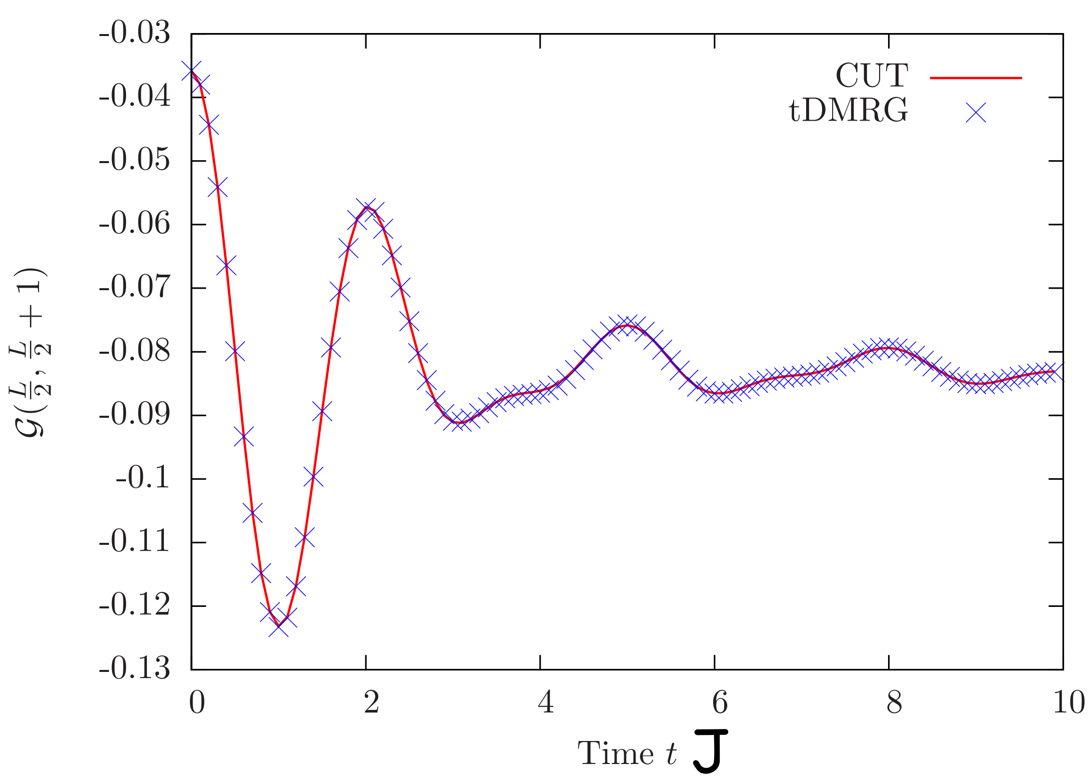
(2) At t=0 quench $\delta_i \longrightarrow \delta_f \quad U_i = 0 \longrightarrow U_f > 0$

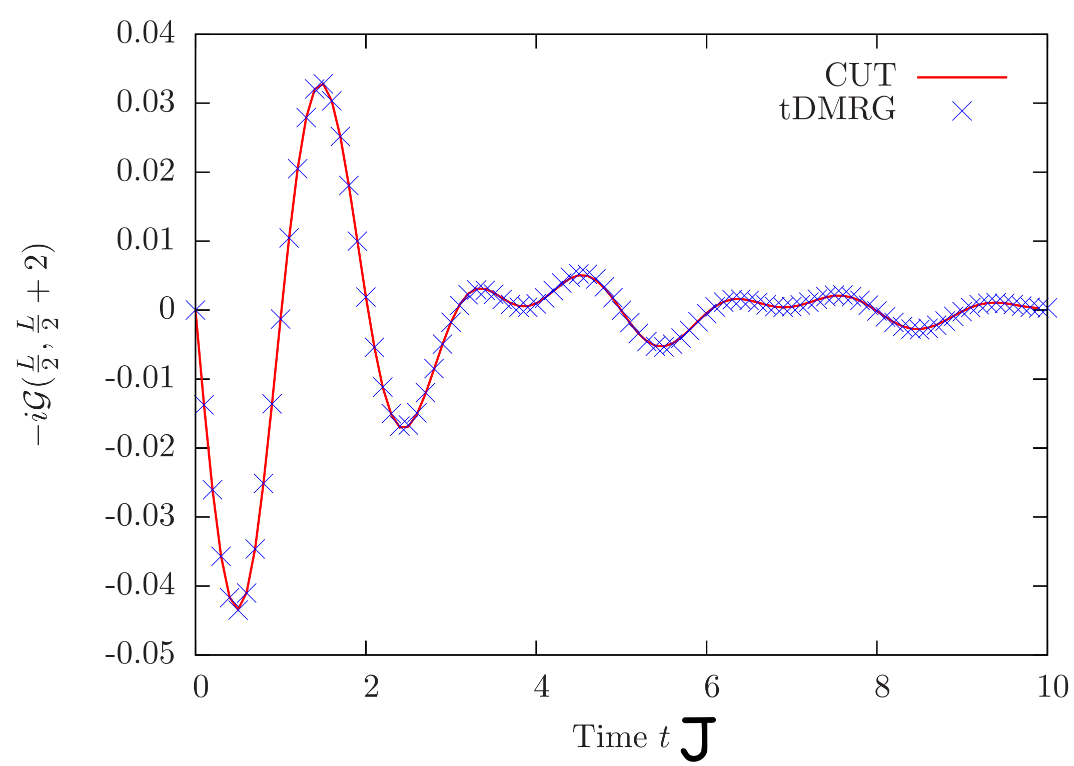
(3) Calculate Green's function $G(j,\ell,t) = \langle \Psi_0(t)|c_j^\dagger c_\ell|\Psi_0(t)\rangle$

using t-DMRG and non-equilibrium CUT method

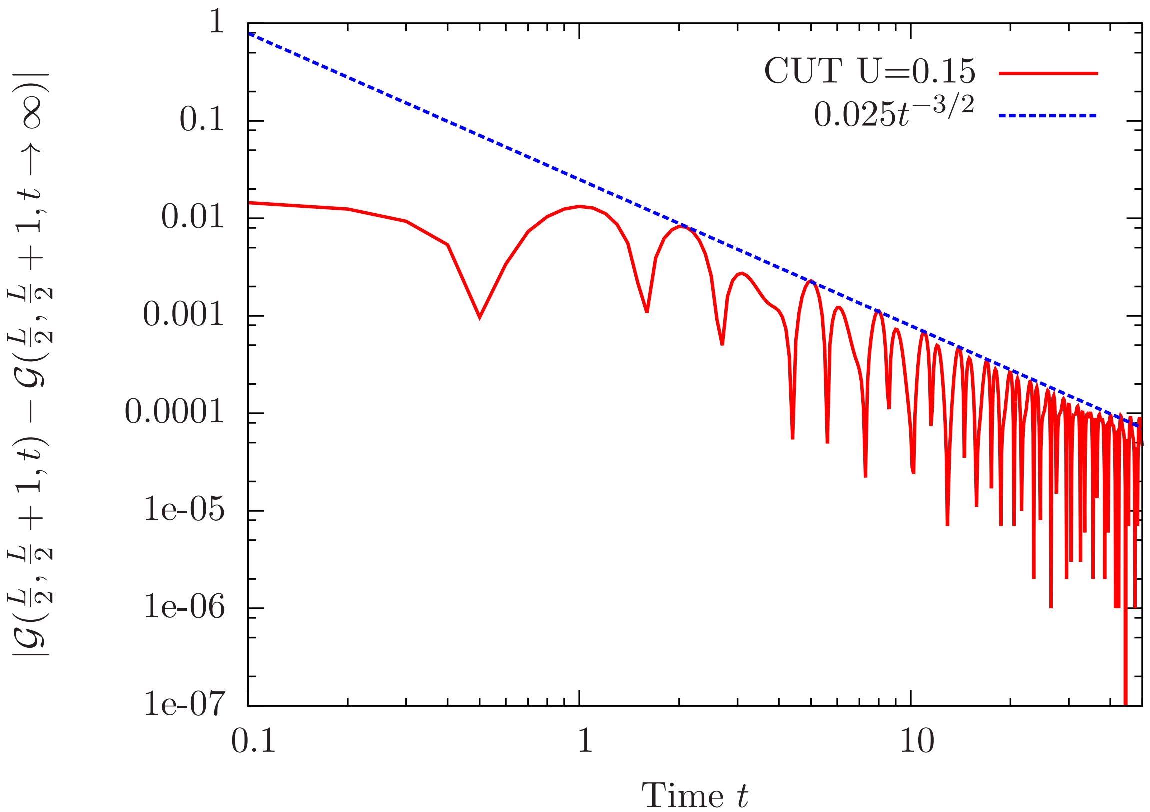
Moeckel&Kehrein '08

$$\delta_i = 0.75, \quad \delta_f = 0.5, \quad U = 0.15J$$





Observe $t^{-3/2}$ power-law decay to constant values



Thermalization?

Constant values are neither thermal nor GGE

→ "Prethermalization Plateaux"

Moeckel&Kehrein '08
Kollar et al '11

Statistical Ensemble describing the pre-thermalization plateau

Essler, Kehrein, Manmana &Robinson '14

Construct operators

$$\mathcal{Q}_\alpha(k) = a^\dagger_\alpha(k) a_\alpha(k) - U \sum_{q_j>0} N^\gamma_{\alpha\alpha}(\mathbf{q}|k,k,B=\infty)\ a^\dagger_{\gamma_1}(q_1) a_{\gamma_2}(q_2) a^\dagger_{\gamma_3}(q_3) a_{\gamma_4}(q_4)$$

$$+\mathcal{O}(U^2)$$

Physical interpretation as **quasiparticle occupation numbers**

Commutation relations:

$$[\mathcal{Q}_\alpha(k), \mathcal{Q}_\beta(p)] = \mathcal{O}(U^2). \qquad [\mathcal{Q}_\alpha(k), H(\delta_f, U)] = \mathcal{O}(U),$$

$\rightarrow$ charges not (perturbatively) conserved at the operator level, but

$$\mathrm{Tr}\left(\rho(t)\mathcal{Q}_\alpha(k)\right) - \langle\Psi_0|\mathcal{Q}_\alpha(k)|\Psi_0\rangle = \mathcal{O}(U^2)$$

Define a density matrix ("deformed GGE") by

$$\varrho_{\rm PT} = \frac{1}{Z_{\rm PT}} \exp\left(\sum_{k,\alpha} \lambda_k^{(\alpha)} \mathcal{Q}_\alpha(k)\right).$$

fix Lagrange multipliers by $\quad {\rm tr}\left[\varrho_{\rm PT}\ \mathcal{Q}_\alpha(k)\right] = \langle\Psi_0|\mathcal{Q}_\alpha(k)|\Psi_0\rangle.$

ρ_{PT} reproduces the prethermalization plateaux values to $O(U^2)$ for both two-point and 4-point functions.

Belief: this can be extended to higher orders in U

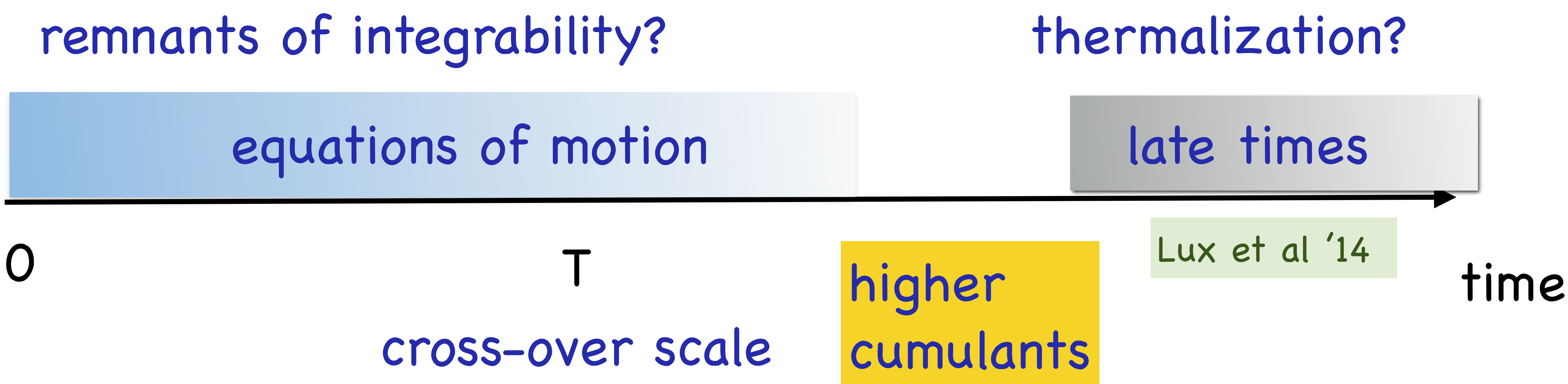
Going beyond the prethermalization plateaux

Bertini, Essler, Groha & Robinson '15

Prepare system in density matrix $\rho(0)$ s.t. Wick's thm holds

Study time-evolution using **equation of motion methods (BBGKY)**

cf Stark & Kollar '13
Nessi & Iucci '15



Equations of motion for $\hat{n}_{\alpha\beta}(q,t) = a^\dagger_\alpha(q,t) a_\beta(q,t)$

(1) $$\frac{\partial}{\partial t}\hat{n}_{\alpha\beta}(k,t) = i\left[H, \hat{n}_{\alpha\beta}(k,t)\right]$$

$$= i\left[\epsilon_\alpha(k,\delta) - \epsilon_\beta(k,\delta)\right]\hat{n}_{\alpha\beta}(k,t) + iU \sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{q}>0} Y^{\boldsymbol{\alpha}}_{\alpha\beta}(k,\boldsymbol{q}) \hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t) \ ,$$

$$\hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t) = a^\dagger_{\alpha_1}(q_1,t) a^\dagger_{\alpha_2}(q_2,t) a_{\alpha_3}(q_3,t) a_{\alpha_4}(q_4,t)$$

(2) $$\frac{\partial}{\partial t}\hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t) = i\left[H, \hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t)\right] = iE_{\boldsymbol{\alpha}}(\boldsymbol{q})\hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t) + iU \sum_{\boldsymbol{\gamma}} \sum_{\boldsymbol{p}>0} V_{\boldsymbol{\gamma}}(\boldsymbol{p}) \left[\hat{A}_{\boldsymbol{\gamma}}(\boldsymbol{p},t), \hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t)\right]$$

Integrate (2) in time, then take expectation values wrt $\rho(0)$

$$\dot{n}_{\alpha\beta}(k,t) = i\left[\epsilon_\alpha(k,\delta) - \epsilon_\beta(k,\delta)\right] n_{\alpha\beta}(k,t) + iU \sum_{\boldsymbol{\alpha}} \sum_{\boldsymbol{q}>0} Y^{\boldsymbol{\alpha}}_{\alpha\beta}(k,\boldsymbol{q}) \langle \hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},0)\rangle e^{itE_{\boldsymbol{\alpha}}(\boldsymbol{q})}$$
$$- U^2 \int_0^t ds \sum_{\boldsymbol{\alpha},\boldsymbol{\gamma}} \sum_{\boldsymbol{q},\boldsymbol{p}>0} \langle \hat{A}_{\boldsymbol{\gamma}}(\boldsymbol{p},s)\hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},s)\rangle \left[Y^{\boldsymbol{\alpha}}_{\alpha\beta}(k,\boldsymbol{q}) e^{i(t-s)E_{\boldsymbol{\alpha}}(\boldsymbol{q})} V_{\boldsymbol{\gamma}}(\boldsymbol{p}) - (\boldsymbol{\alpha},\boldsymbol{q}) \to (\boldsymbol{\gamma},\boldsymbol{p})\right]$$

Drop terms involving 4,6,... particle cumulants:

$$\langle \hat{A}_{\boldsymbol{\gamma}}(\boldsymbol{p},t)\hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t)\rangle = f(\{n_{\alpha\beta}(k,t)\}) + \mathcal{C}[\langle \hat{A}_{\boldsymbol{\gamma}}(\boldsymbol{p},t)\hat{A}_{\boldsymbol{\alpha}}(\boldsymbol{q},t)\rangle]\,,$$

disconnected parts (retained)

higher cumulants (dropped)

"Equations of motion"

$$\dot{n}_{\alpha\beta}(k,t) = i\epsilon_{\alpha\beta}(k)n_{\alpha\beta}(k,t) + 4iUe^{it\epsilon_{\alpha\beta}(k)}\sum_{\gamma_1} J_{\gamma_1\alpha}(k;t)n_{\gamma_1\beta}(k,0) - J_{\beta\gamma_1}(k;t)n_{\alpha\gamma_1}(k,0)$$

$$-U^2\int_0^t \mathrm{d}t' \sum_{\gamma}\sum_{k_1,k_2>0} K^{\gamma}_{\alpha\beta}(k_1,k_2;k;t-t')n_{\gamma_1\gamma_2}(k_1,t')n_{\gamma_3\gamma_4}(k_2,t')$$

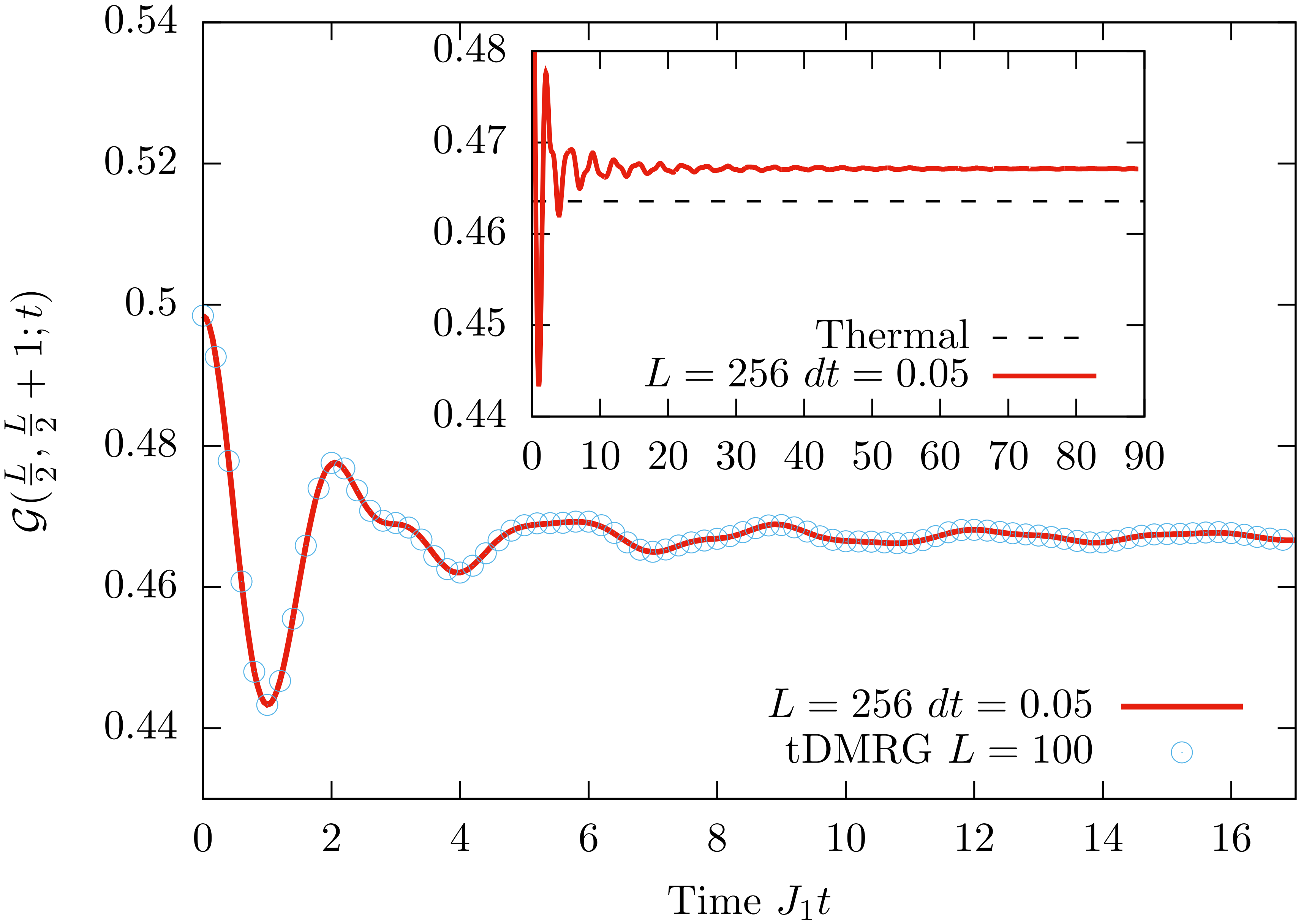
$$-U^2\int_0^t \mathrm{d}t' \sum_{\underline{\gamma}}\sum_{k_1,k_2,k_3>0} L^{\underline{\gamma}}_{\alpha\beta}(k_1,k_2,k_3;k;t-t')n_{\gamma_1\gamma_2}(k_1,t')n_{\gamma_3\gamma_4}(k_2,t')n_{\gamma_5\gamma_6}(k_3,t'),$$

Can be integrated numerically for large system sizes (L=360)

Can eliminate time-integrals for late t

$\Rightarrow$ quantum Boltzmann-like equations for $\delta_f=0$

Results for $\delta_i = 0.8 \quad \delta_f = 0.4 \quad U = 0.4$



- Excellent agreement with tDMRG & CUT
- Nice prethermalization plateaux up to late times

To be able to **tune** the "duration" of the plateau **at fixed U** study

$$H(J_2,\delta,U) = -J_1\sum_{l=1}^{L}\left[1+(-1)^l\delta\right]\left(c_l^\dagger c_{l+1} + \text{H.c.}\right) - J_2\sum_{l=1}^{L}\left[c_l^\dagger c_{l+2} + \text{H.c.}\right] + U\sum_{l=1}^{L} n_l n$$

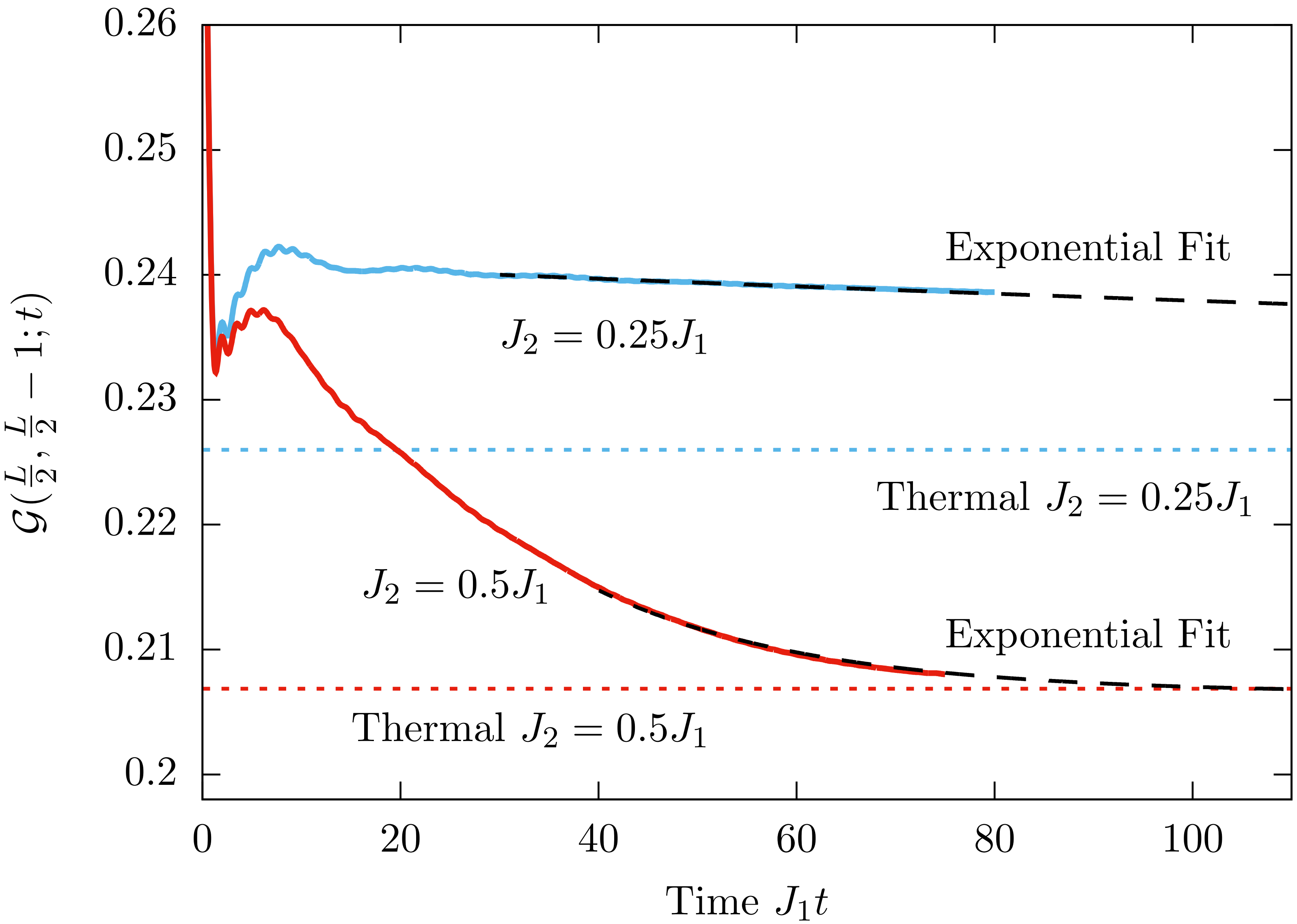
Extra term modifies U=0 band structure, opens additional scattering channels

Initial density matrix: $$\rho(\beta,J_2,\delta) = \frac{e^{-\beta H(J_2,\delta,0)}}{\text{Tr}(e^{-\beta H(J_2,\delta,0)})}$$

Thermal state, Wick's thm holds

Quench to $H(J_2, \delta = 0.1, U = 0.4)$

Initial state $\rho(\beta = 2, J_2 = 0, \delta_i = 0)$ (finite T free fermions)



Slow (exponential) drift takes us off the prethermalization plateau

Time scale over which prethermalization plateaux disappear

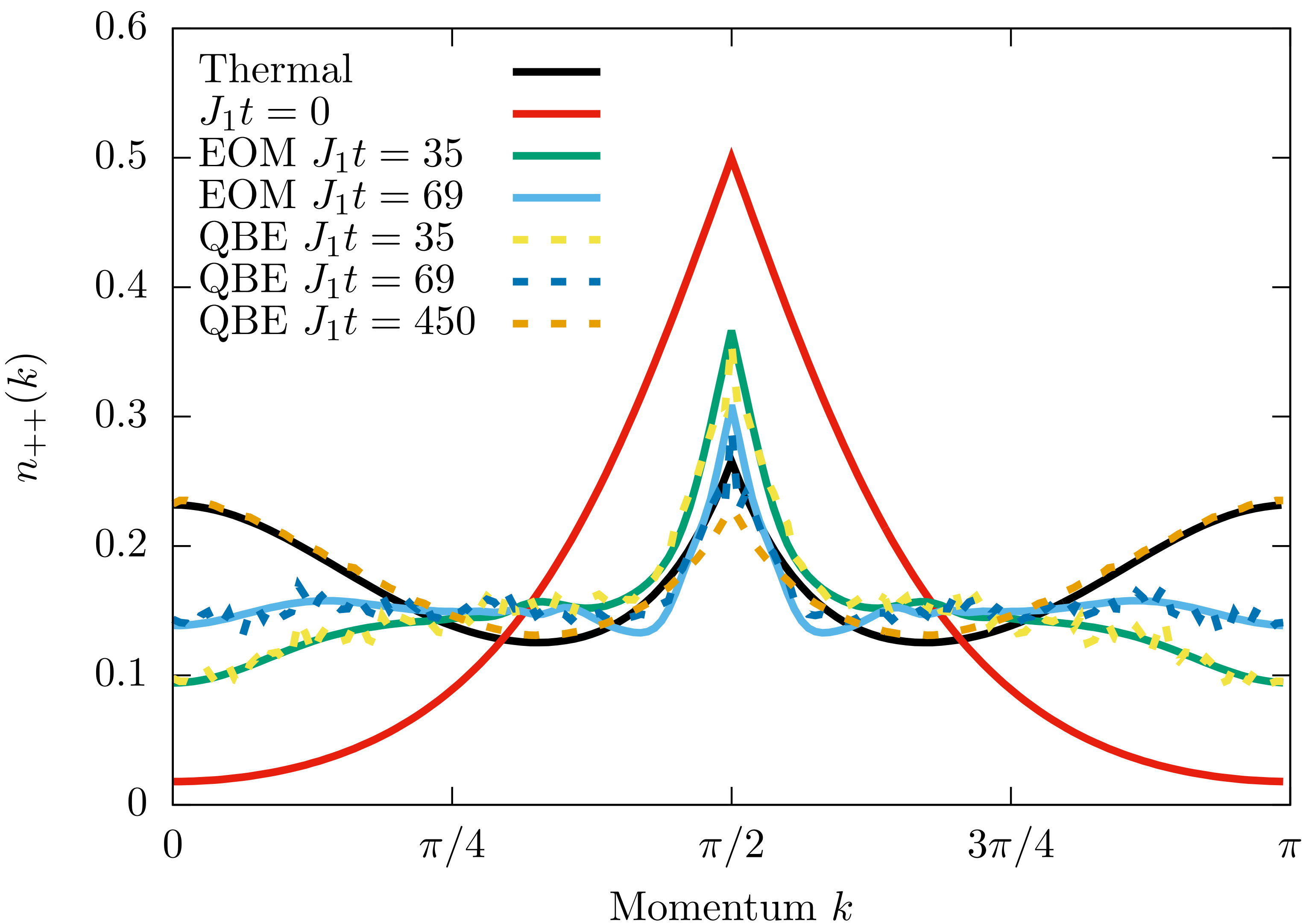
$$\mathcal{G}(i,j;t) \sim \mathcal{G}(i,j)_{\mathrm{th}} + A_{ij}(J_2,\delta,U)e^{-t/\tau_{ij}(J_2,\delta,U)}$$

where $\tau_{ij}(J_2,\delta,U) \propto U^2$

Mode occupation numbers approach thermal values:

Quench to $H(J_2 = 2, \delta_f = 0, U = 0.4)$

Initial state $\rho(\beta = 2, J_2 = 0, \delta_i = 0.5)$ (finite T free fermions)



Summary

- Integrable models have unusual (non-thermal) steady states after quantum quenches.
- Weak integrability breaking leads to interesting transients: prethermalization plateaux (PP).
- Expectation values of local operators slowly "drift off" the PP towards thermal values.
- Can one take into account 4-particle cumulants to access very late times and quantitatively describe thermalization?
- Weak integrability breaking for **strongly interacting** models?

$$H = J_1 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+1} + J_2 \sum_j \mathbf{S}_j \cdot \mathbf{S}_{j+2} \qquad 0 < J_2 \ll J_1$$