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Around the time-time correlations in KPZ growth models

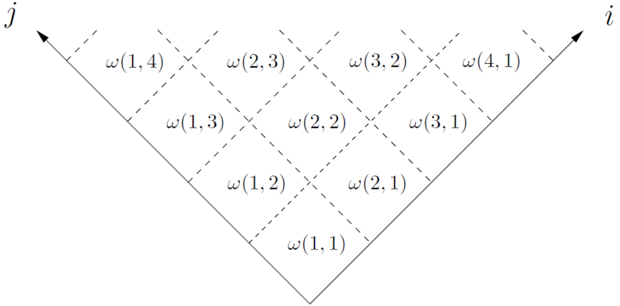
Patrik L. Ferrari
jointly with Herbert Spohn

arXiv:1602.00486

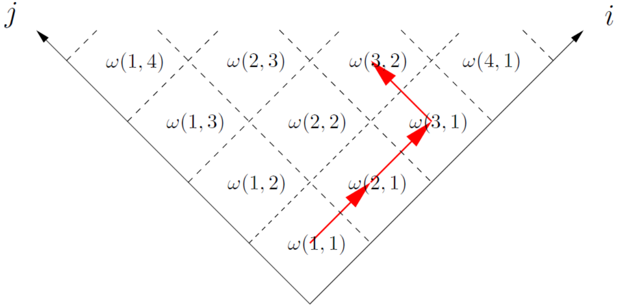
universitätbonn iam

http://wt.iam.uni-bonn.de/∼ferrari

- $\omega(i,j)$ (i.i.d.) random variables



- $\omega(i,j)$ (i.i.d.) random variables
- Directed polymer: path π composed by $\nearrow$ and $\nwarrow$
- Length of π: $\ell(\pi) = \sum_{(i,j)\in\pi} \omega(i,j)$



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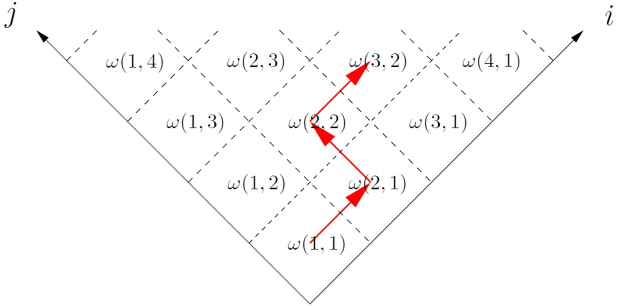
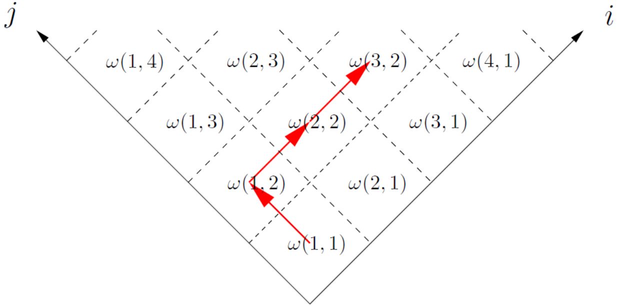


- $\omega(i,j)$ (i.i.d.) random variables
- Directed polymer: path π composed by $\nearrow$ and $\nwarrow$
- Length of π: $\ell(\pi) = \sum_{(i,j)\in\pi} \omega(i,j)$
- Maximal length: $L_{(m,n)} = \max_{\pi:(1,1)\to(m,n)} \ell(\pi)$

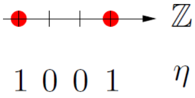


TASEP with step initial conditions

- TASEP: Totally Asymmetric Simple Exclusion Process

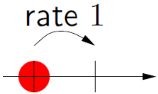
- Configurations

  $\eta = \{\eta_j\}_{j\in\mathbb{Z}}$, $\eta_j = \begin{cases} 1, & \text{if } j \text{ is occupied,} \\ 0, & \text{if } j \text{ is empty.} \end{cases}$



- Dynamics

  Independently, particles jump on the right site with rate 1, provided the right is empty.
  $\Leftrightarrow$ Waiting time $\exp(1)$-distributed



- Particles are ordered: position of particle k is $x_k(t)$
- Initial condition: $x_k(0) = -k$, $k = 1, 2, \ldots$.

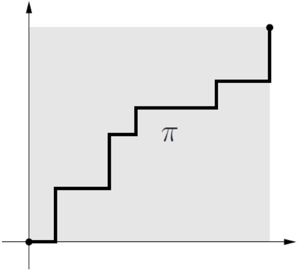
- $\omega(i,j) \sim \exp(1)$ is the waiting time (once allowed) of the particle j to do its ith jump (= from $-j+i-1$ to $-j+i$)

$\Rightarrow$ $L_{(m,n)}$ is the time when particle n reaches site $-n+m$

$$\mathbb{P}(L_{(m,n)} \leq t) = \mathbb{P}(x_n(t) + n \geq m)$$

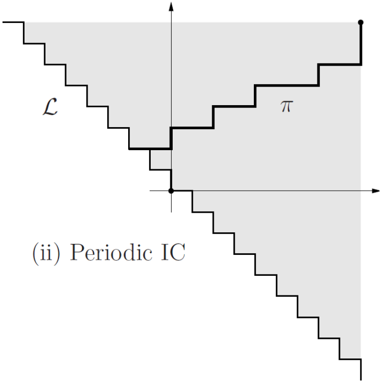
- Similarly, one can define LPP between two sets of points as well as TASEP with other initial conditions.

- We will discuss the following three geometries (LPP) / initial conditions (TASEP):
- Point-to-point problem / step initial conditions



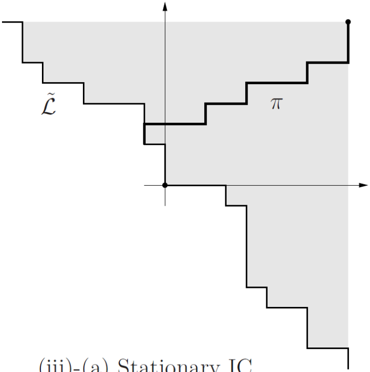
(i) Step IC

- We will discuss the following three geometries (LPP) / initial conditions (TASEP):
- Point-to-point problem / step initial conditions
- Point-to-line problem / flat initial conditions

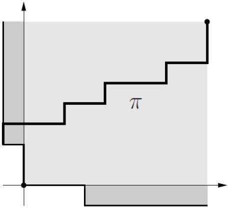


(ii) Periodic IC

- We will discuss the following three geometries (LPP) / initial conditions (TASEP):
- Point-to-point problem / step initial conditions
- Point-to-line problem / flat initial conditions
- Point-to-random walk line / stationary initial conditions



(iii)-(a) Stationary IC



(iii)-(b) Equivalent to stationary IC

Different space-time cuts

- Cut at fixed vertical coordinate equal to t: TASEP configuration at time t
- Cut at $j=n$ with n fixed: trajectory of a tagged particle
- Cut at $j=i$: integrated current at the origin

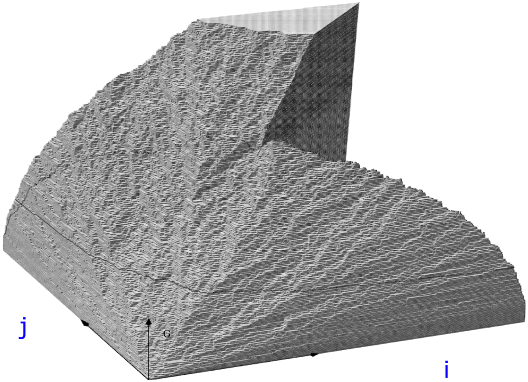


Figure by Michael Prähofer

- A simulation of tagged particle vs. integrated current for TASEP with initial half-flat initial conditions

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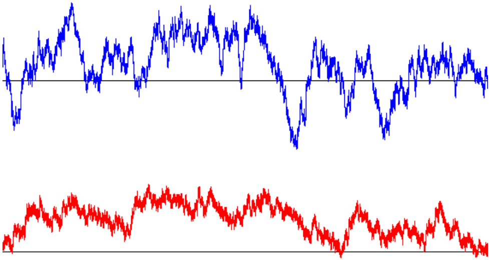
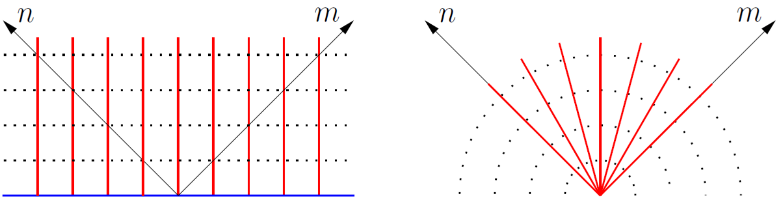


- The space-time is non-trivially fibred: spatial correlation length is $\mathcal{O}(t^{2/3})$, but in space-time, there are some directions (characteristic lines) with scaling exponent 1: slow decorrelations phenomenon

Ferrari'08;Corwin,Ferrari,Péché'10

- More generally (flat IC case): process to be studied

$$(\tau, w) \mapsto \frac{L_{(\mathcal{L},0)\to(\tau t - w t^{2/3}, \tau t + w t^{2/3})} - 4\tau t}{t^{1/3}}$$



Characteristic lines for the step IC (left) and for flat IC (right)

- We consider flat and stationary with density $1/2$ so that $\{(0,t), t \geq 0\}$ is a characteristic line
- From now on, we study the integrated current through the origin during time $[0,t]$, $J(t)$
- Rescaled process

$$\tau \mapsto \mathcal{X}(\tau) = \lim_{t\to\infty} -t^{-1/3}\big(J(\tau t) - \tfrac{1}{4}\tau t\big).$$

- One-point distribution

$$\mathbb{P}\big(\mathcal{X}^{\rm step}(1) \leq 2^{-4/3} s\big) = F_{\rm GUE}(s),$$
$$\mathbb{P}\big(\mathcal{X}^{\rm flat}(1) \leq 2^{-1} s\big) = F_{\rm GOE}(2s),$$
$$\mathbb{P}\big(\mathcal{X}^{\rm stat}(1) \leq 2^{-4/3} s\big) = F_{\rm BR}(s).$$

Baik,Rains'99,'00;Johansson'00;Prähofer,Spohn'02

- Experimental results by Kazumasa Takeuchi Takeuchi,Sano'12
  See also his talk of last week: http://online.kitp.ucsb.edu/online/randomkpz16/takeuchi/
- Formula for $\mathbb{P}\left(\mathcal{X}(\tau) \le s_1, \mathcal{X}(1) \le s_2\right)$ using replica approach in a polymer model

  Dotsenko'13
- "Finite time" formula for the two-time distribution in a semi-directed polymer model and rigorous limiting formula for $\mathbb{P}\left(\mathcal{X}(\tau) \le s_1, \mathcal{X}(1) \le s_2\right)$

  Johansson'15

- Covariance behavior as $\tau \to 0$:

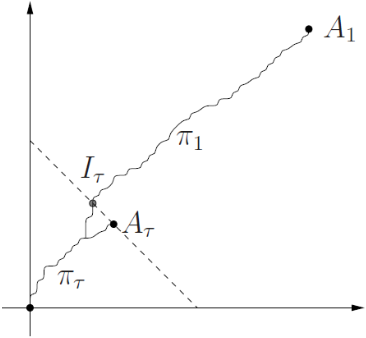
$$C^{\text{step}}(\tau) = \mathcal{O}\left(\tau^{2/3}\right),$$
$$C^{\text{flat}}(\tau) = \mathcal{O}\left(\tau^{4/3}\right),$$
$$C^{\text{stat}}(\tau) = \mathcal{O}\left(\tau^{2/3}\right).$$

- Covariance behavior as $\tau \to 1$:

$$C^{\text{step}}(\tau) = C^{\text{step}}(1) - \mathcal{O}((1-\tau)^{2/3}),$$
$$C^{\text{flat}}(\tau) = C^{\text{flat}}(1) - \mathcal{O}((1-\tau)^{2/3}),$$
$$C^{\text{stat}}(\tau) = C^{\text{stat}}(1) - \mathcal{O}((1-\tau)^{2/3}).$$

- We are going to give a heuristic argument (no proofs) that explains the above behaviors.

- Set $A_\tau = (\tau t/4, \tau t/4)$ and $I_\tau(u) = A_\tau + u(\tau t)^{2/3}(1, -1)$.



(i) Step IC

- Set $A_\tau = (\tau t/4, \tau t/4)$ and $I_\tau(u) = A_\tau + u(\tau t)^{2/3}(1,-1)$. Then as $t \to \infty$ one has Johansson'03+Corwin,Ferrari,Péché'10

$$\frac{L_{0\to A_\tau} - \tau t}{t^{1/3}} \simeq \tau^{1/3}\mathcal{A}_2(0),$$
$$\frac{L_{0\to I_\tau(u)} - \tau t}{t^{1/3}} \simeq \tau^{1/3}\left(\mathcal{A}_2(u) - u^2\right),$$
$$\frac{L_{I_\tau(u)\to A_1} - (1-\tau)t}{t^{1/3}} \simeq (1-\tau)^{1/3}\left[\tilde{\mathcal{A}}_2\left(u\hat{\tau}^{2/3}\right) - \left(u\hat{\tau}^{2/3}\right)^2\right],$$

  where $\mathcal{A}_2$ and $\tilde{\mathcal{A}}_2$ are two independent Airy$_2$ processes.
- We have

$$\mathcal{X}^{\text{step}}(\tau) = \tau^{1/3}\mathcal{A}_2(0)$$

  and, using $L_{0\to A_1} = \max_u\left(L_{0\to I_\tau(u)} + L_{I_\tau(u)\to A_1}\right)$, also

$$\mathcal{X}^{\text{step}}(1) = \tau^{1/3}\max_{u\in\mathbb{R}}\left\{\mathcal{A}_2(u) - u^2 + \hat{\tau}^{-1/3}\tilde{\mathcal{A}}_2\left(u\hat{\tau}^{2/3}\right) - u^2\hat{\tau}\right\}$$

  with $\hat{\tau} = \tau/(1-\tau)$.

- Using $\hat{\tau}^{-1/3}\tilde{\mathcal{A}}_2\big(u\hat{\tau}^{2/3}\big) \simeq B(u)$ a Brownian motion

Hägg'07,Corwin,Hammond'11

- For small τ,

$$C^{\text{step}}(\tau) = \text{Cov}\left(\mathcal{X}^{\text{step}}(\tau), \mathcal{X}^{\text{step}}(1)\right) \simeq \tau^{2/3}\,\text{Cov}\left(\mathcal{A}_2(0), \max_{u\in\mathbb{R}}\left\{\mathcal{A}_2(u) - u^2 + B(u)\right\}\right),$$

- Conditioning on B

$$C^{\text{step}}(\tau) \simeq \tau^{2/3}\mathbb{E}\Big[\text{Cov}\left(\mathcal{A}_2(0), \max_{u\in\mathbb{R}}\left\{\mathcal{A}_2(u) - u^2 + B(u)\right\}\Big|B\right)\Big].$$

- For typical realizations of B, the maximum is reached for u of order 1, where the last covariance if of order 1, leading to

$$C^{\text{step}}(\tau) = \mathcal{O}(\tau^{2/3})$$

- In this case the maximum is reached for $u = \mathcal{O}((1-\tau)^{2/3})$. Set $u = v(1-\tau)^{2/3}/\tau^{2/3}$. As $\tau \to 1$, and conditioning on $\tilde{\mathcal{A}}_2$,

$$C^{\rm step}(\tau) = \mathrm{Cov}\left(\mathcal{X}^{\rm step}(\tau), \mathcal{X}^{\rm step}(1)\right)$$

$$\simeq \mathbb{E}\Big[\mathrm{Cov}\Big(\mathcal{A}_2(0), \max_{v\in\mathbb{R}}\left\{\mathcal{A}_2(v(1-\tau)^{2/3}) + (1-\tau)^{1/3}(\tilde{\mathcal{A}}_2(v) - \tilde{\mathcal{A}}_2(0) - v^2)\right\}\big|\tilde{\mathcal{A}}_2\Big)\Big]$$

- In this case the maximum is reached for $u = \mathcal{O}((1-\tau)^{2/3})$. Set $u = v(1-\tau)^{2/3}/\tau^{2/3}$. As $\tau \to 1$, and conditioning on $\tilde{\mathcal{A}}_2$,

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- For typical realizations of the process $\tilde{\mathcal{A}}_2$, the maximum is reached for v of order 1. Also, $\mathcal{A}_2$ is locally Brownian, in particular

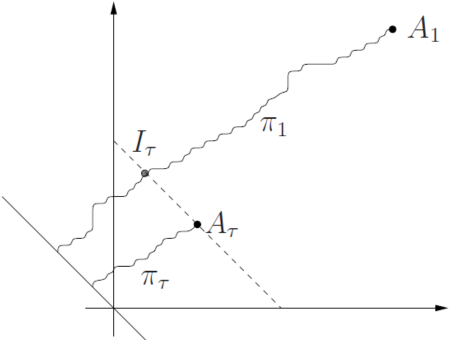
$$\mathrm{Cov}\left(\mathcal{A}_2(0), \mathcal{A}_2(v(1-\tau)^{2/3}\right) \simeq \mathrm{Var}\left(\mathcal{A}_2(0)\right) - |v|(1-\tau)^{2/3}$$

Prähofer,Spohn'02,Widom'03

- Using the independence of $\tilde{\mathcal{A}}_2$ and $\mathcal{A}_2$, we thus expects that

$$C^{\rm step}(\tau) = \mathrm{Var}\left(\mathcal{A}_2(0)\right) - \mathcal{O}((1-\tau)^{2/3}).$$

- Set $A_\tau = (\tau t/4, \tau t/4)$ and $I_\tau(u) = A_\tau + u(\tau t)^{2/3}(1,-1)$.



(ii) Periodic IC

- Set $A_\tau = (\tau t/4, \tau t/4)$ and $I_\tau(u) = A_\tau + u(\tau t)^{2/3}(1,-1)$. Then as $t \to \infty$ one has

  Borodin,Ferrari,Prähofer,Sasamoto'07+Corwin,Ferrari,Péché'10

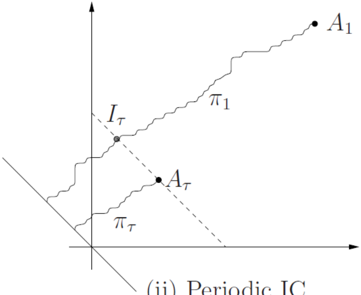
$$\frac{L_{\mathcal{L}\to A_\tau} - \tau t}{t^{1/3}} \simeq \tau^{1/3}\mathcal{A}_1(0),$$
$$\frac{L_{\mathcal{L}\to I_\tau(u)} - \tau t}{t^{1/3}} \simeq \tau^{1/3}\mathcal{A}_1(\tilde{c}u),$$
$$\frac{L_{I_\tau(u)\to A_1} - (1-\tau)t}{t^{1/3}} \simeq (1-\tau)^{1/3}\big[\tilde{\mathcal{A}}_2\big(u\hat{\tau}^{2/3}\big) - \big(u\hat{\tau}^{2/3}\big)^2\big],$$

  where the Airy$_1$ process $\mathcal{A}_1$ is independent of the Airy$_2$ process $\tilde{\mathcal{A}}_2$.
- We have
$$\mathcal{X}^{\text{flat}}(\tau) = \tau^{1/3}\mathcal{A}_1(0)$$
  and
$$\mathcal{X}^{\text{flat}}(1) = \max_{u\in\mathbb{R}}\big\{\tau^{1/3}\mathcal{A}_1(u) + (1-\tau)^{1/3}\tilde{\mathcal{A}}_2\big(u\hat{\tau}^{2/3}\big) - u^2\tau^{1/3}\hat{\tau}\big\}$$

- For small τ, the maximum over u is typically taken for $u \sim \mathcal{O}(\tau^{-2/3})$
- Since the covariance of the Airy_1 process decays superexponentially in u Bornemann,Ferrari,Prähofer'08 their contribution to

$$C^{\text{flat}}(\tau) = \text{Cov}(\mathcal{X}(\tau), \mathcal{X}(1))$$

is negligible.



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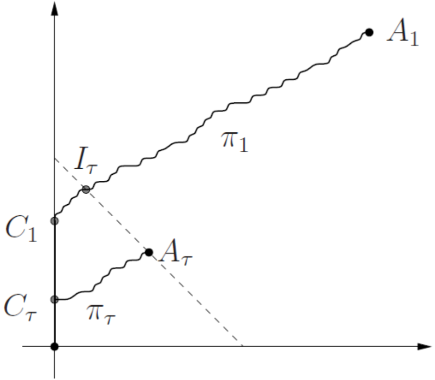
$$C^{\mathrm{flat}}(\tau) = \mathrm{Cov}(\mathcal{X}(\tau), \mathcal{X}(1))$$

  is negligible.
- With probability $\mathcal{O}(\tau^{2/3})$, the maximum is take for $u = \mathcal{O}(1)$. In this case the heuristic is the same as for the point-to-point case. Thus,

$$C^{\mathrm{flat}}(\tau) = \mathcal{O}(\tau^{2/3} C^{\mathrm{step}}(\tau)) = \mathcal{O}(\tau^{4/3})$$

- Set $A_\tau = (\tau t/4, \tau t/4)$ and $I_\tau(u) = A_\tau + u(\tau t)^{2/3}(1,-1)$.



- Set $A_\tau = (\tau t/4, \tau t/4)$ and $I_\tau(u) = A_\tau + u(\tau t)^{2/3}(1,-1)$. Then as $t \to \infty$ one has

  Imamura,Sasamoto'05;Baik,Ferrari,Péché'10

$$\frac{L_{0\to A_\tau} - \tau t}{t^{1/3}} \simeq \tau^{1/3}\mathcal{A}_{\rm stat}(0),$$
$$\frac{L_{0\to I_\tau(u)} - \tau t}{t^{1/3}} \simeq \tau^{1/3}\mathcal{A}_{\rm stat}(u),$$
$$\frac{L_{I_\tau(u)-A_1} - (1-\tau)t}{t^{1/3}} \simeq (1-\tau)^{1/3}\big[\tilde{\mathcal{A}}_2\big(u\hat\tau^{2/3}\big) - \big(u\hat\tau^{2/3}\big)^2\big],$$

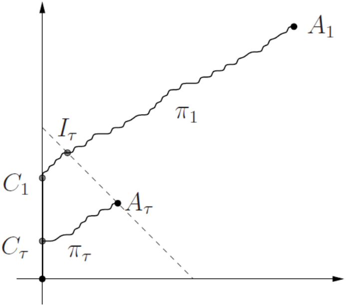
  where the processes $\mathcal{A}_{\rm stat}$ and $\tilde{\mathcal{A}}_2$ are independent.
- We have

$$\mathcal{X}^{\rm stat}(\tau) = \tau^{1/3}\mathcal{A}_{\rm stat}(0)$$

  and

$$\mathcal{X}^{\rm stat}(1) = \max_{u\in\mathbb{R}}\big\{\tau^{1/3}\mathcal{A}_{\rm stat}(u) + (1-\tau)^{1/3}\tilde{\mathcal{A}}_2\big(u\hat\tau^{2/3}\big) - u^2\tau^{1/3}\hat\tau^{-1}\big\}$$

- For small τ, the maximum over u is typically taken for $u \sim \mathcal{O}(\tau^{-2/3})$, the maximizers to A_τ and A_1 uses then different noises (independent) except for the noise on the axis



- For small τ, the maximum over u is typically taken for $u \sim \mathcal{O}(\tau^{-2/3})$, the maximizers to A_τ and A_1 uses then different noises (independent) except for the noise on the axis
- Thus we expect

$$\begin{aligned} C^{\mathrm{stat}}(\tau) &= \mathrm{Cov}(\mathcal{X}^{\mathrm{stat}}(\tau), \mathcal{X}^{\mathrm{stat}}(1)) \\ &\simeq \mathcal{O}(t^{-2/3})\, \mathrm{Cov}\left(L_{(-1,-1)\to C_\tau}, L_{(-1,-1)\to C_1}\right). \end{aligned}$$

- The sums of random variables in the LPP problem between the origin and C_1 (and C_τ) are asympotically Brownian motions. Thus implies that

$$C^{\mathrm{stat}}(\tau) = \mathcal{O}\left(\tau^{2/3}\right)$$

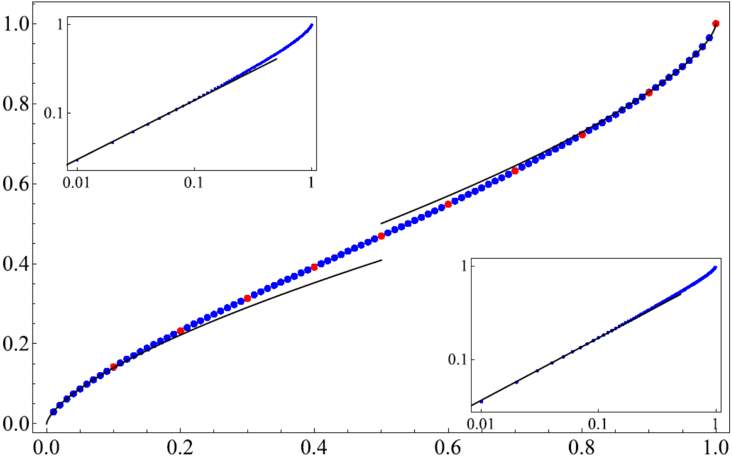


Figure: Plot of $\tau \mapsto \mathrm{Cov}(\mathcal{X}^{\mathrm{step}}(\tau), \mathcal{X}^{\mathrm{step}}(1)) / \mathrm{Var}(\mathcal{X}^{\mathrm{step}}(1))$. The top-left (resp. right-bottom) inset is the log-log plot around $\tau = 0$ (resp. $\tau = 1$).

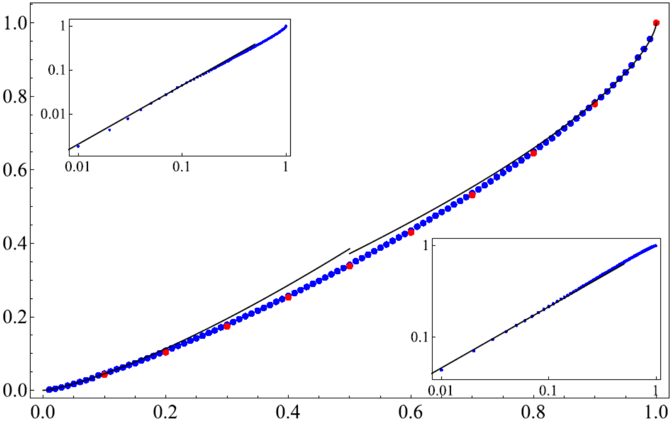


Figure: Plot of $\tau \mapsto \mathrm{Cov}(\mathcal{X}^{\mathrm{flat}}(\tau), \mathcal{X}^{\mathrm{flat}}(1)) / \mathrm{Var}(\mathcal{X}^{\mathrm{flat}}(1))$. The top-left (resp. right-bottom) inset is the log-log plot around $\tau = 0$ (resp. $\tau = 1$).

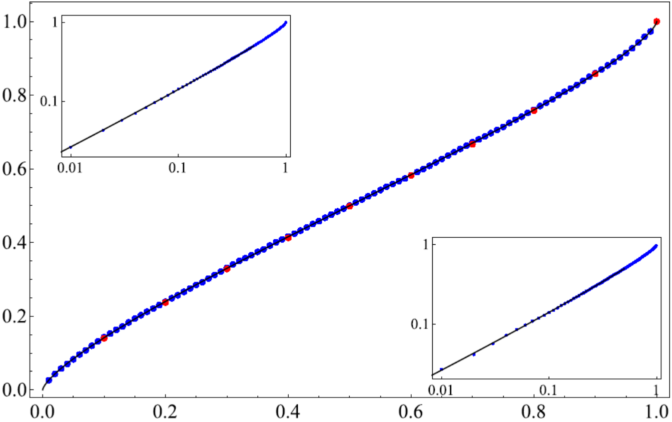


Figure: Plot of $\tau \mapsto \mathrm{Cov}(\mathcal{X}^{\mathrm{stat}}(\tau), \mathcal{X}^{\mathrm{stat}}(1)) / \mathrm{Var}(\mathcal{X}^{\mathrm{stat}}(1))$. The top-left inset is the log-log plot around $\tau = 0$ and the right-bottom inset is the log-log plot around $\tau = 1$. The fit is made with the function $\tau \mapsto \frac{1}{2}(1 + \tau^{2/3} - (1-\tau)^{2/3})$.

- TASEP with stationary initial conditions
- $\mathrm{j}(t)$ is the empirical current across the bond $(0,1)$ from which

$$J(t) = \int_0^t ds\, \mathrm{j}(s)$$

- Two-point function (stationary covariance)

$$S(j,t) = \mathbb{E}\big(\eta_j(t)\eta_0(0)\big) - \rho^2.$$

- A sum rule:

$$\mathrm{Var}(J(t)) = \sum_{j\in\mathbb{Z}} |j| S(j,t) - \sum_{j\in\mathbb{Z}} |j| S(j,0)$$

- $j(\rho) = \rho(1-\rho)$ is the expected current with respect to the stationary initial condition with density ρ
- A small perturbation of the steady state will propagate with velocity $v(\rho) = j'(\rho)$, $v(1/2) = 0$.
- The current-current covariance is then given by

$$\mathbb{E}\big(\mathsf{j}(t)\,\mathsf{j}(t')\big) - j(\rho)^2 = \rho(1-\rho)\delta(t-t') + h(t-t')$$

- The smooth part $h(t-t')$ is given by

$$h(t) = -\langle (r^{\mathrm{R}}_{0,1} - j(\rho))\mathrm{e}^{L|t|}(r_{0,1} - j(\rho))\rangle_\rho,$$

where for TASEP $r_{0,1}(\eta) = \eta_0(1-\eta_1)$, $r^{\mathrm{R}}_{0,1}(\eta) = -(1-\eta_0)\eta_1$; $\langle\cdot\rangle_\rho$ is the average with respect to the stationary measure with density ρ; L is the backwards generator of TASEP.

- According to the KPZ scaling theory:

$$S(j,t) \simeq \chi(\Gamma t)^{-2/3} f_{\mathrm{KPZ}}((\Gamma t)^{-2/3} j)$$

where in the special case of TASEP case $\chi = \rho(1-\rho)$, and $\Gamma = \chi^2$. Krug,Meakin,Halpin-Healy'92

- Using the sum rule

$$\chi \int_{\mathbb{R}} dx |x| f_{\mathrm{KPZ}}(x) (\Gamma t)^{2/3} \simeq -2 \int_0^t ds \int_s^\infty du h(u),$$

which implies

$$h(t) \simeq -c_0 t^{-4/3}, \quad c_0 = \tfrac{1}{9} \Gamma^{2/3} \chi \int_{\mathbb{R}} dx |x| f_{\mathrm{KPZ}}(x).$$

- Covariance of integrate current

$$\mathrm{Cov}\big(J(t),J(\tau t)\big) = -\int_0^t\int_0^{\tau t} ds ds' \Big(\int_{\mathbb{R}} du\, h(u)\delta(s-s') - h(s-s')\Big)$$
$$= -\int_0^{\tau t} ds \Big(2\int_s^{\infty} ds' h(s') - \int_{\tau t - s}^{t-s} ds' h(s')\Big)$$

- Using

$$h(t) \simeq -c_0 t^{-4/3}$$

one gets

$$\mathrm{Cov}\big(J(t),J(\tau t)\big) \simeq \big(1+\tau^{2/3}-(1-\tau)^{2/3}\big)(\Gamma t)^{2/3}\chi\int_0^{\infty} dx|x| f_{\mathrm{KPZ}}(x)$$

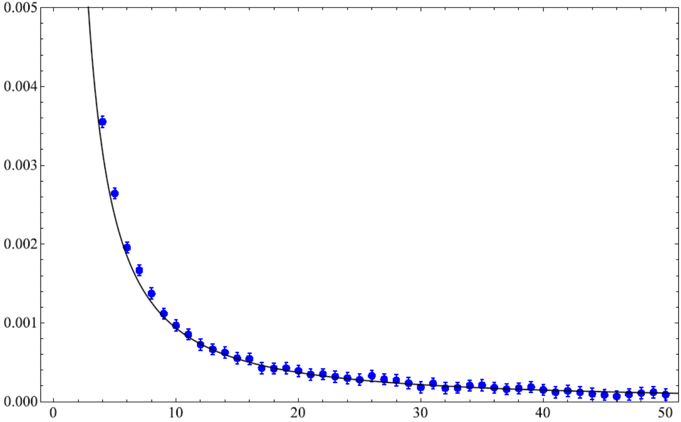


Figure: The smooth part of the current-current correlations for TASEP. We plot $-h(t)$ and the theoretical large time behavior $0.02013 \cdot t^{-4/3}$.

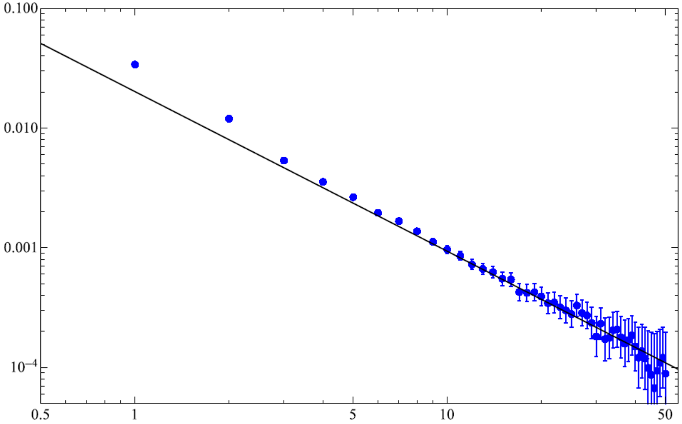


Figure: Log-log plot of the smooth part of the current-current correlation for TASEP.