# The two-time distribution in random growth

Kurt Johansson
KTH Royal Institute of Technology

KITP, February 19, 2016.

# Directed last-passage times

Consider the *last-passage times*

$$G(m,n) = \max_{\pi:(1,1)\nearrow(m,n)} \sum_{(i,j)\in\pi} w(i,j),$$

where $w(i,j)$ are i.i.d. geometric random variables

$$\mathbb{P}[w(i,j)=k] = (1-q)q^k, \quad k \geq 0.$$

# Directed last-passage times

We have the limit theorem

$$\mathbb{P}\left[\frac{G(n,[\lambda n]) - an}{bn^{1/3}} \le \xi\right] \to F_2(\xi)$$

as $n \to \infty$, where

$$F_2(\xi) = \det(I - K_{\mathrm{Ai}})_{L^2(\xi,\infty)}$$

is the *Tracy-Widom distribution*, $K_{\mathrm{Ai}}$ the *Airy kernel*.

# Directed last-passage times

We have the limit theorem

$$\mathbb{P}\left[\frac{G(n,[\lambda n]) - an}{bn^{1/3}} \leq \xi\right] \to F_2(\xi)$$

as $n \to \infty$, where

$$F_2(\xi) = \det(I - K_{\mathrm{Ai}})_{L^2(\xi,\infty)}$$

is the *Tracy-Widom distribution*, $K_{\mathrm{Ai}}$ the *Airy kernel*.

Fluctuation exponent $1/3$.

# Two-point distribution, space direction

Joint distribution of $G(m_1, n_1)$ and $G(m_2, n_2)$ when $m_1 < m_2$ and $n_1 > n_2$.

$$\mathbb{P}\left[\frac{G(n+\nu_1 n^{2/3}, n-\nu_1 n^{2/3}) - an}{bn^{1/3}} \leq \xi_1, \frac{G(n+\nu_2 n^{2/3}, n-\nu_2 n^{2/3}) - an}{bn^{1/3}} \leq \xi_2\right]$$

converges to a Fredholm determinant involving the extended Airy kernel.

Fluctuation exponent $2/3$.

We can think of

$$k \to G(n+k, n-k), \quad -n < k < n,$$

as a one-dimensional interface, *discrete polynuclear growth*, at time t=2n.

# Two-point distribution, time direction

Two-time joint distribution

$$\mathbb{P}\left[G(m_1, m_1) \le v_1, G(m_2, m_2) \le v_2\right]$$

We expect non-trivial fluctuations if

$$\frac{m_1}{m_2 - m_1} \sim c, \quad c > 0.$$

Exponent 1

*Slow de-correlation phenomenon*

(P. L. Ferrari and I. Corwin, P.L. Ferrari, S. Péché)

# More general problem

Consider the discrete "surface"

$$(m, n) \to G(m, n)$$

for $(m, n) \in \mathbb{Z}_+^2$.

We would like to prove that, under the appropriate scalings, we have a limiting random surface in the sense of convergence in finite-dimensional distributions.

# Zero temperature Brownian semi-discrete directed polymer

Last-passage time

$$H(\mu, n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}).$$

Distributed as the largest eigenvalue of a GUE-matrix

$$\mathbb{P}[H(\mu, n) \leq \xi] = \frac{1}{Z_{\mu,n}} \int_{(-\infty,\xi]^n} \prod_{1\leq j<k\leq n} (x_k - x_j)^2 \prod_{j=1}^{p} e^{-\frac{x_j^2}{2\mu}} \, d^n x.$$

(Gravner, Tracy, Widom and Baryshnikov).

# Zero temperature Brownian semi-discrete directed polymer

Last-passage time

$$H(\mu, n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}).$$

Distributed as the largest eigenvalue of a GUE-matrix

$$\mathbb{P}[H(\mu, n) \le \xi] = \frac{1}{Z_{\mu,n}} \int_{(-\infty,\xi]^n} \prod_{1\le j<k\le n} (x_k - x_j)^2 \prod_{j=1}^{p} e^{-\frac{x_j^2}{2\mu}} \, d^n x.$$

Limit of $G(m, n)$

$$\frac{G([\mu T], n) - \frac{q}{1-q}[\mu T]}{\frac{\sqrt{q}}{1-q}\sqrt{T}} \to H(\mu, n)$$

in distribution as $T \to \infty$.
(Gravner, Tracy, Widom and Baryshnikov).

# Main theorem

## Theorem

*Let* $0 < t_1 < t_2$, $\eta_1, \eta_2, \nu_1, \nu_2 \in \mathbb{R}$ *be given. Set*

$$\alpha = (t_1/(t_2 - t_1))^{1/3}.$$

*Introduce the scaling,* $i = 1, 2$,

$$\mu_i = t_i M - \nu_i (t_i M)^{2/3},\ n_i = t_i M + \nu_i (t_i M)^{2/3}, \quad \xi_i = 2 t_i M + (\eta_i - \nu_i^2)(t_i M)^{1/3}.$$

$i = 1, 2$. *With this scaling,*

$$\lim_{M \to \infty} \mathbb{P}[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2] = F_{tt}(\eta_1, \eta_2; \alpha, \nu_1, \nu_2).$$

There is an explicit, but rather complicated, formula for the *two-time distribution* $F_{\text{tt}}$.

A different formula was derived non-rigorously by V. Dotsenko using the replica method.

# A Markov chain

Let

$$\mathbf{G}(m) = (G(m,1), \ldots, G(m,n))$$

and

$$w_m(x) = (1-q)^m \binom{x+m-1}{x} q^x 1(x \geq 0).$$

For $\mathbf{x}, \mathbf{y} \in W_n = \{\mathbf{x} \in \mathbb{Z}^n \,;\, x_1 \leq x_2 \leq \cdots \leq x_n\}$ and $m > \ell \geq 0$,

$$\mathbb{P}[\mathbf{G}(m) = \mathbf{y} \,|\, \mathbf{G}(\ell) = \mathbf{x}] = \det \left( \Delta^{j-i} w_{m-\ell}(y_j - x_i) \right)_{1 \leq i,j \leq n}.$$

In particular

$$\mathbb{P}[\mathbf{G}(m) = \mathbf{x}] = \det \left( \Delta^{j-i} w_m(x_j) \right)_{1 \leq i,j \leq n}.$$

(Inspired by similar results by J. Warren.)

# Joint distribution

We get the formula

$$\mathbb{P}[G(m_1, n_1) \leq v_1, G(m_2, n_2) \leq v_2]$$
$$= \sum_{u=-\infty}^{v_1} \sum_{\substack{x \in W_{n_2} \\ x_{n_1} = u}} \sum_{\substack{y \in W_{n_2} \\ y_{n_2} \leq v_2}} \det\left(\Delta^{j-i} w_{m_1}(x_j)\right)_{1 \leq i,j \leq n_2} \det\left(\Delta^{j-i} w_{m_2 - m_1}(y_j - x_i)\right)_{1 \leq i,j \leq n_2},$$

# Joint distribution

We get the formula

$$\mathbb{P}[G(m_1,n_1) \le v_1, G(m_2,n_2) \le v_2]$$
$$= \sum_{u=-\infty}^{v_1} \sum_{\substack{x\in W_{n_2}\\ x_{n_1}=u}} \sum_{\substack{y\in W_{n_2}\\ y_{n_2}\le v_2}} \det\left(\Delta^{j-i} w_{m_1}(x_j)\right)_{1\le i,j\le n_2} \det\left(\Delta^{j-i} w_{m_2-m_1}(y_j - x_i)\right)_{1\le i,j\le n_2},$$

Into this formula we can insert

$$\Delta^k w_m(x) = \frac{(1-q)^m}{2\pi i}\int_{\gamma_r} \frac{(1-z)^k dz}{(1-qz)^m z^{x+k+1}},$$

where $0 < r < 1$.

# Joint distribution

After a non-trivial computation we obtain

$$\mathbb{P}[G(m_1,n_1)\le v_1, G(m_2,n_2)\le v_2]$$
$$= \sum_{u=-\infty}^{v_1} \frac{(1-q)^{m_2 n_2}(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2n_2} n_1!^2 (\Delta n)!^2} \int_{\gamma_{s_1}^{n_1}} d^{n_1}z \int_{\gamma_{s_2}^{\Delta n}} d^{\Delta n}z \int_{\gamma_{r_1}^{n_1}} d^{n_1}w \int_{\gamma_{r_2}^{\Delta n}} d^{\Delta n}w$$
$$\times \det\left(z_j^{i-1}\right)_{1\le i,j\le n_2} \det\left(w_j^{i-1}\right)_{1\le i,j\le n_2} \det\left(\frac{1}{w_j-z_i}\right)_{1\le i,j\le n_1} \det\left(\frac{1}{z_j-w_i}\right)_{n_1<i,j\le n_2}$$
$$\times \prod_{j=n_1+1}^{n_2}\frac{1-z_j}{1-w_j}\left(1-\prod_{j=1}^{n_1}\frac{z_j}{w_j}\right)\prod_{j=1}^{n_2}\frac{w_j^{u-v_2-\Delta n}}{z_j^{u+n_1}(1-z_j)^{\Delta n}(1-qz_j)^{m_1}(1-w_j)^{n_1}(1-qw_j)^{\Delta m}},$$

Here $\Delta n = n_2 - n_1$, $\Delta m = m_2 - m_1$, $0 < s_1 < r_1 < 1$, $0 < r_2 < s_2 < 1$.

# Two identities

An important ingredient in the above derivation are two symmetrization identities:

$$\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{j=1}^{n} \left(\frac{1-w_{\sigma(j)}}{w_{\sigma(j)}}\right)^{j} \frac{1}{(1-w_{\sigma(1)})(1-w_{\sigma(1)}w_{\sigma(2)})\cdots(1-w_{\sigma(1)}\cdots w_{\sigma(n)})}$$

$$= (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \frac{1}{w_j^n} \det\left(w_j^{i-1}\right)_{1\le i,j\le n},$$

(Tracy-Widom ASEP identity) and

$$\sum_{\sigma_1,\sigma_2\in S_n} \operatorname{sgn}(\sigma_1\sigma_2) \prod_{j=1}^{n} \left(\frac{w_{\sigma_2(j)}(1-z_{\sigma_1(j)})}{z_{\sigma_1(j)}(1-w_{\sigma_2(j)})}\right)^{j}$$

$$\times \frac{1}{\left(1-\frac{z_{\sigma_1(1)}}{w_{\sigma_2(1)}}\right)\left(1-\frac{z_{\sigma_1(1)}z_{\sigma_1(2)}}{w_{\sigma_2(1)}w_{\sigma_2(2)}}\right)\cdots\left(1-\frac{z_{\sigma_1(1)}\cdots z_{\sigma_1(n)}}{w_{\sigma_2(1)}\cdots w_{\sigma_2(n)}}\right)}$$

$$= \prod_{j=1}^{n} \frac{w_j^{n+1}(1-z_j)^n}{z_j^n(1-w_j)^n} \det\left(\frac{1}{w_j-z_i}\right)_{1\le i,j\le n}.$$

# Limit to the Brownian directed polymer

Taking the appropriate limit to the Brownian directed polymer we obtain

$$\frac{\partial}{\partial \xi_1}\mathbb{P}\left[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2\right] = \frac{\partial}{\partial h}\bigg|_{h=0} Q(h),$$

where we recall that

$$H(\mu, n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}).$$

# Limit to the Brownian directed polymer

Taking the appropriate limit to the Brownian directed polymer we obtain

$$\frac{\partial}{\partial \xi_1} \mathbb{P}\left[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2\right] = \left.\frac{\partial}{\partial h}\right|_{h=0} Q(h),$$

where

$$Q(h) = \frac{(-1)^{\frac{n_2(n_2-1)}{2}}}{(2\pi i)^{2n_2} n_1! \Delta n!}$$
$$\times \int_{\Gamma_{d_1}^{n_1}} d^{n_1} z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n} z \int_{\Gamma_{d_3}^{n_1}} d^{n_1} w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n} w \det\left(z_j^{i-1}\right)_{1\leq i,j\leq n_2} \det\left(w_j^{i-1}\right)_{1\leq i,j\leq n_2}$$
$$\times \prod_{j=1}^{n_1} \frac{e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{z_j^{\Delta n} w_j^{n_1}} \left(\frac{1}{z_j - w_j} - h\right) \prod_{j=n_1+1}^{n_2} \frac{e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{z_j^{\Delta n - 1} w_j^{n_1+1}(w_j - z_j)}.$$

$d_1 < d_3 < 0$ , $d_4 < d_2 < 0$ and $\Delta\xi = \xi_2 - \xi_1$ , $\Delta\mu = \mu_2 - \mu_1$.

# Limit to the Brownian directed polymer

$$Q(h) = \frac{(-1)^{\frac{n_2(n_2-1)}{2}}}{(2\pi i)^{2n_2} n_1! \Delta n!}$$
$$\times \int_{\Gamma_{d_1}^{n_1}} d^{n_1} z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n} z \int_{\Gamma_{d_3}^{n_1}} d^{n_1} w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n} w \det\left(z_j^{i-1}\right)_{1\le i,j\le n_2} \det\left(w_j^{i-1}\right)_{1\le i,j\le n_2}$$
$$\times \prod_{j=1}^{n_1} \frac{e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{z_j^{\Delta n} w_j^{n_1}} \left(\frac{1}{z_j - w_j} - h\right) \prod_{j=n_1+1}^{n_2} \frac{e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{z_j^{\Delta n - 1} w_j^{n_1+1}(w_j - z_j)}.$$

$d_1 < d_3 < 0$ , $d_4 < d_2 < 0$ and $\Delta\xi = \xi_2 - \xi_1$ , $\Delta\mu = \mu_2 - \mu_1$.

In order to understand the asymptotics of $Q(h)$ we need to rewrite it further so that we can do an appropriate expansion.

# Limit to the Brownian directed polymer

$$
\begin{aligned}
Q(h) = {} & \frac{1}{(2\pi i)^{4n_2} n_1!(\Delta n)!} \int_{\Gamma_{d_1}^{n_1}} d^{n_1} z \int_{\Gamma_{d_2}^{\Delta n}} d^{\Delta n} z \int_{\Gamma_{d_3}^{n_1}} d^{n_1} w \int_{\Gamma_{d_4}^{\Delta n}} d^{\Delta n} w \int_{\gamma_{\tau_1}^{n_2}} d^{n_2} \zeta \int_{\gamma_{\tau_2}^{n_2}} d^{n_2} \omega \\
& \times \det\left(\frac{1}{\zeta_j^i}\right)_{1\le i,j\le n_2} \det\left(\frac{1}{\omega^{n_2+1-i}}\right)_{1\le i,j\le n_2} \\
& \times \prod_{j=1}^{n_1} \frac{z_j^{n_1} w_j^{\Delta n} e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{e^{\frac{1}{2}\mu_1 \zeta_j^2 - \xi_1 \zeta_j + \frac{1}{2}\Delta\mu \omega_j^2 - \Delta\xi \omega_j} (\zeta_j - z_j)(\omega_j - w_j)} \left(\frac{1}{z_j - w_j} - h\right) \\
& \times \prod_{j=n_1+1}^{n_2} \frac{z_j^{n_1+1} w_j^{\Delta n - 1} e^{\frac{1}{2}\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2}\Delta\mu w_j^2 - \Delta\xi w_j}}{e^{\frac{1}{2}\mu_1 \zeta_j^2 - \xi_1 \zeta_j + \frac{1}{2}\Delta\mu \omega_j^2 - \Delta\xi \omega_j} (w_j - z_j)(\zeta_j - z_j)(\omega_j - w_j)},
\end{aligned}
$$

where

$$
d_1 < d_3 < -\max(\tau_1, \tau_2) < 0 \;,\; d_4 < d_2 < -\max(\tau_1, \tau_2) < 0.
$$

# Analogous simpler computation

Let

$$Df(x) = f'(x) \quad , \quad D^{-1}f(x) = \int_{-\infty}^{x} f(y)\, dy.$$

# Analogous simpler computation

Let

$$Df(x) = f'(x) \quad , \quad D^{-1}f(x) = \int_{-\infty}^{x} f(y)\,dy.$$

We have the following formula of Warren

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \int_{y_1 \leq \cdots \leq y_n \leq \eta} \det \Big( (D^{k-j}\phi_{1/2})(y_k) \Big)_{1 \leq j,k \leq n} d^n y, \qquad (1)$$

# Analogous simpler computation

Let

$$Df(x) = f'(x) \quad , \quad D^{-1}f(x) = \int_{-\infty}^{x} f(y)\, dy.$$

We have the following formula of Warren

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \int_{y_1 \leq \cdots \leq y_n \leq \eta} \det \Big( (D^{k-j}\phi_{1/2})(y_k) \Big)_{1 \leq j,k \leq n} d^n y, \qquad (1)$$

where

$$\phi_{1/2}(t) = \frac{1}{\sqrt{\pi}} e^{-t^2} = \frac{1}{\pi i} \int_{\Gamma_{-2}} e^{w^2 - 2tw} dw.$$

Here $\Gamma_{-2}$ : Im $w = -2$.

# Analogous simpler computation

We have the following formula of Warren

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \int_{y_1 \leq \cdots \leq y_n \leq \eta} \det \left( (D^{k-j} \phi_{1/2})(y_k) \right)_{1 \leq j,k \leq n} d^n y, \qquad (1)$$

where

$$\phi_{1/2}(t) = \frac{1}{\sqrt{\pi}} e^{-t^2} = \frac{1}{\pi i} \int_{\Gamma_{-2}} e^{w^2 - 2tw} dw.$$

Here $\Gamma_{-2}$ : $\mathrm{Im}\, w = -2$.
This is an alternative form for the largest eigenvalue distribution of an $n \times n$ GUE matrix.

# Analogous simpler computation

We have the following formula of Warren

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \int_{y_1 \leq \cdots \leq y_n \leq \eta} \det\left((D^{k-j}\phi_{1/2})(y_k)\right)_{1 \leq j,k \leq n} d^n y, \qquad (1)$$

We have that

$$D^{k-j}\phi_{1/2}(y_k) = \frac{(-2)^{k-j}}{\pi i} \int_{\Gamma_{-2}} w^{k-j} e^{w^2 - 2y_k w} dw.$$

Insert this formula into Warren's formula.

# Analogous simpler computation

We have the following formula of Warren

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \int_{y_1 \leq \cdots \leq y_n \leq \eta} \det\left((D^{k-j}\phi_{1/2})(y_k)\right)_{1 \leq j,k \leq n} d^n y, \qquad (1)$$

We have that

$$D^{k-j}\phi_{1/2}(y_k) = \frac{(-2)^{k-j}}{\pi i} \int_{\Gamma_{-2}} w^{k-j} e^{w^2 - 2y_k w} dw.$$

Insert this formula into Warren's formula.

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta]$$
$$= \frac{1}{(\pi i)^n} \int_{\Gamma_{-2}^n} \left(\prod_{k=1}^n e^{w_k^2} w_k^k\right) \det\left(\frac{1}{w_k^j}\right) \left(\int_{y_1 \leq \cdots \leq y_n \leq \eta} \prod_{k=1}^n e^{-2y_k w_k} d^n y\right) d^n w.$$

# Analogous simpler computation

We have that

$$D^{k-j}\phi_{1/2}(y_k) = \frac{(-2)^{k-j}}{\pi i}\int_{\Gamma_{-2}} w^{k-j}e^{w^2-2y_k w}dw.$$

Insert this formula into Warren's formula.

$$\begin{aligned}
&\mathbb{P}[H(\frac{1}{2},n)\leq \eta] \\
&= \frac{1}{(2\pi i)^n}\int_{\Gamma_{-2}^n}\left(\prod_{k=1}^n e^{w_k^2}w_k^k\right)\det\left(\frac{1}{w_k^j}\right)\left(\int_{y_1\leq\cdots\leq y_n\leq\eta}\prod_{k=1}^n e^{-2y_k w_k}d^n y\right)d^n w \\
&= \frac{(-1)^n}{(\pi i)^n}\int_{\Gamma_{-2}^n}\prod_{k=1}^n e^{w_k^2-2\eta w_k}w_k^k\frac{1}{w_1(w_1+w_2)\ldots(w_1+\cdots+w_n)}\det\left(\frac{1}{w_k^j}\right).
\end{aligned}$$

# Analogous simpler computation

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta]$$

$$= \frac{(-1)^n}{(2\pi i)^n} \int_{\Gamma^n_{-2}} \prod_{k=1}^{n} e^{w_k^2 - 2\eta w_k} w_k^k \frac{1}{w_1(w_1 + w_2) \dots (w_1 + \dots + w_n)} \det \left( \frac{1}{w_k^j} \right) d^n w.$$

Symmetrize by permuting $w_1, \dots, w_n$ and use the identity

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} w_{\sigma(k)}^k \frac{1}{w_\sigma(1)(w_\sigma(1) + w_\sigma(2)) \dots (w_\sigma(1) + \dots + w_\sigma(n))} = \det(w_k^{j-1}).$$

# Analogous simpler computation

Thus

$$\mathbb{P}[H(\frac{1}{2},n)\leq\eta]=\frac{(-1)^n}{(2\pi i)^n n!}\int_{\Gamma_{-2}^n}\prod_{k=1}^{n}e^{w_k^2-2\eta w_k}\det\left(w_k^{j-1}\right)\det\left(\frac{1}{w_k^j}\right)d^n w.$$

# Analogous simpler computation

Thus

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \frac{(-1)^n}{(2\pi i)^n n!} \int_{\Gamma_{-2}^n} \prod_{k=1}^{n} e^{w_k^2 - 2\eta w_k} \det\left(w_k^{j-1}\right) \det\left(\frac{1}{w_k^j}\right) d^n w.$$

Now,

$$\det\left(\frac{1}{w_k^j}\right) = (-1)^n \det\left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{e^{2\eta z - z^2}}{z - w_k} \frac{dz}{z^j}\right),$$

where $\gamma_1 : |z| = 1$. Insert this into the previous formula and use the Cauchy-Binet (Andrieff) identity.

# Analogous simpler computation

We get the determinantal formula

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \det\left( \frac{1}{(2\pi i)^2} \int_{\Gamma_{-2}} dw \int_{\gamma_1} dz \frac{e^{w^2 - z^2 + 2\eta(z-w)}}{z - w} \frac{w^{k-1}}{z^j} \right).$$

# Analogous simpler computation

We get the determinantal formula

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \det\left(\frac{1}{(2\pi i)^2}\int_{\Gamma_{-2}} dw \int_{\gamma_1} dz \frac{e^{w^2 - z^2 + 2\eta(z-w)}}{z - w}\frac{w^{k-1}}{z^j}\right).$$

Move $\Gamma_{-2}$ to $\Gamma_2$ (motivated by asymptotic analysis).

# Analogous simpler computation

We get the determinantal formula

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \det\left(\frac{1}{(2\pi i)^2}\int_{\Gamma_{-2}} dw \int_{\gamma_1} dz \frac{e^{w^2 - z^2 + 2\eta(z-w)}}{z - w}\frac{w^{k-1}}{z^j}\right).$$

Move $\Gamma_{-2}$ to $\Gamma_2$ (motivated by asymptotic analysis).
Since we cross the pole at $w = z$, this gives

$$\mathbb{P}[H(\frac{1}{2}, n) \leq \eta] = \det\left(\delta_{jk} + \frac{1}{(2\pi i)^2}\int_{\Gamma_2} dw \int_{\gamma_1} dz \frac{e^{w^2 - z^2 + 2\eta(z-w)}}{z - w}\frac{w^{k-1}}{z^j}\right).$$

Can be expanded in a Fredholm expansion. We can then take the limit of individual terms in the expansion.

# The two-time distribution

Recall

$$\lim_{M\to\infty} \mathbb{P}[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2] = F_{\mathrm{tt}}(\eta_1, \eta_2; \alpha).$$

where

$$\mu_i = t_i M = n_i = t_i M, \quad \xi_i = 2t_i M + \eta_i (t_i M)^{1/3},$$

$i = 1, 2$, and

$$\alpha = (t_1/(t_2 - t_1))^{1/3}.$$

# The two-time distribution

We have the formula

$$
\begin{aligned}
&F_{\mathrm{tt}}(\eta_1^*, \eta_2; \alpha) \\
&= F_2(\eta_2) - \sum_{r,s,t=0}^{\infty} \frac{1}{(r!)^2 s! t!} \\
&\times \int_{\eta_1^*}^{\infty} d\eta_1 \int_{(-\infty,0]^r} d^r x \int_{(-\infty,0]^s} d^s x' \int_{[0,\infty)^r} d^r y \int_{[0,\infty)^t} d^t y' \, W^{(1)}_{r,s,r,t}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}') \\
&- \sum_{r=1}^{\infty} \sum_{s,t=0}^{\infty} \frac{1}{r!(r-1)! s! t!} \\
&\times \int_{\eta_1^*}^{\infty} d\eta_1 \int_{(-\infty,0]^r} d^r x \int_{(-\infty,0]^s} d^s x' \int_{[0,\infty)^{r-1}} d^{r-1} y \int_{[0,\infty)^t} d^t y' \, W^{(2)}_{r,s,r-1,t}(\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}'),
\end{aligned}
$$

# The two-time distribution

Block determinant

$$W^{(1)}_{r_1,s,r_2,t}(\mathbf{x},\mathbf{x}',\mathbf{y},\mathbf{y}') = \begin{vmatrix} \psi(\mathbf{x},\mathbf{x}) & \psi(\mathbf{x},\mathbf{x}') & \psi(\mathbf{x},0) & \psi(\mathbf{x},\mathbf{y}) & \psi(\mathbf{x},\mathbf{y}') \\ \phi(\mathbf{x}',\mathbf{x}) & \phi(\mathbf{x}',\mathbf{x}') & \phi(\mathbf{x}',0) & \phi(\mathbf{x}',\mathbf{y}) & \phi(\mathbf{x}',\mathbf{y}') \\ \psi(0,\mathbf{x}) & \psi(0,\mathbf{x}') & \psi(0,0) & \psi(0,\mathbf{y}) & \psi(0,\mathbf{y}') \\ \phi(\mathbf{y},\mathbf{x}) & \phi(\mathbf{y},\mathbf{x}') & \phi(\mathbf{y},0) & \phi(\mathbf{y},\mathbf{y}) & \phi(\mathbf{y},\mathbf{y}') \\ \psi(\mathbf{y}',\mathbf{x}) & \psi(\mathbf{y}',\mathbf{x}') & \psi(\mathbf{y}',0) & \psi(\mathbf{y}',\mathbf{y}) & \psi(\mathbf{y}',\mathbf{y}') \end{vmatrix}$$

$\mathbf{x} \in \mathbb{R}^{r_1}$, $\mathbf{x}' \in \mathbb{R}^{s}$, $\mathbf{y} \in \mathbb{R}^{r_2}$, $\mathbf{y}' \in \mathbb{R}^{t}$ and $0 \in \mathbb{R}$.

$W^{(2)}_{r_1,s,r_2,t}$ is very similar ($\psi$ in middle row replaced by $\phi$).

# The two-time distribution

The functions $\phi$ and $\psi$ are defined by

$$\phi(x,y) = \phi_1(x,y) + 1(y \geq 0)\phi_2(x,y) - 1(x < 0)\phi_3(x,y),$$

and

$$\psi(x,y) = -\psi_1(x,y) - 1(y > 0)\phi_2(x,y) + 1(x \leq 0)\phi_3(x,y),$$

where

$$\phi_1(x,y) = -\alpha \int_0^\infty K_{\mathrm{Ai}}\left(\eta_1 - \tau, \eta_1 - y\right) K_{\mathrm{Ai}}\left(\Delta\eta + \alpha\tau, \Delta\eta + \alpha x\right) d\tau,$$

$$\psi_1(x,y) = \alpha \int_0^\infty K_{\mathrm{Ai}}\left(\eta_1 + \tau, \eta_1 - y\right) K_{\mathrm{Ai}}\left(\Delta\eta - \alpha\tau, \Delta\eta + \alpha x\right) d\tau,$$

$$\phi_2(x,y) = \alpha K_{\mathrm{Ai}}\left(\Delta\eta + \alpha x, \Delta\eta + \alpha y\right),$$

$$\phi_3(x,y) = K_{\mathrm{Ai}}\left(\eta_1 - x, \eta_1 - y\right).$$

and

$$\Delta\eta = \eta_2 \left(\frac{t_2}{\Delta t}\right)^{1/3} - \eta_1 \left(\frac{t_1}{\Delta t}\right)^{1/3}.$$

Thank you!