Generalized Toda hamiltonians acting on partition functions

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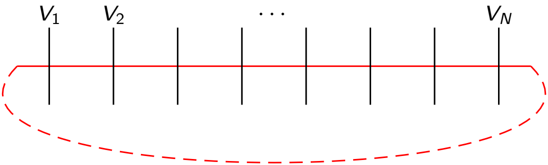
Outline

1. Combinatorial partition functions
2. Difference operators and the quantum (toroidal) algebra
3. q-Difference Hamiltonian

Combinatorial partition functions

- A set of symmetric polynomials coming from the combinatorics of the Bethe ansatz solutions of the generalized inhomogeneous XXX spin chain:



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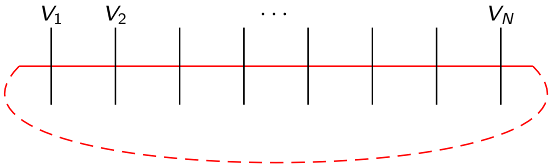
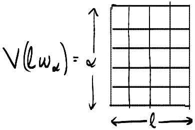


- $\{V_i\}$ are special $\mathfrak{g} = \mathfrak{sl}_{r+1}$-representations and the Hilbert space is $\mathcal{H} = \otimes V_i$.
- Combinatorial partition function $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ depends only on highest weights of $\{V_i\}$.

The functions $\chi_{\vec{\lambda}}(\mathbf{z}; q)$

- The multi-partition $\vec{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(r)})$ parameterizes the combinatorial data in the set $\{V_i\}$: $\lambda^{(\alpha)}$ is a partition with $n_\ell^{(\alpha)}$ parts of length ℓ,

$$n_\ell^{(\alpha)} := \#\{V_i = V(\ell\omega_\alpha)\}$$



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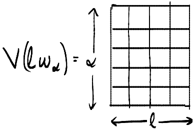

- For fixed r, $\{\chi_{\vec{\lambda}}(\mathbf{z}; q)\}$ are symmetric polynomials $\{\chi_{\vec{\lambda}}(\mathbf{z}; q)\}$ with coefficients in $\mathbb{N}[q]$, with $\mathbf{z} = (z_1, ..., z_{r+1})$.

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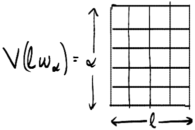

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- Each polynomial $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ is a partition function of the linearized spectrum:
  - $\chi_{\vec{\lambda}}(1; 1) = \dim \mathcal{H}$ is the dimension of the Hilbert space.
  - The coefficient of $s_\mu(\mathbf{z})$ in $\chi_{\vec{\lambda}}(\mathbf{z}; 1)$ is the dimension of the "spin sector" μ in the Bethe ansatz solution.
  - The polynomial $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ is a partition function of the linearized spectrum of the model.

Algebras and difference Hamiltonians acting on $\{\chi_{\vec{\lambda}}(\mathbf{z}; q)\}$

In this talk:

- We switch points of view: $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ are considered as states of a 1-dimensional particle system.
- Creation operators: $\chi_{\vec{\lambda}}(\mathbf{z})$ can be constructed by the action of elements in the nilpotent subalgebra of $U_v(\widehat{\mathfrak{sl}}_2)$ with $v = \sqrt{q}$. [In the polynomial rep]
- The set $\{\chi_{\vec{\lambda}}(\mathbf{z}; q)\}$ is closed under the action of q-difference Hamiltonians generalizing the q-difference quantum Toda family. (Related to Cartan currents).
- Special cases: q-Whittaker functions for $U_q(\mathfrak{sl}_{r+1})$, modified Hall-Littlewood polynomials.
- There is a natural t-deformation helps to see the structure of the algebra which generates $\chi_{\vec{\lambda}}(\mathbf{z}; q)$: Quantum toroidal algebra.

Explicit combinatorial formula for $\chi_{\vec{\lambda}}(\mathbf{z}; q)$

$$\chi_{\vec{\lambda}}(\mathbf{z}; q) = \sum_{\vec{\mu}} q^{\frac{1}{2}F(\vec{\mu})} \prod_{\alpha,i} \begin{bmatrix} p_i^{(\alpha)} + m_i^{(\alpha)} \\ m_i^{(\alpha)} \end{bmatrix}_q s_{\lambda - C\mu}(\mathbf{z})$$

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where

- The sum is over multi-partitions $\vec{\mu} = (\mu^{(1)}, ..., \mu^{(r)})$;
- $F(\vec{\mu}) = \sum \mu_i^{(\alpha)} C_{\alpha,\beta} \mu_i^{(\beta)}$, $C =$ Cartan matrix;
- $\mathbf{m} = \{m_i^{(\alpha)}\}$ with $m_i^{(\alpha)}$ the number of columns of $\mu^{(\alpha)}$ of length i.
- The integers $p_i^{(\alpha)}$: Sum over the first i columns of the composition $\lambda^{(\alpha)} - (C\vec{\mu})^{(\alpha)}$.
- $s_{\lambda - C\mu}(\mathbf{z})$ is the Schur function corresponding to $\sum_i (\lambda_i^{(\alpha)} - \sum_\beta C_{\alpha,\beta} \mu_i^{\beta}) \omega_\alpha$.

Relation to Bethe ansatz of generalized Heisenberg chain

- The polynomial $\chi_{\vec{\lambda}}(\mathbf{z}; q = 1)$ is the character of the $\mathfrak{g}$-module

$$\mathcal{H} = \otimes V(i\omega_\alpha)^{\otimes n_i^{(\alpha)}}$$

where $n_i^{(\alpha)}$ is the number of parts of $\lambda^{(\alpha)}$ of length i: The space of states of the periodic, inhomogeneous spin chain with a representation of type $V(i\omega_\alpha)$ at each site.

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  where $n_i^{(\alpha)}$ is the number of parts of $\lambda^{(\alpha)}$ of length i: The space of states of the periodic, inhomogeneous spin chain with a representation of type $V(i\omega_\alpha)$ at each site.
- When q is arbitrary, the graded character can be defined using the representation theory of $\widehat{\mathfrak{sl}}_{r+1}$ or $U_q(\widehat{\mathfrak{sl}}_{r+1})$.
- Given a solution to the BAE parameterized by a set of integers, the power of q keeps track of the sum of these integers.
- The summation is over all sets of Bethe integers corresponding to solutions of BAE.

Special case: “Level 1”

Choose all representations to be fundamental representations with highest weight ω_α for various α.

- The partitions $\lambda^{(\alpha)} = (1^{n^{(\alpha)}})$ have one column each.
- The functions $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ are polynomial versions of q-Whittaker functions.
- Satisfy q-difference version of relativistic Toda equation on the open chain of length r.

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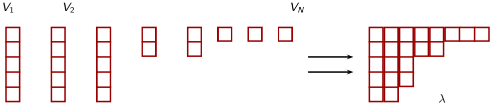
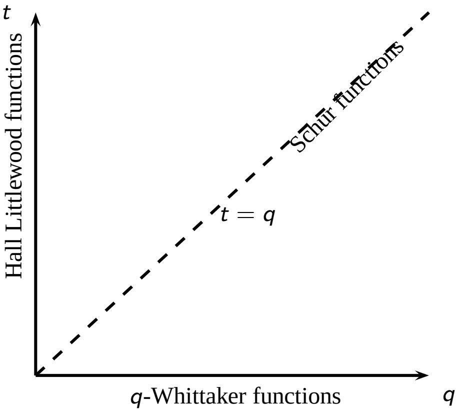
- Satisfy q-difference version of relativistic Toda equation on the open chain of length r.
- In terms of Macdonald polynomials,

$$\chi_{\vec{\lambda}}(\mathbf{z}; q) = P_\lambda(\mathbf{z}; q, 0)$$

where λ is the partition with $n_1^{(\alpha)}$ columns of length α.



Macdonald symmetric functions $P_\lambda(\mathbf{z}; q, t)$



Special case: Symmetric power representations

Take all representations V_i to be symmetric power representations, with highest weight $\ell_i\omega_1$. Only $\lambda^{(1)}$ is non-trivial, and it has $n_i^{(1)}$ rows of length i.

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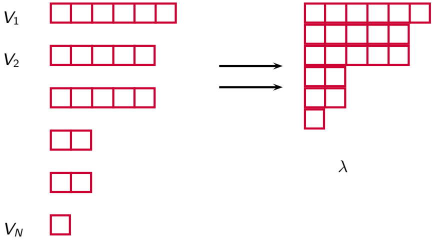
- The functions $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ are related to Hall-Littlewood symmetric functions by a plethysm.
- Satisfy q-difference Toda on the semi-infinite lattice.
- A specialization of the modified Macdonald polynomial

$$\chi_{\vec{\lambda}}(\mathbf{z}; t) = \widetilde{H}_\lambda(\mathbf{z}; 0, t).$$



Conformal field theory limit

The polynomial $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ becomes (up to normalization) the graded character of an affine $\mathfrak{sl}_{r+1}$-module of level $k \in \mathbb{N}$:

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- Take $V_i = V(k\omega_1)$ for all i.
- Take $N(r+1)$ sites in the quantum spin chain.
- There is a well-defined limit $N \to \infty$ which gives an integrable module character corresponding to the vacuum module.
- Coefficients of $s_\mu(\mathbf{z})$ are (normalized) Virasoro characters for a WZW model in CFT.

Creation operators

How to generate the symmetric polynomials $\chi_{\vec{\lambda}}(\mathbf{z};q)$ using difference operators:

- We have an operator $\chi_{\vec{\lambda}}(\mathbf{z};q) \mapsto \chi_{\vec{\lambda}'}(\mathbf{z};q)$, where the multipartition $\vec{\lambda}'$ differs from $\vec{\lambda}$ in having one more row of length k in $\lambda^{(\alpha)}$. $(n_k^{(\alpha)} \mapsto n_k^{(\alpha)} + 1)$.

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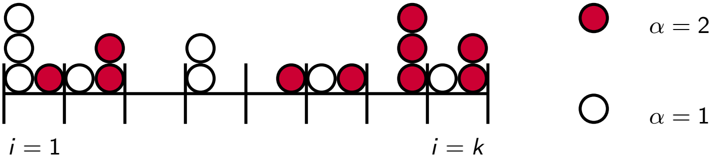
- **Theorem:** If $\lambda_1^{(\alpha)} \leq k$ for all α, adding 1 to $n_k^{(\alpha)}$ corresponds to acting with a q-difference operator on $\chi_{\vec{\lambda}}(\mathbf{z};q)$:

$$\chi_{\vec{\lambda}'}(\mathbf{z};q^{-1}) = q^{\#} M_k^{(\alpha)} \chi_{\vec{\lambda}}(\mathbf{z};q^{-1})$$

where

$$M_k^{(\alpha)} = q^{\alpha k/2} \sum_{\substack{I \subset [1,r+1] \\ |I|=\alpha}} \prod_{i \in I} z_i^k \prod_{j \notin I} \frac{z_i}{z_i - z_j} \prod_{i \in I} q^{\delta_i}.$$

Particles in one dimension picture



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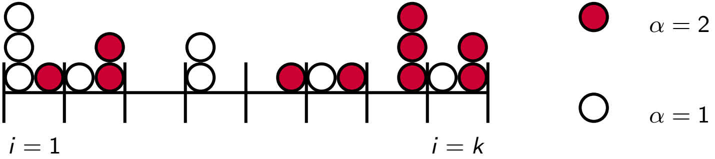

$n_1^{(1)} = 3$, $n_1^{(2)} = 1$,... and $M_k^{(\alpha)}$ creates a particle of color α in box k.

$$\chi_{\vec{\lambda}}(\mathbf{z}; q^{-1}) \propto \prod_{i=k}^{1} (M_i^{(\alpha)})^{n_i^{(\alpha)}}(1).$$

Relation to quantum affine algebra

$$M_n^{(\alpha)} = q^{n\alpha/2} \sum_{\substack{I \subset [1,r+1] \\ |I|=\alpha}} \prod_{i \in I} z_i^n \prod_{j \notin I} \frac{z_i}{z_i - z_j} \prod_{i \in I} q^{\delta_i}.$$

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- **Theorem 1:** The operators $M_k^{(\alpha)}$ with $\alpha > 1$ are polynomials of degree α (iterated q-commutators) in the generators $\{M_n := M_n^{(1)} : n \in \mathbb{Z}\}$:

$$M_n^{(\alpha)} \propto [[\cdots[M_{n-\alpha+1}, M_{n-\alpha+3}]_{q^2}, M_{n-\alpha+5}]_{q^3}, \cdots, M_{n+\alpha-1}]_{q^\alpha}.$$

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  So that $M^{(1)}$ generate the algebra of creation operators.

- **Theorem:** In terms of the generating function $e(x) = \sum_{k\in\mathbb{Z}} M_k x^k$,

$$M_n^{(\alpha)} = \frac{1}{n!} CT_{x_1,\ldots,x_\alpha}\left[\left(\prod_{1\le i<j\le\alpha} \frac{(x_i - x_j)(x_i - qx_j)}{x_i x_j}\right) \frac{e(x_1)\cdots e(x_\alpha)}{(x_1\cdots x_\alpha)^n}\right]$$

Quadratic relations

- **Theorem 2:** The currents

$$e(x) = \sum_n z^n M_n$$

satisfy the exchange relation

$$\frac{x - qy}{x - y} e(x)e(y) = \frac{y - qx}{y - x} e(y)e(x).$$

Same as the Drinfeld generators of the nilpotent subalgebra of the quantum affine algebra of $\mathfrak{sl}_2$, $U_{\sqrt{q}}(\widehat{\mathfrak{n}}_+)$.

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- Warning: The algebras are not isomorphic: For $r < \infty$ fixed, there is one further relation:

$$M_n^{(r+2)} = 0.$$

- **Next:** What is the role of the rest of the quantum affine algebra?

Ding-Iohara or Drinfeld algebras

The Drinfeld presentation of the quantum affine algebra at level 0 has the form of commutations of generating functions $\psi^{\pm}(z), x^{\pm}(z)$, where $\psi^{\pm}(z)$ are commuting power series in $z^{\pm 1}$, and relations

$$\begin{aligned} g^{\epsilon\epsilon'}(z,w)\psi^{\epsilon}(z)x^{\epsilon'}(w) &= g^{\epsilon\epsilon'}(w,z)x^{\epsilon'}(w)\psi^{\epsilon}(z) \\ g^{\epsilon}(z,w)x^{\epsilon}(z)x^{\epsilon}(w) &= g^{\epsilon}(w,z)x^{\epsilon}(w)x^{\epsilon}(z) \\ \left[x^{+}(z),x^{-}(w)\right] &= \frac{\delta(z/w)}{g(1,1)}(\psi^{+}(z)-\psi^{-}(z)) \\ &+ \text{Serre} \end{aligned}$$

where $\epsilon, \epsilon' \in \pm 1$.

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- If $g(z,w) = \frac{z-q^2 w}{z-w}$ this is the Drinfeld presentation of $U_q(\widehat{sl}_2)$ at level 0.
- If $g(z,w) = \frac{(z-qw)(z-t^{-1}w)(z-tq^{-1}w)}{z-w}$, this is currently called the quantum toroidal algebra of $\widehat{\mathfrak{gl}}_1$

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- Let $\theta = \sqrt{t}$ with $t \in \mathbb{C}$ and define

$$M_k^{(\alpha)}(\mathbf{z}; q, t) := q^{\alpha k/2} \sum_{\substack{I \subset [1,r+1] \\ |I|=\alpha}} \prod_{i \in I} z_i^k \prod_{j \notin I} \frac{\theta z_i - z_j/\theta}{z_i - z_j} \prod_{i \in I} q^{\delta_i}.$$

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- When $k = 0$ these are Macdonald operators.
- When $t \to \infty$, we get the generators $M_k^{(\alpha)}$ from the previous slide.
- **Theorem 1':** The operators $M_k^{(\alpha)}(\mathbf{z}; q, t)$ are polynomials in the generators $M_n := M_n^{(1)}(\mathbf{z}; q, t)$.

Exchange relations for the quantum toroidal algebra

- **Theorem 2':** The generating functions

$$e(x) := \sum_{k\in\mathbb{Z}} M_k(\mathbf{z}; q, t) x^k$$

satisfy the exchange relation

$$g(x,y)e(x)e(y) = g(y,x)e(y)e(x),$$

where $g(x,y) = \frac{(x-qy)(x-t^{-1}y)(x-q^{-1}ty)}{x-y}$, plus Serre.

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- These would generate the "positive part" of the quantum toroidal algebra.
- Warning: The algebra depends on r, it is a quotient by a (q,t) quantum determinant of "size" $r+2$ (automatically satisfied when there is only a finite number of variables z_i).

Other currents in the quantum toroidal algebra

We have a difference operator realization of

$$x^+(z) := \frac{q^{1/2}}{1-q} e(z; q, t)$$

in the (quotient of) the quantum toroidal algebra. Where can we find the negative currents $x^-(z)$ and the Cartan currents $\psi^\pm(z)$?

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Answer: For r finite, construct the difference operator

$$x^-(z) = \frac{q^{-1/2}}{1-q^{-1}} e(z; q^{-1}, t^{-1})$$

and define $\psi^\pm(w)$ from $[x^+(z), x^-(w)]$:

$$\psi^\pm(w) = \prod_{i=1}^{r+1} \frac{(1 - q^{-\frac{1}{2}} t (z_i w)^{\pm 1})(1 - q^{\frac{1}{2}} t^{-1} (z_i w)^{\pm 1})}{(1 - q^{-\frac{1}{2}} (z_i w)^{\pm 1})(1 - q^{\frac{1}{2}} (z_i w)^{\pm 1})}$$

Limit $t \to \infty$

In the limit $t \to \infty$, $x^{\pm}(z) \sim \theta^r$,

$$[x^+(z), x^-(w)] = \frac{\delta(z/w)}{g(1,1)}(\psi^+(w) - \psi^-(w))$$

scales as t^r and $\psi^{\pm}(w)$ as t^{r+1}. The limit of the rescaled Cartan current is

$$\psi^{\pm}(w;q) = q^{-\frac{r+1}{2}} \prod_{i=1}^{r+1} \frac{(z_i w)^{\pm 1}}{(1 - q^{\frac{1}{2}}(z_i w)^{\pm 1})(1 - q^{-\frac{1}{2}}(z_i w)^{\pm 1})}$$

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- The non-zero coefficients of each $w^n\ \psi^+(w;q)$ can be expressed in terms of elementary symmetric functions.
- Acting on $\chi_{\vec{\lambda}}(\mathbf{z};q)$ with $\psi^+(w;q)$ gives Pieri-type rules.

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In the limit $t \to \infty$, $x^{\pm}(z) \sim \theta^r$,

$$[x^+(z), x^-(w)] = \frac{\delta(z/w)}{g(1,1)}(\psi^+(w) - \psi^-(w))$$

scales as t^r and $\psi^{\pm}(w)$ as t^{r+1}. The limit of the rescaled Cartan current is

$$\psi^{\pm}(w;q) = q^{-\frac{r+1}{2}} \prod_{i=1}^{r+1} \frac{(z_i w)^{\pm 1}}{(1 - q^{\frac{1}{2}}(z_i w)^{\pm 1})(1 - q^{-\frac{1}{2}}(z_i w)^{\pm 1})}$$

- Note that the current has valuation $w^{\pm(r+1)}$.
- The non-zero coefficients of each $w^n\ \psi^+(w;q)$ can be expressed in terms of elementary symmetric functions.
- Acting on $\chi_{\vec{\lambda}}(\mathbf{z};q)$ with $\psi^+(w;q)$ gives Pieri-type rules.
- The resulting difference equations for $\chi_{\vec{\lambda}}(\mathbf{z};q)$ can be read as (commuting) q-difference Hamiltonians acting on the partition functions.

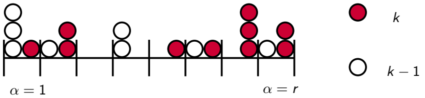
Difference equations: Level-k

Fix $k \in \mathbb{N}$ and

- Consider the special case of the function $\chi_{\mathbf{m},\mathbf{n}}(\mathbf{z}; q)$ for the Hilbert space

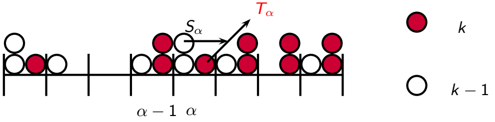
$$\bigotimes_{\alpha} V((k-1)\omega_\alpha)^{\otimes m_\alpha} \otimes V(k\omega_\alpha)^{\otimes n_\alpha}$$

- Interpret each polynomial $\chi_{\mathbf{m},\mathbf{n}}(\mathbf{z}; q)$ as a wave function of a finite chain with
  - An infinite numbers of particles at sites 0 and $r+1$;
  - m_α particles of color $k-1$ at site α,
  - n_α particles of color k at site α.



Hamiltonian

Let S_α denote the hopping of a particle of color $k-1$ from site α to $\alpha+1$; Let T_α be the same for particles of color k.



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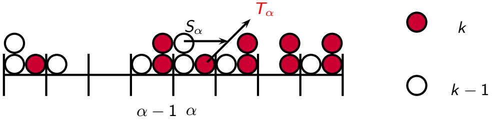


Theorem: The Hamiltonian

$$H = \sum_{\alpha=1}^{r+1} S_{\alpha-1}^{-1} T_{\alpha-1} - q^{k-1} \sum_{\alpha=1}^{r} q^{-(k-1)m_\alpha - kn_\alpha} S_{\alpha-1}^{-1} T_\alpha$$

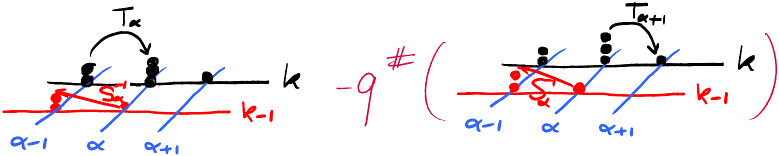
acts on $\chi_{\mathbf{m},\mathbf{n}}(\mathbf{z};q)$ as

$$H\chi_{\mathbf{m},\mathbf{n}}(\mathbf{z};q) = e_1(\mathbf{z})\chi_{\mathbf{m},\mathbf{n}}(\mathbf{z};q),$$

where $e_1(\mathbf{z})$ is the elementary symmetric function in $r+1$ variables.

The Hamiltonian

$$H = \sum_{\alpha=1}^{r+1} S_{\alpha-1}^{-1} T_{\alpha-1} - q^{k-1} \sum_{\alpha=1}^{r} q^{-(k-1)m_\alpha - kn_\alpha} S_{\alpha-1}^{-1} T_\alpha$$



Special case: q-difference Toda

The level-k difference Hamiltonian can be read as a generalization of the q-difference Toda: When $k = 1$, there is no action of S_α, and the Hamiltonian simplifies:

$$H = \sum_{\alpha=1}^{r} (1 - q^{-n_\alpha}) T_\alpha + T_0.$$

We call this a quantum Toda Hamiltonian: A specialization of Etingof's hamiltonian acting on q-Whittaker functions corresponding to $U_v(\mathfrak{sl}_{r+1})$ with $v^2 = q^{-1}$.

Special case 2: Symmetric power representations

In the case where

$$\mathcal{H} = \otimes V(\ell\omega_1)^{n_\ell}$$

(so that the eigenfunctions are modified Hall-Littlewood polynomials) we have another q difference Toda:

$$\lambda = (1^{n_1}, ..., k^{n_k}).$$

We interpret χ_λ to be the wave function for n_i particles at site $i \in \mathbb{N}$ and an infinite number of particles at site 0.

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Define S_i to be the hopping term from i to $i+1$ and H is the Toda Hamiltonian on the semi-infinite line:

$$H = S_0 + \sum_{i=1}^{\infty}(1 - q^{n_i})S_i.$$

$$H\overline{\chi}_\lambda(\mathbf{z}; q^{-1}) = e_1(\mathbf{z})\overline{\chi}_\lambda(\mathbf{z}; q^{-1})$$

where $\overline{\chi}_\lambda = M_k^{n_k} \cdots M_1^{n_1}(1)$.

The "AGT" philosophy

- The function $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ can be interpreted in two ways: As a graded counting function ("partition function") or as a wave function.
- The algebra acting on $\chi_{\vec{\lambda}}(\mathbf{z}; q)$ is the $t \to \infty$ limit of the quantum toroidal algebra.
- The quantum toroidal algebra enters the AGT correspondence: Instanton counting partition function of supersymmetric 5-dimensional gauge theory vs. inner product of deformed Gaiotto states, degenerate Whittaker vectors for the q, t-Virasoro algebra (a subalgebra).
- The 4-dimensional theory corresponds to the rational (Yangian) degeneration of the quantum toroidal algebra: $t = q^{\beta}$, $q \to 1$.
- Here we have a different limit, $t \to \infty$, which is a quantum algebra.
- Major difference: Finite rank r gives us a quotient of the q.t. algebra.