z-measures and the non-linear Luttinger liquid

KITP, February 2016

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(with Tom Price and Dima Kovrizhin)
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Discussions: Steve Simon

The spectral function

We'd like to calculate

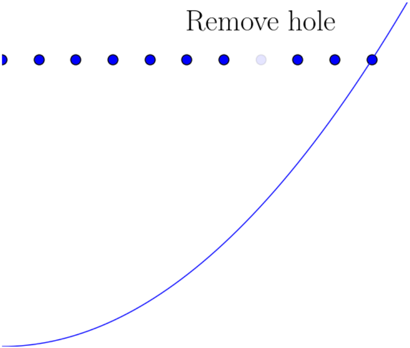
$$A(p,\omega) = \sum_{\lambda} | \langle \lambda | \hat{\psi}_p | 0 \rangle |^2 \delta(\omega - E_\lambda)$$

- Remove a particle of momentum p from ground state
- Create excited state of $N-1$ particle system $|\lambda\rangle$
- $A(p,\omega)$ is energy and momentum resolved rate (Golden Rule)
- Example: 1D Fermi gas

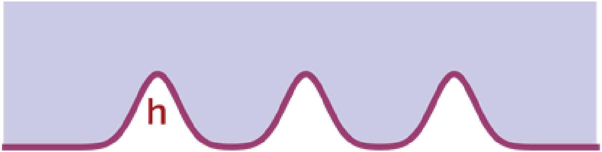
$$A(p,\omega) = \theta(-\omega)\delta(\omega - \xi(p))$$

The spectral function

$$A(p,\omega) = \theta(-\omega)\delta(\omega - \xi(p))$$



Spectral function of a FQHE edge



- Prediction of *Chiral Luttinger Liquid* (χLL) theory

$$A(p,\omega) \propto \omega^{\nu^{-1}-1}\delta(\omega - cp)\theta(-\omega)$$

Wen (1990)

- Spectral function still has δ-support

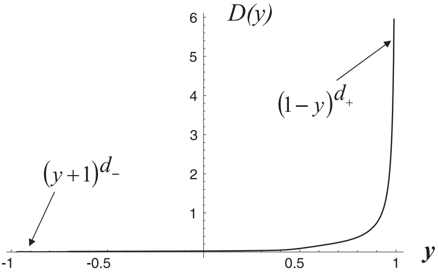
Recent developments

- χLL theory has degeneracies that will be generically lifted

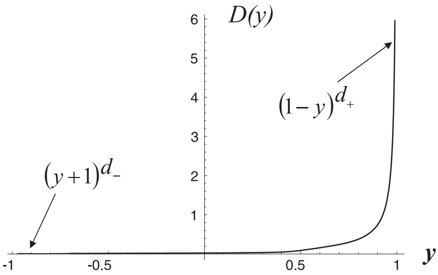
  "fine structure"

- Corrections are *universal* — Imambekov & Glazman, 2009

$$A(p, \omega) \propto D\left(\frac{\omega - cp}{p^2}\right)$$



Recent developments



- Depends on a single parameter δ_+

$$d_+ = \left(\frac{\delta_+}{2\pi}\right)^2 - 1, \qquad d_- = \left(2 - \frac{\delta_+}{2\pi}\right)^2 - 1$$

- Full function: only numerical evaluation available so far!

Recent developments

Basic approach:

- Low energy spectrum of 1D quantum fluid is (free) *fermionic* (phenomenology + exact solutions)
- Spectral function

$$A(p,\omega)=\sum_{\lambda}|\langle\lambda|\hat{\psi}_p|0\rangle|^2\delta(\omega-E_\lambda)$$

fixed by *measure* $|\langle\lambda|\hat{\psi}_p|0\rangle|^2$ on fermionic states

Outline

Bosonized viewpoint: $\hat{\psi}$ as vertex operator

Ground state of Fermi gas is Vandermonde determinant

$$|0\rangle = \Delta_N \equiv \prod_{j<k}^{N} (z_j - z_k)$$

$$z_i = e^{i\theta_i}$$

(after appropriate boost)

Remove particle at Z

$$\hat{\psi}(Z)\,|0\rangle = \prod_{i}^{N-1} (Z - z_i)\Delta_{N-1}$$

$$\prod_{i}^{N-1} (Z - z_i) = Z^{N-1} \exp\left(\sum_i \log\left[1 - z_i/Z\right]\right) = \exp\left(-\sum_n p_n Z^{-n}\right)$$

$$p_n \equiv \textstyle\sum_j z_j^n$$

Bosonized viewpoint: $\hat{\psi}$ as vertex operator

Define chiral boson field

$$\phi(z) = \sum_{k>0} \left[p_{-k} z^k - p_k z^{-k}\right] = \phi^+(z) + \phi^-(z)$$

where $p_{-k} = p_k^\dagger$ and

$$[p_k, p_l] = k\delta_{k+l,0}$$

$$\psi(z) = \exp\left[\phi(z)\right], \qquad \psi^\dagger(z) = \exp\left[-\phi(z)\right]$$

reproduce correct algebra and

$$\hat{\psi}(Z)\left|0\right\rangle = \prod_i^{N-1}(Z - z_i)\Delta_{N-1} = e^{\phi(Z)}\left|0\right\rangle = e^{\phi^-(Z)}\left|0\right\rangle$$

Interacting systems

To summarize 40+ years' work...

- Low energy properties of 1D Quantum Fluid are described by

$$\psi(Z) \to \exp\left[\eta\phi(z)\right]$$

- Giving density matrix

$$\langle\psi^\dagger(\bar{z}_1)\psi(z_2)\rangle = (1 - \bar{z}_1 z_2)^{-\eta^2}$$

- η may be related to observables, or calculated in exactly soluble models

Dynamics: linear dispersion

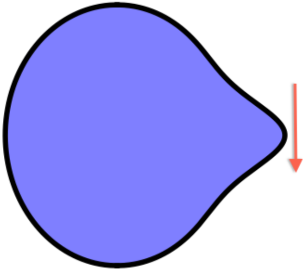
Simplest Hamiltonian

$$H_2 = \frac{\omega}{2}\int_0^{2\pi} j(\theta)^2 \frac{d\theta}{2\pi}$$

- Equation of motion

$$\partial_t j(\theta, t) = i\left[H_2, j(\theta, t)\right] = -\omega\partial_\theta j(\theta, t)$$

- FQHE: any disturbance to the edge just rotates at ω



Dynamics: linear dispersion

$$\langle \psi^\dagger(\bar{z}_1)\psi(z_2)\rangle = (1-\bar{z}_1 z_2)^{-\eta^2}$$

with dynamics given by

$$H_2 = \frac{\omega}{2}\int_0^{2\pi} j(\theta)^2 \frac{d\theta}{2\pi}$$

recovers Wen's result

$$A(p,\omega) \propto \omega^{\nu^{-1}-1}\delta(\omega - cp)\theta(-\omega)$$

Quadratic dispersion: nonlinear Luttinger Liquid

Phenomenological Hamiltonian

$$H_3 \propto \frac{1}{3}\int_0^{2\pi} j(\theta)^3 \frac{d\theta}{2\pi}$$

Hamiltonian is of free fermion form

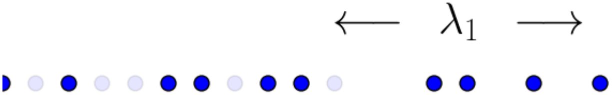
$$H_3 \propto \int \frac{d\theta}{2\pi} \partial_\theta \psi^\dagger(\theta) \partial_\theta \psi(\theta) \cdots$$

Note that these are *not* fundamental fermions!
Although spectrum is simple, *spectral function* is not!

Outline

Slater determinants

- Eigenstates of free fermion Hamiltonian



$$\Psi_\lambda = \det \begin{pmatrix} z_1^{\lambda_1+N-1} & z_2^{\lambda_1+N-1} & \cdots & z_N^{\lambda_1+N-1} \\ z_1^{\lambda_2+N-2} & z_2^{\lambda_2+N-2} & \cdots & z_N^{\lambda_2+N-2} \\ \cdots & \cdots & \cdots & \cdots \\ z_1^{\lambda_N} & z_2^{\lambda_N} & \cdots & z_N^{\lambda_N} \end{pmatrix}$$

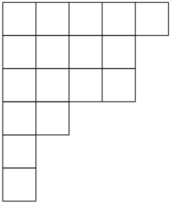
$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_N$$

- $s_\lambda = \Psi_\lambda / \Delta$ are *Schur polynomials*
- Momentum and energy

$$P = \frac{2\pi}{L} \sum_j \lambda_j, \qquad \mathcal{E}_2 \sim \left(\frac{2\pi}{L}\right)^2 \sum_j (\lambda_j - j + N)^2$$

Free fermions and partitions

- A partition λ is represented by a *Young diagram*



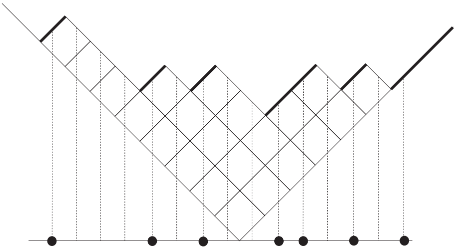
$$\lambda = (5, 4, 4, 2, 1, 1)$$

- Size of partition is denoted $|\lambda|$ (17 here)
- Momentum $P = \frac{2\pi}{L}|\lambda|$
- (i, j) denotes box in row i and column j

Free fermions and partitions

$$\Psi_\lambda = \det \begin{pmatrix} z_1^{\lambda_1+N-1} & z_2^{\lambda_1+N-1} & \dots & z_N^{\lambda_1+N-1} \\ z_1^{\lambda_2+N-2} & z_2^{\lambda_2+N-2} & \dots & z_N^{\lambda_2+N-2} \\ \dots & \dots & \dots & \dots \\ z_1^{\lambda_N} & z_2^{\lambda_N} & \dots & z_N^{\lambda_N} \end{pmatrix}$$

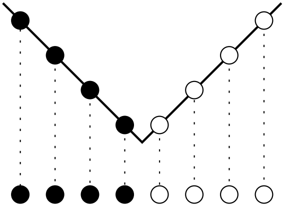
- Read off the fermionic occupancies from the Young diagram

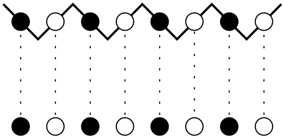


- Diagonal coordinates $\{\lambda_j - j + N\}$ give momenta of fermions

Free fermions and partitions

Examples





Fermion basis

Effect of annihilation is

$$\exp(\zeta\phi(Z))\,|0\rangle = \prod_j (Z - z_j)^\zeta \Delta$$

Need to express $\prod_j (Z - z_j)^\zeta$ in terms of Schurs

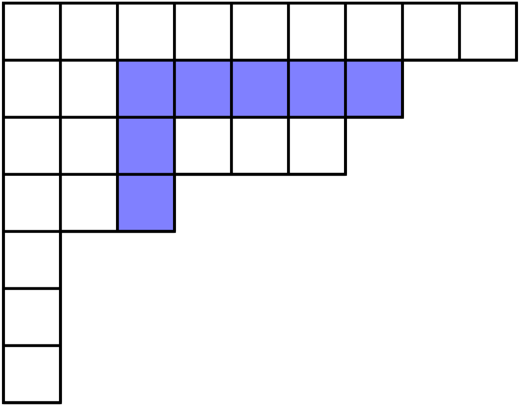
$$\prod_{i=1}^{N-1} (Z - z_i)^\zeta = \sum_\lambda a_\lambda(Z,\zeta) s_\lambda(z)$$

Result is

$$a_\lambda(Z,\zeta) = Z^{(N-1)\zeta - |\lambda|} \prod_{\square\in\lambda} \frac{c(\square) - \zeta}{h(\square)}$$

Anatomy of a partition

- $h(i, j)$ denotes the *hook length* associated with box (i, j)



Anatomy of a partition

$c(\square) = j - i$ denotes the *content* of box $\square = (i, j)$

0	1	2	3
−1	0	1	2
−2	−1	0	
−3			

$$a_\lambda(Z, \eta) = Z^{(N-1)\eta - |\lambda|} \prod_{\square \in \lambda} \frac{c(\square) - \eta}{h(\square)}$$

Form factors from Cauchy identity

Fundamental identity in symmetric function theory

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \prod_{i,j}(1-x_i y_j)^{-1}$$

Now note

$$\prod_{i=1}^{N-1}(Z-\tilde{z}_i)^{-m} = Z^{-(N-1)m}\sum_\lambda s_\lambda(\overbrace{Z^{-1},Z^{-1},\ldots,}^{m\text{ times}}0,\ldots)s_\lambda(z_i)$$

$$= Z^{-(N-1)m}\sum_\lambda Z^{-|\lambda|}s_\lambda(\overbrace{1,1,\ldots,}^{m\text{ times}}0,\ldots)s_\lambda(z_i)$$

(From homogeneity of Schurs)

$$s_\lambda(\overbrace{1,1,\ldots,}^{m\text{ times}}0,\ldots) = \prod_{\square\in\lambda}\frac{m+c(\square)}{h(\square)}.$$

'Generalized binomial coefficient' depends on *shape* of partition λ

Form factors from Cauchy identity

Thus

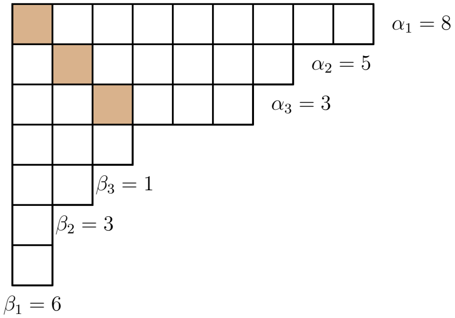
$$\prod_{i=1}^{N-1} (Z - z_i)^{\zeta} = \sum_{\lambda} a_{\lambda}(Z, \zeta) s_{\lambda}(z)$$

Where

$$a_{\lambda}(Z, \eta) = Z^{(N-1)\eta - |\lambda|} \prod_{\square \in \lambda} \frac{c(\square) - \eta}{h(\square)}$$

Frobenius coordinates

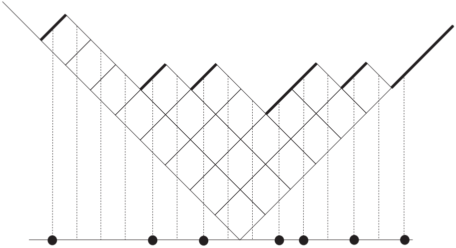
$$\lambda = (9, 7, 6, 3, 2, 1, 1) = (8, 5, 3|6, 3, 1)$$



Modified Frobenius coordinates

$$\{x_i\} = \left\{\alpha_1 + \frac{1}{2}, \dots, \alpha_d + \frac{1}{2}, -\beta_1 - \frac{1}{2}, \dots, -\beta_d - \frac{1}{2}\right\} \in \mathbb{Z}' = \mathbb{Z} - \frac{1}{2},$$

Positions of particles *above* and holes *below* the Fermi surface



Form factors in terms of Frobenius coordinates

$$\langle\lambda|\exp\left[\eta\phi(Z)\right]|0\rangle = Z^{(N-1)\eta-|\lambda|}\prod_{\square\in\lambda}\frac{c(\square)-\eta}{h(\square)}$$

z-measures are *determinantal point processes*

- The probability of a configuration $\{x_i\}$ of N points is

$$\mathbb{P}(\{x_i\}) = \frac{\det\left[L(x_i, x_j)\right]_{i,j=1}^{N}}{\det\left[1 + L\right]}$$

for known L, depending on z, z'

- n-point correlation function at points $\{y_i\}$ is likewise

$$\det\left[K(y_i, y_j)\right]_{i,j=1}^{n}$$

where

$$K = L(1 + L)^{-1}$$

is *resolvent* of L

Continuum limit (large partitions)

- $L(x,y) \to \mathcal{L}(x,y)$, $x, y \in \mathbb{R}$

$$\mathcal{L}(x,y) = \begin{cases} 0 & xy > 0 \\ \frac{|\sin \pi z|}{\pi} \frac{(x/|y|)^{\mathrm{Re}\, z} e^{(y-x)/2}}{x-y} & x > 0, y < 0 \\ \frac{|\sin \pi z|}{\pi} \frac{(y/|x|)^{\mathrm{Re}\, z} e^{(x-y)/2}}{x-y} & x < 0, y > 0 \end{cases}$$

- Corresponding resolvent $\mathcal{K}$ known explicitly: *Whittaker kernel*

Borodin (1998)

"Orthogonality Catastrophe"

VOLUME 18, NUMBER 24 PHYSICAL REVIEW LETTERS 12 JUNE 1967

INFRARED CATASTROPHE IN FERMI GASES WITH LOCAL SCATTERING POTENTIALS

P. W. Anderson
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received 27 March 1967)

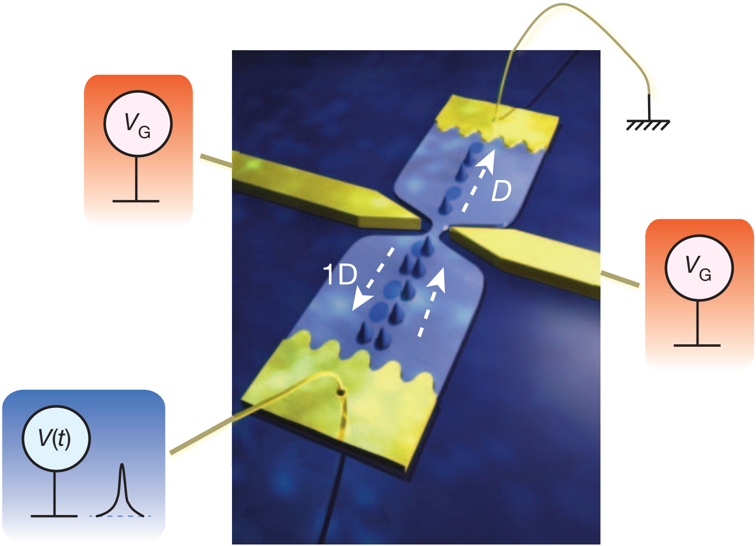
We prove that the ground state of a system of N fermions is orthogonal to the ground state in the presence of a finite range scattering potential, as $N\to\infty$. This implies that the response to application of such a potential involves only emission of excitations into the continuum, and that certain processes in Fermi gases may be blocked by orthogonality in a low-T, low-energy limit.

Distribution of particles (α_i) and holes $(\{\beta_i\})$ obeys

$$\lim_{k\to\infty}\alpha_k^{1/k}=\lim_{k\to\infty}\beta_k^{1/k}=\exp\left(-\frac{\pi^2}{\sin^2\pi\zeta}\right)$$

Borodin & Olshanski, 1998

Voltage pulses and counting statistics



Voltage pulses and counting statistics

PRL **97,** 116403 (2006) PHYSICAL REVIEW LETTERS week ending 15 SEPTEMBER 2006

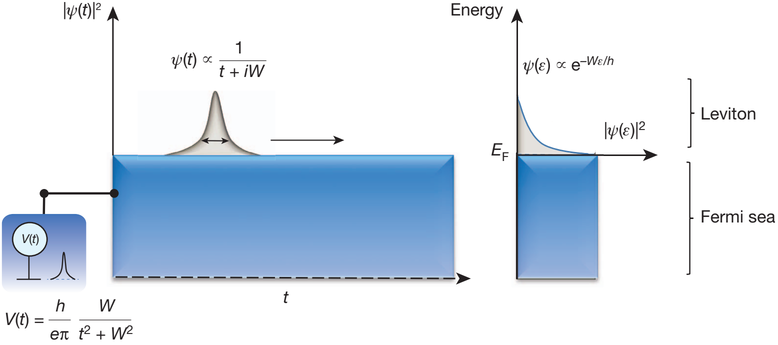
Minimal Excitation States of Electrons in One-Dimensional Wires

J. Keeling,[1] I. Klich,[2] and L. S. Levitov[1]
[1]*Department of Physics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, Massachusetts 02139, USA*
[2]*Department of Physics, California Institute of Technology, Pasadena, California 91125, USA*
(Received 1 April 2006; published 14 September 2006)

A strategy is proposed to excite particles from a Fermi sea in a noise-free fashion by electromagnetic pulses with realistic parameters. We show that by using quantized pulses of simple form one can suppress the particle-hole pairs which are created by a generic excitation. The resulting many-body states are characterized by one or several particles excited above the Fermi surface accompanied by no disturbance below it. These excitations carry charge which is integer for noninteracting electron gas and fractional for Luttinger liquid. The operator algebra describing these excitations is derived, and a method of their detection which relies on noise measurement is proposed.

Minimal excitation pulse = *Leviton*

Voltage pulses and counting statistics



Dubois *et al.*, 2013

Voltage pulse corresponds to vertex operator

$$\exp\left[\int d\theta \varphi(\theta)\phi(\theta)\right], \qquad \varphi(t) = e\int^t V(t')dt'$$

Thus usual $\exp[\phi(Z)]$ for $|Z| < 1$ is *Lorentzian*

Outline

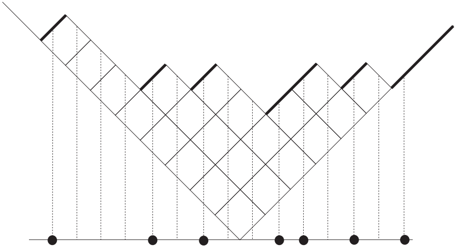
Result for spectral function

$$A(p,\omega) = \sum_{|\lambda|=p} \delta(\omega - \mathcal{E}_\lambda) \left| \binom{\eta}{\lambda} \right|^2$$

With

$$\binom{\eta}{\lambda} = \prod_{(i,j)\in\lambda} \frac{\eta + j - i}{h(i,j)}$$

and energy $\mathcal{E}_\lambda = \sum_j (\lambda_j - j)^2$



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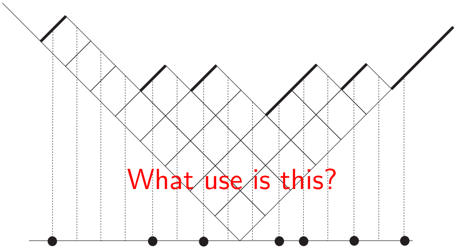


Example 1: The Linear χLL

Consider density matrix

$$\langle\psi^\dagger(\bar{z}_1)\psi(z_2)\rangle = \sum_\lambda \langle 0|\psi^\dagger(z_1)|\lambda\rangle \langle\lambda|\psi(z_2)|0\rangle$$
$$= \sum_\lambda (\bar{z}_1 z_2)^{|\lambda|} \left|\binom{\eta}{\lambda}\right|^2$$

Recall

$$s_\lambda(\overbrace{1,1,\ldots}^{m \text{ times}},0,\ldots) = \prod_{\square\in\lambda} \frac{m+c(\square)}{h(\square)},$$

and apply Cauchy to give

$$\langle\psi^\dagger(\bar{z}_1)\psi(z_2)\rangle = (1-\bar{z}_1 z_2)^{-\eta^2}$$

With *linear* dispersion

$$\mathcal{E}_\lambda = \sum_j (\lambda_j - j) = |\lambda| + \text{const.},$$

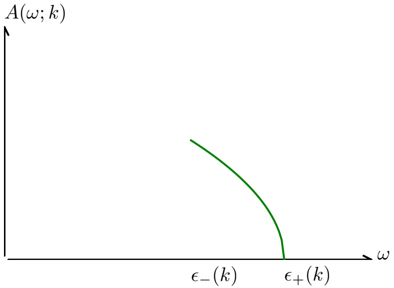
this coincides with Wen's result by Lorentz invariance

Example 2: $\nu = 1/4$

Recall $\eta = \nu^{-1/2}$

$$\binom{\eta}{\lambda} = \prod_{(i,j)\in\lambda} \frac{\eta + j - i}{h(i,j)}$$

- For $\nu = 1/4$, $j \leq 2$. Only two columns
- Analytic calculation is possible



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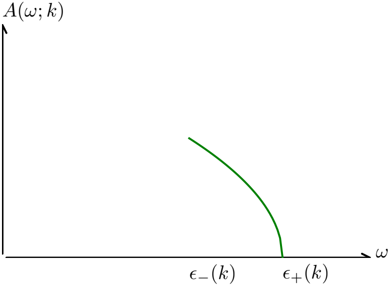
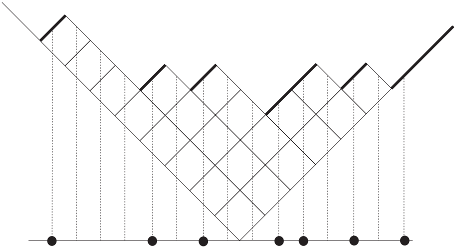


What about the general case?

Example 3: Close to threshold



Have to satisfy energy and momentum conservation

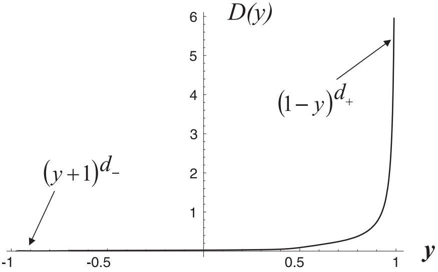
$$P = \frac{2\pi}{L} \sum_j (\lambda_j - j)\,, \qquad \mathcal{E} = \left(\frac{2\pi}{L}\right)^2 \sum_j (\lambda_j - j)^2$$

- To get $P \sim O(1)$ requires $\lambda_j, j \sim O(L^{1/2})$
- But then $\mathcal{E} \sim O(L^{-1/2})$

Partition gets a long ($O(L)$) leg or arm (quantum impurity)

Example 3: Close to threshold

- Amputation of leg or arm shifts $\zeta \to \zeta \pm 1$



- Depends on a single parameter δ_+

$$d_+ = \left(\frac{\delta_+}{2\pi}\right)^2 - 1, \qquad d_- = \left(2 - \frac{\delta_+}{2\pi}\right)^2 - 1$$

$$\zeta = 1 + \frac{\delta}{2\pi}$$

z-measures on partitions

- Fourier transform of spectral function

$$A(p,t) = \sum_{|\lambda|=p} \left| \binom{\eta}{\lambda} \right|^2 e^{-i\mathcal{E}_\lambda t}$$

- View this as generating function of $\mathcal{E}_\lambda$ for some *measure*

$$A(p,t) = \mathbb{E}\left[e^{-i\mathcal{E}_\lambda t} \right]$$

- Example of *z-measure*[1]

$$M_{z,z'}(\lambda) = \mathcal{N}_{z,z'}(\lambda) \prod_{(i,j)\in\lambda} \frac{(j-i+z)(j-i+z')}{h^2(i,j)},$$

$$z = z' = \eta$$

- ... many nice properties

[1] Borodin, Kerov, Olshanski, Vershik,...

z-measures are *determinantal point processes*

- The probability of a configuration $\{x_i\}$ of N points is

$$\mathbb{P}(\{x_i\}) = \frac{\det\left[L(x_i, x_j)\right]_{i,j=1}^{N}}{\det\left[1+L\right]}$$

for known L, depending on z, z'

- n-point correlation function at points $\{y_i\}$ is likewise

$$\det\left[K(y_i, y_j)\right]_{i,j=1}^{n}$$

where

$$K = L(1+L)^{-1}$$

is *resolvent* of L

Continuum limit (large partitions)

- $L(x,y) \to \mathcal{L}(x,y)$, $x, y \in \mathbb{R}$

$$\mathcal{L}(x,y) = \begin{cases} 0 & xy > 0 \\ \frac{|\sin \pi z|}{\pi} \frac{(x/|y|)^{\operatorname{Re} z} e^{(y-x)/2}}{x-y} & x > 0, y < 0 \\ \frac{|\sin \pi z|}{\pi} \frac{(y/|x|)^{\operatorname{Re} z} e^{(x-y)/2}}{x-y} & x < 0, y > 0 \end{cases}$$

- Corresponding resolvent $\mathcal{K}$ known explicitly: *Whittaker kernel*

Borodin (1998)

Continuum limit (large partitions)

- Spectral function takes form of Fredholm determinant[2]

$$A(p,t) = \mathbb{E}\left[e^{-i\mathcal{E}_\lambda t}\right] = \frac{\det(1+e^{-i\mathcal{E}_x t}\mathcal{L})}{\det(1+\mathcal{L})}$$

$$\mathcal{E}_x = x^2 \operatorname{sgn}(x)$$

- Expressed as solution to matrix Riemann–Hilbert problem

[2] Bettelheim, Abanov and Wiegmann (2006)

Conclusions

- Spectral function of FQHE edge in harmonic + quartic trap coincides with Imambekov–Glazman universal $D(y)$ function[3].

[3] Subject to assumption on absence of BO term

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[3] Subject to assumption on absence of BO term

Conclusions

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- $D(y)$ can be expressed in terms of Fredholm determinant with integrable kernel.

[3] Subject to assumption on absence of BO term

Conclusions

- Spectral function of FQHE edge in harmonic + quartic trap coincides with Imambekov–Glazman universal $D(y)$ function[3]. What is this function?
- $D(y)$ can be expressed in terms of Fredholm determinant with integrable kernel.
- Connection to asymptotic form factor calculations in exactly soluble models by Kitanine, Kozlowski, Maillet, N. Slavnov, and V. Terras: see recent work of Kozlowski and Maillet, arXiv:1501.07711

[3] Subject to assumption on absence of BO term

Conclusions

- Spectral function of FQHE edge in harmonic + quartic trap coincides with Imambekov–Glazman universal $D(y)$ function[3]. What is this function?
- $D(y)$ can be expressed in terms of Fredholm determinant with integrable kernel.
- Connection to asymptotic form factor calculations in exactly soluble models by Kitanine, Kozlowski, Maillet, N. Slavnov, and V. Terras: see recent work of Kozlowski and Maillet, arXiv:1501.07711
- New Numerical techniques: up-down Markov chains...

[3]Subject to assumption on absence of BO term