PRINCETON UNIVERSITY

Beta ensembles and the stochastic Airy semigroup

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Outline

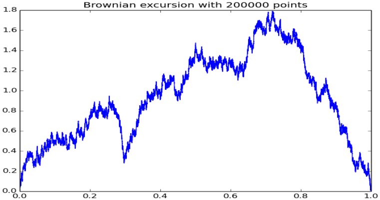
1. GUE eigenvalues and beta ensembles
2. The $\beta = 2$ point of view
3. Tridiagonal models and stochastic Airy operators point of view
4. Our results
5. Some ideas from proofs

A problem for experts

- Consider a standard Brownian excursion $e(t)$, $t \in [0,1]$.
- Let ℓ_a, $a \geq 0$ be its local time on level a.
- Show that

$$\int_0^1 e(t)\,\mathrm{d}t - \frac{1}{2}\int_0^\infty \ell_a^2\,\mathrm{d}a$$

is Gaussian with mean 0 and variance $\frac{1}{12}$.



Gaussian unitary ensemble

- Consider the $N \times N$ Hermitian matrix with normal entries (**GUE**):

$$\begin{pmatrix} A_{1,1} & \frac{A_{1,2}+i\,B_{1,2}}{\sqrt{2}} & \frac{A_{1,3}+i\,B_{1,3}}{\sqrt{2}} & \cdots \\ \frac{A_{1,2}-i\,B_{1,2}}{\sqrt{2}} & A_{2,2} & \frac{A_{2,3}+i\,B_{2,3}}{\sqrt{2}} & \cdots \\ \frac{A_{1,3}-i\,B_{1,3}}{\sqrt{2}} & \frac{A_{2,3}-i\,B_{2,3}}{\sqrt{2}} & A_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Interested in behavior of eigenvalues $\lambda_1(N) \geq \lambda_2(N) \geq \cdots \geq \lambda_N(N)$ as $N \to \infty$.

- Specifically, will look at $\lambda_1(N) \geq \cdots \geq \lambda_k(N)$ for fixed k as $N \to \infty$.

Eigenvalue distribution and beta ensembles

- Joint eigenvalue distribution given by density

$$\frac{1}{Z_{2}(N)} \prod_{1 \leq i < j \leq N} (x_j - x_i)^{2} \prod_{i=1}^{N} e^{-x_i^2/2}.$$

- From the point process point of view: no reason for the 2

$\Longrightarrow$ will replace it by a general parameter $\beta > 0$:

$$\frac{1}{Z_{\beta}(N)} \prod_{1 \leq i < j \leq N} (x_j - x_i)^{\beta} \prod_{i=1}^{N} e^{-x_i^2/2}.$$

Some results for $\beta = 2$

<u>Theorem</u> (Tracy, Widom '94) For $\beta = 2$, the rescaled process of largest eigenvalues

$$\left(N^{2/3}\left(\frac{\lambda_1(N)}{\sqrt{N}} - 2\right),\ N^{2/3}\left(\frac{\lambda_2(N)}{\sqrt{N}} - 2\right), \dots \right)$$

converges to a determinantal point process with kernel

$$\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x - y}$$

where $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$ is the Airy function.

$\Longrightarrow$ In principle, have a full understanding of the limiting process.

Tracktable explicit formulas?

Laplace transforms and a result of Okounkov

- An attempt to understand the Airy process is to consider Laplace transforms $\sum_{k=1}^{\infty} e^{T\mu_k}$, $T > 0$ where $\mu_1 \geq \mu_2 \geq \cdots$ are points of the Airy process.

- To understand the distribution of $\sum_{k=1}^{\infty} e^{T\mu_k}$ can consider moments

$$\mathbb{E}\left[\left(\sum_{k=1}^{\infty} e^{T\mu_k}\right)^{\ell}\right], \quad \mathbb{E}\left[\left(\sum_{k=1}^{\infty} e^{T_1\mu_k}\right)^{\ell_1}\left(\sum_{k=1}^{\infty} e^{T_2\mu_k}\right)^{\ell_2}\right], \quad \ldots$$

- **Okounkov '02** obtained beautiful formulas for such, starting with

$$\mathbb{E}\left[\sum_{k=1}^{\infty} e^{T\mu_k}\right] = \sqrt{\frac{2}{\pi}}\, T^{-3/2} e^{T^3/96}.$$

Some questions

- Which of these results extend to all β?
- Is there a relation between β-ensembles and the Airy function for general β?
- Are there versions of Okounkov's formulas for general β and do they have a probabilistic meaning?
- What is the meaning of Laplace transforms in Okounkov's result? Why are these canonical observables to look at?
- What makes $\beta = 2$ special?

Tridiagonal models

Theorem (Dumitriu, Edelman '02) For $\beta > 0$, the tridiagonal random matrix

$$M(N) := \begin{pmatrix} N(0,2/\beta) & \chi_\beta/\sqrt{\beta} & 0 & \dots \\ \chi_\beta/\sqrt{\beta} & N(0,2/\beta) & \chi_{2\beta}/\sqrt{\beta} & \ddots \\ 0 & \chi_{2\beta}/\sqrt{\beta} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

has a joint eigenvalue distribution given by

$$\frac{1}{Z_\beta(N)} \prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} e^{-\beta x_i^2/4}.$$

Towards the stochastic Airy operator I

- Key feature of the tridiagonal model: the unequal sizes of off-diagonal entries: from order 1 to order $\sqrt{N}$.
- Order $\sqrt{N}$ of largest entries suggests that the fluctuations of $M(N)/\sqrt{N}$ might converge to an operator $-SAO_\beta$ on a suitable infinite-dimensional space.

  $\Longrightarrow$ fluctuations of largest eigenvalues of $M(N)/\sqrt{N}$ should then converge to the eigenvalues of $-SAO_\beta$ (**Edelman, Sutton '07**).
- To make this precise, let

$$\chi_{m\beta}/\sqrt{\beta} =: \sqrt{m} + \xi_\beta(m).$$

Towards the stochastic Airy operator II

- Consider the diagonal noise $M_{NN}(N)$, $M_{(N-1)(N-1)}(N)$, $\ldots$ and the off-diagonal noise $\xi_\beta(N)$, $\xi_\beta(N-1)$, $\ldots$.
- In the limit, one expects these to converge to two independent instances of white noise.
- More precisely:

$$N^{-1/6} \sum_{m=N-\lfloor aN^{1/3} \rfloor}^{N} M_{mm}(N) \to s_{D,\beta} W_D(a),$$

$$N^{-1/6} \sum_{m=N-\lfloor aN^{1/3} \rfloor}^{N} \xi(m) \to s_{OD,\beta} W_{OD}(a)$$

with two independent Brownian motions W_D, W_{OD}.

Towards the stochastic Airy operator III

- Define the combined Brownian motion

  $W_\beta(a) := s_{D,\beta} W_D(a) + s_{OD,\beta} W_{OD},\ a \geq 0.$

- Equipped with W define formally the stochastic Airy operator

$$SAO_\beta = -\frac{\mathrm{d}^2}{\mathrm{d}a^2} + a + W'_\beta(a)$$

  on $L^2([0,\infty))$ with Dirichlet boundary condition at 0.

- **Ramirez, Rider, Virag '11** made rigorous sense of SAO_β and its eigenvalues $-\mu_1 \leq -\mu_2 \leq \cdots$ and proved the following:

General β convergence theorem

Theorem (Ramirez, Rider, Virag '11) The fluctuations of the largest eigenvalues of $M(N)$

$$\left(N^{2/3}\left(\frac{\lambda_1(N)}{\sqrt{N}}-2\right),\ N^{2/3}\left(\frac{\lambda_2(N)}{\sqrt{N}}-2\right),\ldots\right)$$

converge to $\mu_1 \geq \mu_2 \geq \cdots$, with $-\mu_1 \leq -\mu_2 \leq \cdots$ being the eigenvalues of SAO_β on $L^2([0,\infty))$. Moreover, $\mu_k \to -\infty$ and $\sum_{k=1}^{\infty} e^{T\mu_k} < 0$ for all $T > 0$ with probability 1.

$\Longrightarrow$ In principle, a full understanding of the limiting process for all values of $\beta > 0$.

Some questions

- Starting from SAO_β, how do we arrive at the Airy process for $\beta = 2$?
- Where are Okounkov's formulas hidden in SAO_β for $\beta = 2$?
- In general, how can we see the special role of SAO_2?
- Once the $\beta = 2$ results are found in SAO_2, can hope to find the appropriate analogues for general β.

$\Longrightarrow$ the goal of our work was to put the two approaches into one framework, to find the special role of $\beta = 2$, as well as the analogues of the $\beta = 2$ formulas.

Our results I

Unifying object: the random integral kernels $K_\beta(x, y; T)$, $T > 0$:

$$\mathbb{E}_{B^{x,y}}\left[\exp\left(-\frac{(x-y)^2}{2T} - \frac{1}{2}\int_0^T B^{x,y}(t)\,\mathrm{d}t + \int_0^\infty L_a(B^{x,y})\,\mathrm{d}W_\beta(a)\right)\mathbf{1}_{\{B^{x,y}>0\}}\right]$$

acting on $L^2([0, \infty))$, where

- $B^{x,y}$ is a standard Brownian bridge connecting x to y in time T,
- W is a standard Brownian motion independent of the bridge,
- $L_a(B^{x,y})$, $a \geq 0$ are local times of $B^{x,y}$.

Our results II: connection to stochastic Airy operator

Theorem (Gorin, S. '16) The (random) integral operators $U_\beta(T)$, $T > 0$ on $L^2([0,\infty))$ with kernels $\frac{1}{\sqrt{2\pi T}} K_\beta(x, y; T)$, $T > 0$ form a semigroup with probability 1, given by $e^{-T\, SAO_\beta}$, $T > 0$. In particular,

$$\int_0^\infty \frac{1}{\sqrt{2\pi T}} K_\beta(x, x; T)\,\mathrm{d}x = \mathrm{Trace}(U_\beta(T)) = \mathrm{Trace}(e^{-T\, SAO_\beta})$$

$$= \sum_{k=1}^\infty e^{T \mu_k},$$

where $\mu_1 \geq \mu_2 \geq \cdots$ are the eigenvalues of SAO_β.

Our results III: connection to tridiagonal models

- To connect to tridiagonal models/beta ensembles, will view $M(N)$ as a quadratic form on $L^2([0,\infty))$.
- More precisely, for $f \in L^2([0,\infty))$ define its projection on $\mathbb{R}^N$ by

$$p_N f = \left(N^{1/6} \int_{N^{-1/3}(N-i)}^{N^{-1/3}(N-i+1)} f(a)\,\mathrm{d}a : \; i = 1, 2, \ldots, N \right).$$

- Then, can identify every symmetric $N \times N$ matrix $A(N)$ with the quadratic form on $L^2([0,\infty))$:

$$(f,g) \mapsto (p_N f)'\, A(N)\, (p_N g).$$

Our results IV: connection to trigiagonal models cont.

Theorem (Gorin, S. '16) The quadratic form associated with

$$A(N,T) := \frac{1}{2}\left(\left(\frac{M(N)}{2\sqrt{N}}\right)^{\lfloor TN^{2/3}\rfloor} + \left(\frac{M(N)}{2\sqrt{N}}\right)^{\lfloor TN^{2/3}\rfloor - 1}\right)$$

converges to the quadratic form $(f,g) \mapsto (f, U_\beta(T)g)$ in the following sense:

- For any finite family of T's, f's, g's, the random vector of $(p_N f)' \, A(N,T) \, (p_N g)$'s converges to the random vector of $(f, U_\beta(T)g)$'s in distribution and in the sense of moments.

Our results V: connection to tridiagonal models cont.

- The traces converge:

$$\mathrm{Trace}(A(N,T)) \longrightarrow \mathrm{Trace}(U_\beta(T)) = \int_0^\infty \frac{1}{\sqrt{2\pi T}} K_\beta(x,x;T)\,\mathrm{d}x$$

  in distribution and in the sense of moments, for any finitely many T's.

In addition,

$$\lim_{N\to\infty} \sum_{k=1}^{\infty} e^{T\lambda_k(N)} = \lim_{N\to\infty} \mathrm{Trace}(A(N,T)) = \int_0^\infty \frac{1}{\sqrt{2\pi T}} K_\beta(x,x;T)\,\mathrm{d}x$$

in distribution and in the sense of moments, for any finitely many T's.

Special role of $\beta = 2$

- Starting point:

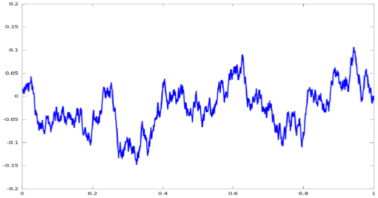
$$\sum_{k=1}^{\infty} e^{T\mu_k} = \int_0^{\infty} \frac{1}{\sqrt{2\pi T}} K_{\beta}(x,x;T)\,\mathrm{d}x$$

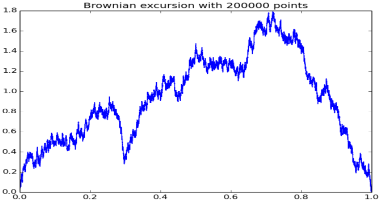
$$= \int_0^{\infty} \frac{1}{\sqrt{2\pi T}} \mathbb{E}_{B^{x,x}}\Big[\exp\Big(-\frac{1}{2}\int_0^T B^{x,x}(t)\,\mathrm{d}t + \int_0^{\infty} L_a(B^{x,x})\,\mathrm{d}W_{\beta}(a)\Big) \mathbf{1}_{\{B^{x,x}>0\}}\Big]\,\mathrm{d}x.$$

- Take the expectation ($\leftrightarrow$ Okounkov's first identity):

$$\int_0^{\infty} \frac{1}{\sqrt{2\pi T}} \mathbb{E}_{B^{x,x}}\Big[\exp\Big(-\frac{1}{2}\int_0^T B^{x,x}(t)\,\mathrm{d}t + \frac{1}{2\beta}\int_0^{\infty} L_a(B^{x,x})^2\mathrm{d}a\Big) \mathbf{1}_{\{B^{x,x}>0\}}\Big]\,\mathrm{d}x.$$

Simplification using Vervaat's transform





Simplification using Vervaat's transform cont.

For our functional

$$\int_0^\infty \frac{1}{\sqrt{2\pi T}} \mathbb{E}_{B^{x,x}} \left[\exp\left(-\frac{1}{2} \int_0^T B^{x,x}(t)\, \mathrm{d}t + \frac{1}{2\beta} \int_0^\infty L_a(B^{x,x})^2 \mathrm{d}a \right) \mathbf{1}_{\{B^{x,x}>0\}} \right] \mathrm{d}x,$$

- write $B^{x,x} = x + B^{0,0}$,
- note $\int_0^\infty L_a(B^{x,x})^2 \mathrm{d}a = \int_{-\infty}^\infty L_a(B^{0,0})^2\, \mathrm{d}a$,
- write the indicator as $\mathbf{1}_{\{x+\min(B^{0,0})>0\}}$ and integrate x out,
- use Vervaat's transform.

Brownian excursion representation and Okounkov

Corollary (Gorin, S. '16) The eigenvalues $\mu_1 \geq \mu_2 \geq \cdots$ of SAO_β satisfy

$$\mathbb{E}\Big[\sum_{k=1}^{\infty} e^{T\mu_k}\Big] = \sqrt{\frac{2}{\pi}}\, T^{-3/2}\, \mathbb{E}\Big[\exp\Big(-\frac{T^{3/2}}{2}\int_0^1 e(t)\,\mathrm{d}t + \frac{T^{3/2}}{2\beta}\int_0^\infty (l_a)^2\,\mathrm{d}y\Big)\Big].$$

In particular, Okounkov's first identity reads

$$\mathbb{E}\Big[\exp\Big(-\frac{T^{3/2}}{2}\int_0^1 e(t)\,\mathrm{d}t + \frac{T^{3/2}}{4}\int_0^\infty (l_a)^2\,\mathrm{d}y\Big)\Big] = e^{T^3/96},$$

i.e.: $\int_0^1 e(t)\,\mathrm{d}t - \frac{1}{2}\int_0^\infty (l_a)^2\,\mathrm{d}a$ Gaussian with mean 0 and variance 12.

No direct proof!!!

A partial result via Jeulin's Theorem

A partial result toward open problem:

Theorem (Csörgö, Shi, Yor '99) The random variables $\int_0^1 e(t)\,\mathrm{d}t$ and $\frac{1}{2}\int_0^\infty (l_a)^2\,\mathrm{d}a$ have the same distribution. In particular, their difference has mean 0.

Proof based on:

Theorem (Jeulin '85) Define $J(a) = \int_0^a l_b\,\mathrm{d}b$ and let J^{-1} be the inverse function. Then, $\tilde{e}(t) := \frac{1}{2}\, l_{J^{-1}(t)}$, $t \in [0,1]$ is a Brownian excursion.

Jeulin's Theorem $\Longrightarrow \frac{1}{2}\int_0^\infty (l_a)^2\,\mathrm{d}a$ is the area under $\tilde{e}$.

Another corollary: $\beta \to \infty$

Corollary Let $-\mu_1 \leq -\mu_2 \leq \cdots$ be the eigenvalues of the deterministic Airy operator $-\frac{\mathrm{d}^2}{\mathrm{d}a^2} + a$. Then,

$$\sum_{k=1}^{\infty} e^{T\mu_k} = \sqrt{\frac{2}{\pi}} T^{-3/2} \mathbb{E}\left[\exp\left(-\frac{T^{3/2}}{2} \int_0^1 e(t)\,\mathrm{d}t \right) \right].$$

In other words, the eigenvalues of the Airy operator describe the Laplace transform of the Brownian excursion area.

Several references for this result can be found in **Janson '07** including **Darling '83**, **Louchard '86**.

$\Longrightarrow$ can view our result as expansion about $\beta = \infty$: $\beta = 2$ special.

Some ideas from proofs: traces of high powers

- Consider entries of high powers $K := \lfloor TN^{2/3} \rfloor$ of $M(N)$:

$$\sum_{i_1,i_2,\dots,i_{K-1}} M_{i,i_1}(N)\, M_{i_1,i_2}(N) \cdots M_{i_K,j}(N).$$

- Sum over paths $|i_2 - i_1| \le 1$, $|i_3 - i_2| \le 1$, $\dots$, i.e. paths of random walk bridges connecting i to j in K steps.
- Interested in $i = \lfloor N - N^{1/3}x \rfloor$, $j = \lfloor N - N^{1/3}y \rfloor$, so pick up only large off-diagonal entries and small diagonal entries.
- Only paths with finitely many diagonal entries will contribute.

Some ideas from proofs: typical behavior

- Typical path of a random walk bridge will visit $O(N^{1/3})$ sites, each $O(N^{1/3})$ times, i.e. pick up $O(N^{1/3})$ different off-diagonal entries, each to power of order $O(N^{1/3})$.
- In the limit, the random walk bridge converges to a Brownian bridge, the occupation times of sites to Brownian bridge local times.
- Main tool: strong invariance principle in the spirit of **Csörgö, Révész '81**, **Borodin '86**, **Khoshnevisan '92**.

Some ideas from proofs: large deviations

Main tool: quantile transform of **Assaf, Forman, Pitman '15**:

THANK YOU
FOR YOUR ATTENTION!