2D Coulomb gas on Riemann surfaces

Riemannian Geometry and/from statistical mechanics/ stochastic processes

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2D Coulomb gas

$$Z_\beta = \int_{\mathbf{C}^N} \prod_{i>j} |z_i - z_j|^{2\beta} \prod_i e^{-\frac{\beta k}{2}|z_i|^2} dz_i d\bar{z}_i$$

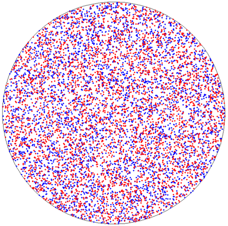


Figure: equilibrium measure

2D Coulomb gas

Particles on a surface interacting through Coulomb force with a neutralizing background

genus-0 surface

$$P = \frac{1}{Z} e^{-\beta E}$$

$$E = -2\pi \sum_{i \neq j}^{N} G(\xi_i, \xi_j)$$

$$-\Delta G(\xi, \xi') = \delta^{(2)}(\xi - \xi') - \frac{1}{V}$$

$$N \to \infty, \quad V \to \infty, \quad N/V = \text{fixed}$$

Complex coordinates on Riemann surface genus - 0

- Complex coordinates

$$ds^2 = g_{ij}\,dx^i dx^j = g_{z\bar{z}} dz d\bar{z}, \quad \sqrt{g} = g_{z\bar{z}}$$

$$z_0 = \infty, \quad g_{z\bar{z}} \to |z|^{-4}$$

- Volume form

$$dV = \sqrt{g}\, dz d\bar{z}$$

- Scalar curvature

$$R = -\frac{1}{\sqrt{g}} \partial\, \partial_{\bar{z}} \log \sqrt{g}$$

- Kähler potential

$$\partial_z \partial_{\bar{z}} K = \sqrt{g}, \quad K \to \frac{\text{Area}}{\pi} \log |z|$$

- Sphere

$$R = 2r^{-2}, \quad \sqrt{g} = (1 + |z|^2/4r^2)^{-2}, \quad K = \frac{V}{\pi} \log(1 + |z|^2/4r^2),$$

Round sphere

$$Z_\beta = \int_{\mathbf{C}^N} \prod_{i>j} |z_i - z_j|^{2\beta} \prod_i \frac{dz_i d\bar{z}_i}{\left(1 + \frac{|z_i|^2}{4r^2}\right)^{k+2}}$$

Boltzmann weight in complex coordinates

$$e^{-\beta E} = |\Psi_\beta|^2,$$

$$\Psi_\beta(z_1, \dots, z_N) \propto \prod_{i>j} (z_i - z_j)^\beta \, e^{-\frac{\beta k}{4} \sum_i^N K(z_i, \bar{z}_i)},$$

k is the neutralizing charge.

The integral

$$Z_N[g] = \int_{\mathbf{C}^N} |\Psi|^2 \prod_i (\sqrt{g}\, dz_i d\bar{z}_i) = \int_M |\Psi|^2 dV$$

converges if

$$\boxed{N \leq \beta^{-1} k + \frac{\chi}{2}}$$

Geometric functional

$$Z_N[g] = \int_{\mathbf{C}^N} \prod_{i>j} |z_i - z_j|^{2\beta} \prod_i e^{-\frac{\beta k}{4} K(z_i, \bar{z}_i)} \sqrt{g} dz_i d\bar{z}_i$$

- If $N < \frac{k}{\beta} + \frac{\chi}{2}$, the support of the equilibrium distribution has a boundary,
- If $N = \frac{k}{\beta} + \frac{\chi}{2}$, the support of the equilibrium distribution is the entire surface,

The functional $Z_N[g]$ depends only on geometry: curvature R and moduli

Analog of Riemann-Roch theorem

Holomorphic sections of line bundle L^k

$$\partial_{\bar{z}} s_i = 0, \quad \langle s_i, s_j \rangle = \int s_i(z)\overline{s_j(z)} e^{-\frac{k}{2}K(z,\bar{z})} \sqrt{g} dz d\bar{z}.$$

Then

$$\Psi(z_1, \ldots, z_N) = \left(\det[s_n(z_i)]\right)^{\beta}$$

The number of Holomorphic sections is

$$\boxed{N = k + \frac{\chi}{2}}$$

Applications

Quantum Hall Effect: Laughlin wave-function,

$$\Psi_{\beta,k}(z_1,\dots,z_N) \propto \prod_{i>j}(z_i - z_j)^{\beta}\, e^{-\frac{\beta k}{4}\sum_i^N K(z_i,\bar{z}_i)}$$

- k - is magnetic field
- $1/\beta$ - filling fraction,
- $\beta = 1$ - Integer QHE
- $\beta =$ integer - Fractional QHE,

Applications

Onsager ensemble:

Stochastic 2D- hydrodynamics of rotating superfluid (quantum turbulence)

- $\beta =$ even integer is the circulation of vortices;
- k - angular velocity of rotation

- $\beta = 1$ Ginibre ensemble

$$\int e^{\mathrm{Tr}\, W(M,M^\dagger)}\, d\mu(M,M^\dagger), \quad M - N \times N \text{ normal matrix}$$

$$W = -\frac{k}{2}\beta K + \log\sqrt{g}$$

$$M = U^{-1}\mathrm{diag}\,(z_1,\ldots z_N)\, U,$$

$$d\mu(M) = d\mu(U)\prod_{i>j}|z_i - z_j|^2 dz_1 \ldots dz_N$$

- possesses a determinantal structure and corresponds τ-functions of integrable hierarchies

Dyson Diffusion on the Riemann surface

Stochastic process

$$dz_i = \frac{k}{4}\partial_{z_i} K(z_i, \bar{z}_i)\, dt + \sum_{j \neq i} \frac{dt}{z_i - z_j} + dB_i,$$

$$dB_i d\bar{B}_j = \frac{\beta}{4}\delta_{ij}\sqrt{g}dt$$

Equilibrium distribution of this process is the Coulomb gas

KIRCHHOFF EQUATIONS

$$\frac{D\omega}{Dt} \equiv \dot{\omega} + \mathbf{u} \cdot \nabla \omega = 0.$$

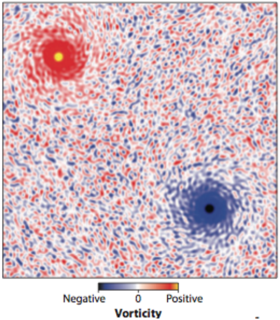
Helmholtz (and later Kirchhoff)

$$\mathrm{u}(z,t) = u_x - \mathrm{i}u_y = \mathrm{i}\sum_{i=1}^{N} \frac{\Gamma_i}{z - z_i(t)}$$

Kirchhoff equations

$$\boxed{\mathrm{i}\,\dot{\bar{z}}_i = \sum_{i \neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)}}$$

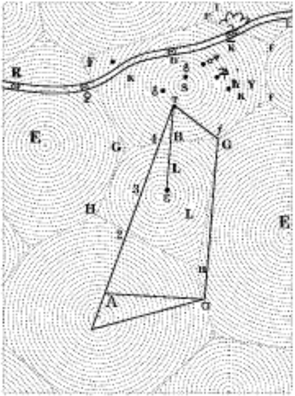
Chiral Flow: Clustering, Rotating Fluid



Chiral Kirchhoff equations $\Gamma_i = \Gamma$

$$\mathrm{i}\,\dot{\bar{z}}_i = \Omega \bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$$

Descartes Cosmogony



Kirchhoff equations

$$\mathrm{i}\,\dot{z}_i = \sum_{i \neq j}^{N} \frac{\Gamma_j}{z_i(t) - z_j(t)}$$

Geometric functionals and variations

- Geometric functionals

$$Z_N[g] = \int_M |\Psi|^2 dV$$

- Consecutive variation over metric

$$T_{ij}\delta g^{ij} = \delta \log Z_N[g]$$

produces all interesting correlation functions

Equilibrium measure: Density as a function of curvature

Can, Laskin, PW, 2014

$$\rho(z) = \int |\Psi(z, z_2, \dots, z_N)|^2 dV_2 \dots dV_N$$

$$\boxed{\rho = \frac{k}{\beta} + \frac{1}{8\pi} R + \frac{1}{k}\frac{c}{48}\Delta R + \dots}$$

$$\boxed{c = 1 - 3\beta}$$

$$S(q) = \int \langle \rho(r)\rho(0)\rangle e^{-iqr} d^2 r = \frac{q^2}{2} - \left(1 - \frac{\beta}{2}\right)\left(\frac{q^2}{2}\right)^2 + \left[\left(1 - \frac{\beta}{2}\right)^2 + \frac{\beta}{12}\right]\left(\frac{q^2}{2}\right)^3 + \dots$$

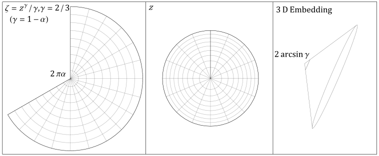
Expansion of the Bergmann kernel $\beta = 1$

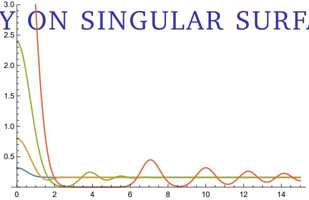
$$\beta = 1: \ \partial_{\bar z} s(z) = 0, \quad \rho = e^{-\frac{k}{2}K(z,\bar z)} \sum_{n=0}^{N-1} |s_n(z)|^2$$

Ferrari, Klevtsov, Zelditch (2012);

$$\rho = k + \frac{1}{8\pi}R + \frac{k^{-1}}{24\pi}\Delta R + \frac{k^{-2}}{16\pi}\left(\frac{1}{8}\Delta^2 R - \frac{5}{48}\Delta(R^2)\right) + \frac{k^{-3}}{32\pi}\left(\frac{29}{720}\Delta(R^3) - \frac{7}{160}\Delta^2(R^2) + \frac{1}{30}\Delta^3 R - \frac{11}{120}\Delta(R\Delta R)\right) + \dots$$

Emergent conformal symmetry on singular surfaces





cone metric $g_{z\bar{z}} = z^{-\alpha}$

$$\text{Density is singular: } k \to \infty \quad m_{2n} = k^{2n} \int r^{2n} \left(\rho(r) - \rho_\infty\right) dV$$

$$\begin{cases} m_0 = \frac{\alpha}{2} \\ m_2 = \Delta_\alpha = \frac{c}{24}(\gamma^{-1} - \gamma), \\ \gamma = 1 - \alpha, \quad c = 1 - 3\beta \end{cases}$$

$-\Delta_\alpha$ is conformal dimension of the "twist" operator in CFT

Geometric functionals: Gravitational "Wess-Zumino"

$$Z[g] = \int_M |\Psi|^2 dV$$

Large N expansion generates geometric invariants

$$\log Z[g] = p_2\, k^2 A^{(2)}[g] + p_1\, k A^{(1)}[g] + p_0\, A^{(0)}[g] + p_{-1} \overbrace{\int R^2 dV}^{\text{local in } R}$$

Geometric functional with non-trivial co-cycle property

$$A^{(2)} = \frac{1}{4V^2} \int K\, dV - \text{Aubin-Yau}, \qquad p_2 = -2\pi/\beta$$

$$A^{(1)} = \frac{2}{V} \int (-\Delta)^{-1} R\, dV, -\text{Mabuchi}, \qquad p_1 = -\frac{1}{4}$$

$$A^{(0)} = \int R(-\Delta)^{-1} R\, dV - \text{Polyakov}, \qquad p_0 = \frac{c}{48\pi}$$

Polyakov's Liouville action of quantum gravity

$$Z[g] = e^{\frac{2\pi}{\beta}k^2 A^{(2)}[g] - \frac{k}{4}A^{(1)}[g]} \left[\det\left(-\Delta_g\right)\right]^{\frac{c}{2}}$$

$$c = 1 - 3\beta$$

$$\log\det\left(-\Delta_g\right) = -\frac{1}{48\pi}\int R(-\Delta)^{-1}R\,dV$$

Scaling on a cone $g \to \lambda g$

$$\det\left(-\Delta_{\lambda g}\right) = \lambda^{-\Delta_\alpha}\det\left(-\Delta_g\right)$$

$$\boxed{\Delta_\alpha = -\tfrac{c}{24}(\gamma^{-1} - \gamma)}$$