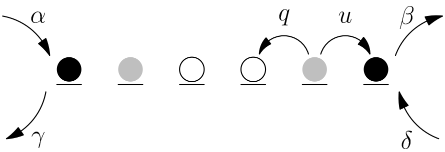
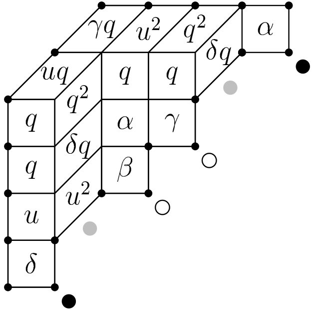
The two-species exclusion process and Koornwinder moments

Lauren K. Williams, UC Berkeley





Program

1. The asymmetric simple exclusion process (ASEP) and its applications
2. Staircase tableaux and steady state probabilities
3. The ASEP with 2 kinds of particles (the 2-species ASEP)
4. Rhombic tableaux and steady state probabilities
5. Orthogonal polynomials (Askey-Wilson and Macdonald-Koornwinder)

Program

1. The asymmetric simple exclusion process (ASEP) and its applications
2. Staircase tableaux and steady state probabilities
3. The ASEP with 2 kinds of particles (the 2-species ASEP)
4. Rhombic tableaux and steady state probabilities
5. Orthogonal polynomials (Askey-Wilson and Macdonald-Koornwinder)

Program

1. The asymmetric simple exclusion process (ASEP) and its applications
2. Staircase tableaux and steady state probabilities
3. The ASEP with 2 kinds of particles (the 2-species ASEP)
4. Rhombic tableaux and steady state probabilities
5. Orthogonal polynomials (Askey-Wilson and Macdonald-Koornwinder)

Program

1. The asymmetric simple exclusion process (ASEP) and its applications
2. Staircase tableaux and steady state probabilities
3. The ASEP with 2 kinds of particles (the 2-species ASEP)
4. Rhombic tableaux and steady state probabilities
5. Orthogonal polynomials (Askey-Wilson and Macdonald-Koornwinder)

Program

1. The asymmetric simple exclusion process (ASEP) and its applications
2. Staircase tableaux and steady state probabilities
3. The ASEP with 2 kinds of particles (the 2-species ASEP)
4. Rhombic tableaux and steady state probabilities
5. Orthogonal polynomials (Askey-Wilson and Macdonald-Koornwinder)

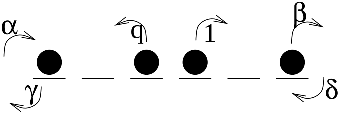
Program

1. The asymmetric simple exclusion process (ASEP) and its applications
2. Staircase tableaux and steady state probabilities
3. The ASEP with 2 kinds of particles (the 2-species ASEP)
4. Rhombic tableaux and steady state probabilities
5. Orthogonal polynomials (Askey-Wilson and Macdonald-Koornwinder)

The asymmetric simple exclusion process (ASEP)

Fix a $1D$ lattice of N sites, which can be occupied by particles. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$ between 0 and 1.

(Usually $u = 1$. Sometimes set $\gamma = \delta = 0$.)



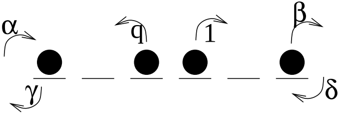
- New particles can enter and exit the lattice from the left at rates α, γ, and particles can exit and enter from the right at rates β, δ.
- A particle can hop right at rate u and left at rate q.
Model is *asymmetric*: we don't require $u = q$.
- *Exclusion:* at most one particle on each site

Depict particles as • or 1 and "holes" as ∘ or 0.

- Question: what happens as time $t \to \infty$?

The asymmetric simple exclusion process (ASEP)

Fix a $1D$ lattice of N sites, which can be occupied by particles. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$ between 0 and 1.
(Usually $u = 1$. Sometimes set $\gamma = \delta = 0$.)



- New particles can enter and exit the lattice from the left at rates α, γ, and particles can exit and enter from the right at rates β, δ.

- A particle can hop right at rate u and left at rate q.
Model is *asymmetric*: we don't require $u = q$.

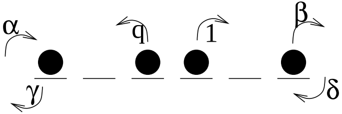
- *Exclusion:* at most one particle on each site

Depict particles as • or 1 and "holes" as ○ or 0.

- Question: what happens as time $t \to \infty$?

The asymmetric simple exclusion process (ASEP)

Fix a $1D$ lattice of N sites, which can be occupied by particles. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$ between 0 and 1.
(Usually $u = 1$. Sometimes set $\gamma = \delta = 0$.)



• New particles can enter and exit the lattice from the left at rates α, γ, and particles can exit and enter from the right at rates β, δ.

• A particle can hop right at rate u and left at rate q.
Model is *asymmetric*: we don't require $u = q$.

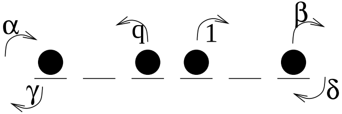
• *Exclusion:* at most one particle on each site

Depict particles as • or 1 and "holes" as ○ or 0.

• Question: what happens as time $t \to \infty$?

The asymmetric simple exclusion process (ASEP)

Fix a $1D$ lattice of N sites, which can be occupied by particles. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$ between 0 and 1.
(Usually $u = 1$. Sometimes set $\gamma = \delta = 0$.)



• New particles can enter and exit the lattice from the left at rates α, γ, and particles can exit and enter from the right at rates β, δ.

• A particle can hop right at rate u and left at rate q.
Model is *asymmetric*: we don't require $u = q$.

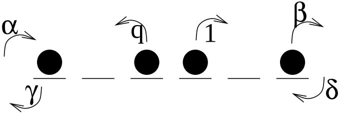
• *Exclusion:* at most one particle on each site

Depict particles as • or 1 and "holes" as ∘ or 0.

• Question: what happens as time $t \to \infty$?

The asymmetric simple exclusion process (ASEP)

Fix a $1D$ lattice of N sites, which can be occupied by particles. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$ between 0 and 1.
(Usually $u = 1$. Sometimes set $\gamma = \delta = 0$.)



• New particles can enter and exit the lattice from the left at rates α, γ, and particles can exit and enter from the right at rates β, δ.

• A particle can hop right at rate u and left at rate q.
Model is *asymmetric*: we don't require $u = q$.

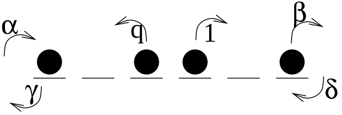
• *Exclusion:* at most one particle on each site

Depict particles as • or 1 and "holes" as ∘ or 0.

• Question: what happens as time $t \to \infty$?

The asymmetric simple exclusion process (ASEP)

Fix a $1D$ lattice of N sites, which can be occupied by particles. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$ between 0 and 1.
(Usually $u = 1$. Sometimes set $\gamma = \delta = 0$.)



• New particles can enter and exit the lattice from the left at rates α, γ, and particles can exit and enter from the right at rates β, δ.

• A particle can hop right at rate u and left at rate q.
Model is *asymmetric*: we don't require $u = q$.

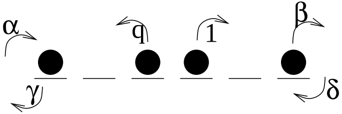
• *Exclusion:* at most one particle on each site

Depict particles as • or 1 and "holes" as ∘ or 0.

• Question: what happens as time $t \to \infty$?

The asymmetric simple exclusion process (ASEP)

Fix a $1D$ lattice of N sites, which can be occupied by particles. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$ between 0 and 1.
(Usually $u = 1$. Sometimes set $\gamma = \delta = 0$.)



• New particles can enter and exit the lattice from the left at rates α, γ, and particles can exit and enter from the right at rates β, δ.

• A particle can hop right at rate u and left at rate q.
Model is *asymmetric*: we don't require $u = q$.

• *Exclusion:* at most one particle on each site

Depict particles as • or 1 and "holes" as ∘ or 0.

• Question: what happens as time $t \to \infty$?

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, . . . , many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, . . . , many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, . . . , many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, . . . , many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, . . . , many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, ..., many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, ..., many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, . . . , many people here!

• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B \bullet$ and $Y = B \circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

The ASEP

• Introduced by biologists (MacDonald, Gibbs, Pipkin) in 1968, and independently by a mathematician (Spitzer) in 1970.
Enormous amount of study, e.g. by Liggett, Derrida, Spohn, Sasamoto, Borodin, Ferrari, Seppalainen, Tracy-Widom, Essler, de Gier, Johansson, Corwin, Peche, Gorin, Shkolnikov, Imamura, . . . , many people here!

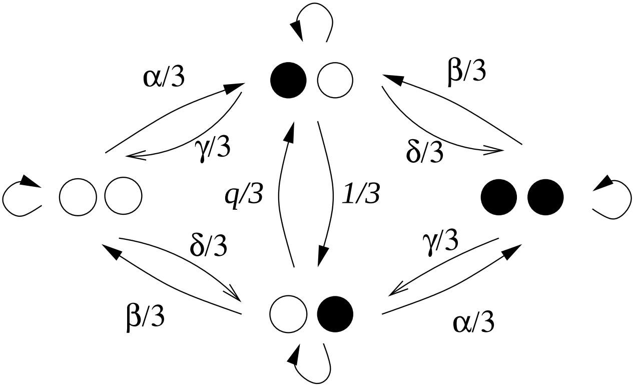
• Let B_N be the set of all 2^N words of length N on letters $\{\circ, \bullet\}$.

• The ASEP is the Markov chain on B_N with transition probabilities:

- If $X = A \bullet \circ B$ and $Y = A \circ \bullet B$ then $P_{X,Y} = \frac{u}{N+1}$ and $P_{Y,X} = \frac{q}{N+1}$.
- If $X = \circ B$ and $Y = \bullet B$ then $P_{X,Y} = \frac{\alpha}{N+1}$ and $P_{Y,X} = \frac{\gamma}{N+1}$.
- If $X = B\bullet$ and $Y = B\circ$ then $P_{X,Y} = \frac{\beta}{N+1}$ and $P_{X,Y} = \frac{\delta}{N+1}$.
- Otherwise $P_{X,Y} = 0$ for $Y \neq X$ and $P_{X,X} = 1 - \sum_{X \neq Y} P_{X,Y}$.

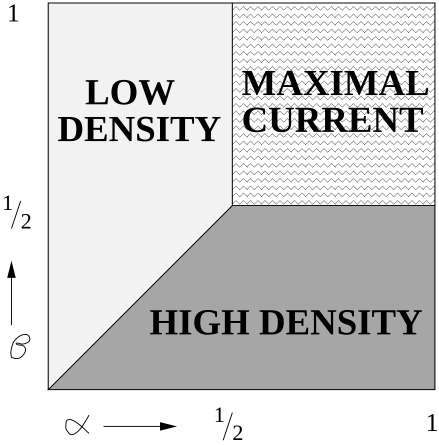
The ASEP

The state diagram of the ASEP for $N = 2$.

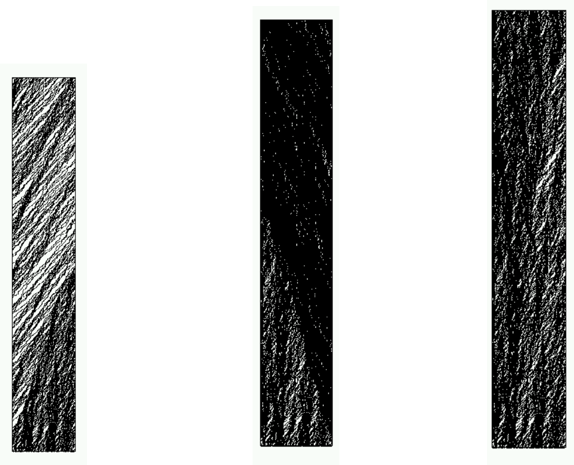


Some features of the ASEP

The ASEP exhibits boundary-induced *phase transitions*. (Here, $q = 0$.)



This picture from paper of Sasamoto. Phase diagram also appeared in e.g. works of Liggett.



(a) $\alpha = 0.2,\ \beta = 1$ (b) $\alpha = 1,\ \beta = 0.2$ (c) $\alpha = \beta = 1$

http://front.math.ucdavis.edu/9910.0270 (Sasamoto)

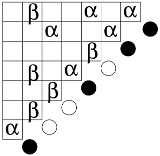
Tableaux formulas for probabilities when $\gamma = \delta = 0$.

Theorem (Corteel-Williams):

There is an explicit combinatorial formula for all steady state probabilities of the ASEP using *staircase tableaux*.

Def. (C.–W.) An α/β *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with α, β, such that:

- all boxes above an α are empty.
- all boxes left of a β are empty.
- all boxes on the southeast border are nonempty.



Its *type* is the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α and a $\circ$ to a β.

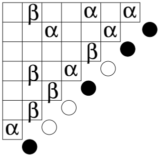
Tableaux formulas for probabilities when $\gamma = \delta = 0$.

Theorem (Corteel-Williams):

There is an explicit combinatorial formula for all steady state probabilities of the ASEP using *staircase tableaux*.

Def. (C.–W.) An α/β *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with α, β, such that:

- all boxes above an α are empty.
- all boxes left of a β are empty.
- all boxes on the southeast border are nonempty.



Its *type* is the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α and a $\circ$ to a β.

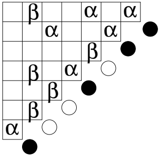
Tableaux formulas for probabilities when $\gamma = \delta = 0$.

Theorem (Corteel-Williams):

There is an explicit combinatorial formula for all steady state probabilities of the ASEP using *staircase tableaux*.

Def. (C.–W.) An α/β *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with α, β, such that:

- all boxes above an α are empty.
- all boxes left of a β are empty.
- all boxes on the southeast border are nonempty.



Its *type* is the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α and a $\circ$ to a β.

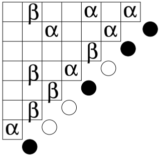
Tableaux formulas for probabilities when $\gamma = \delta = 0$.

Theorem (Corteel-Williams):

There is an explicit combinatorial formula for all steady state probabilities of the ASEP using *staircase tableaux*.

Def. (C.–W.) An α/β *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with α, β, such that:

- all boxes above an α are empty.
- all boxes left of a β are empty.
- all boxes on the southeast border are nonempty.



Its *type* is the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α and a $\circ$ to a β.

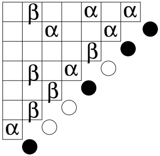
Tableaux formulas for probabilities when $\gamma = \delta = 0$.

Theorem (Corteel-Williams):

There is an explicit combinatorial formula for all steady state probabilities of the ASEP using *staircase tableaux*.

Def. (C.–W.) An α/β *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with α, β, such that:

- all boxes above an α are empty.
- all boxes left of a β are empty.
- all boxes on the southeast border are nonempty.



Its *type* is the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α and a $\circ$ to a β.

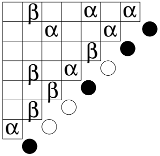
Tableaux formulas for probabilities when $\gamma = \delta = 0$.

Theorem (Corteel-Williams):

There is an explicit combinatorial formula for all steady state probabilities of the ASEP using *staircase tableaux*.

Def. (C.–W.) An α/β *staircase tableau* of size N is a Young diagram of shape $(N, \dots, 2, 1)$, whose boxes are empty or filled with α, β, such that:

- all boxes above an α are empty.
- all boxes left of a β are empty.
- all boxes on the southeast border are nonempty.



Its *type* is the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α and a $\circ$ to a β.

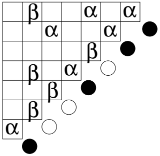
Tableaux formulas for probabilities when $\gamma = \delta = 0$.

Theorem (Corteel-Williams):

There is an explicit combinatorial formula for all steady state probabilities of the ASEP using *staircase tableaux*.

Def. (C.–W.) An α/β *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with α, β, such that:

- all boxes above an α are empty.
- all boxes left of a β are empty.
- all boxes on the southeast border are nonempty.

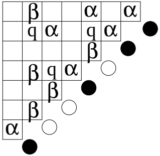


Its *type* is the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α and a $\circ$ to a β.

Tableaux formulas for probabilities

Assign q to each blank box with an α to the right and a β below it.

Define the *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

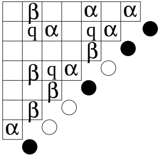
Consider the ASEP with parameters α, β, q general, and $\gamma = \delta = 0$. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities

Assign q to each blank box with an α to the right and a β below it.
Define the *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

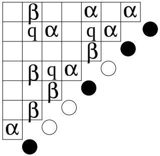
Consider the ASEP with parameters α, β, q general, and $\gamma = \delta = 0$. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities

Assign q to each blank box with an α to the right and a β below it.
Define the *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

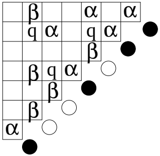
Consider the ASEP with parameters α, β, q general, and $\gamma = \delta = 0$. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities

Assign q to each blank box with an α to the right and a β below it. Define the *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

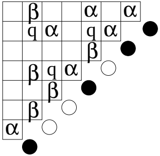
Consider the ASEP with parameters α, β, q general, and $\gamma = \delta = 0$. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities

Assign q to each blank box with an α to the right and a β below it. Define the *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

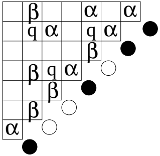
Consider the ASEP with parameters α, β, q general, and $\gamma = \delta = 0$. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities

Assign q to each blank box with an α to the right and a β below it.
Define the *weight* $\text{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

Consider the ASEP with parameters α, β, q general, and $\gamma = \delta = 0$.
The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \text{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:

Let $Z_2 = \alpha^2 + \alpha\beta(\alpha+\beta+q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:

Let $Z_2 = \alpha^2 + \alpha\beta(\alpha + \beta + q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:

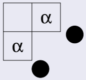
Let $Z_2 = \alpha^2 + \alpha\beta(\alpha + \beta + q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:



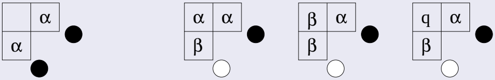
Let $Z_2 = \alpha^2 + \alpha\beta(\alpha + \beta + q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:



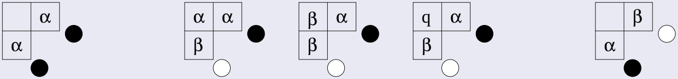
Let $Z_2 = \alpha^2 + \alpha\beta(\alpha + \beta + q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha + \beta + q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:



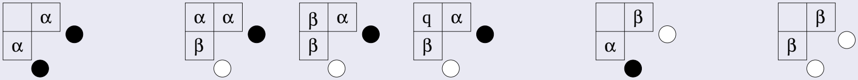
Let $Z_2 = \alpha^2 + \alpha\beta(\alpha+\beta+q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:



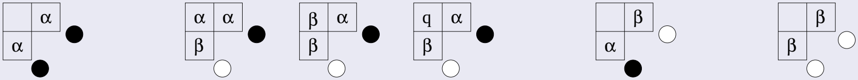
Let $Z_2 = \alpha^2 + \alpha\beta(\alpha + \beta + q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:



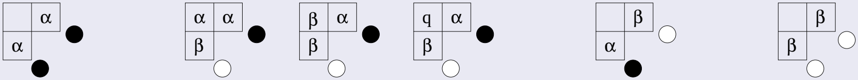
Let $Z_2 = \alpha^2 + \alpha\beta(\alpha+\beta+q) + \alpha\beta + \beta^2$. By the Theorem, we have that

$$\Pr(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \Pr(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \Pr(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \Pr(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities: example

What are the probabilities of the different states for $N = 2$?

The tableaux of the various types are:



Let $Z_2 = \alpha^2 + \alpha\beta(\alpha + \beta + q) + \alpha\beta + \beta^2$. By the Theorem, we have that

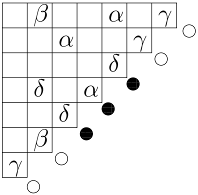
$$\mathrm{Pr}(\bullet\bullet) = \frac{\alpha^2}{Z_2}, \mathrm{Pr}(\bullet\circ) = \frac{\alpha\beta(\alpha+\beta+q)}{Z_2}, \mathrm{Pr}(\circ\bullet) = \frac{\alpha\beta}{Z_2}, \mathrm{Pr}(\circ\circ) = \frac{\beta^2}{Z_2}.$$

Tableaux formulas for probabilities in the general case.

Previous slides were for $\gamma = \delta = 0$. But we can remove this hypothesis with a slightly more general definition of staircase tableaux.

Def. (C.-W.) A *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.



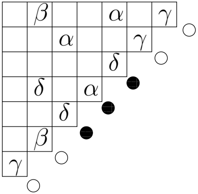
Define its *type* to be the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Tableaux formulas for probabilities in the general case.

Previous slides were for $\gamma = \delta = 0$. But we can remove this hypothesis with a slightly more general definition of staircase tableaux.

Def. (C.-W.) A *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.



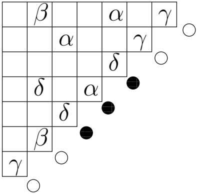
Define its *type* to be the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Tableaux formulas for probabilities in the general case.

Previous slides were for $\gamma = \delta = 0$. But we can remove this hypothesis with a slightly more general definition of staircase tableaux.

Def. (C.-W.) A *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.



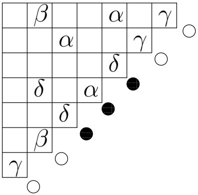
Define its *type* to be the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Tableaux formulas for probabilities in the general case.

Previous slides were for $\gamma = \delta = 0$. But we can remove this hypothesis with a slightly more general definition of staircase tableaux.

Def. (C.-W.) A *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.



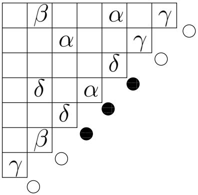
Define its *type* to be the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Tableaux formulas for probabilities in the general case.

Previous slides were for $\gamma = \delta = 0$. But we can remove this hypothesis with a slightly more general definition of staircase tableaux.

Def. (C.-W.) A *staircase tableau* of size N is a Young diagram of shape $(N, \dots, 2, 1)$, whose boxes are empty or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.



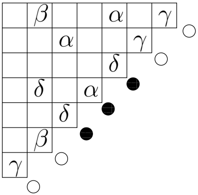
Define its *type* to be the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Tableaux formulas for probabilities in the general case.

Previous slides were for $\gamma = \delta = 0$. But we can remove this hypothesis with a slightly more general definition of staircase tableaux.

Def. (C.-W.) A *staircase tableau* of size N is a Young diagram of shape $(N, \ldots, 2, 1)$, whose boxes are empty or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.



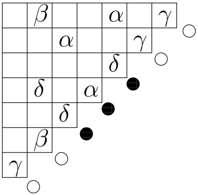
Define its *type* to be the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Tableaux formulas for probabilities in the general case.

Previous slides were for $\gamma = \delta = 0$. But we can remove this hypothesis with a slightly more general definition of staircase tableaux.

Def. (C.-W.) A *staircase tableau* of size N is a Young diagram of shape $(N, \dots, 2, 1)$, whose boxes are empty or filled with $\alpha, \beta, \gamma, \delta$, such that:

- all boxes above an α or γ are empty.
- all boxes left of a β or δ are empty.
- all boxes on the southeast border are nonempty.

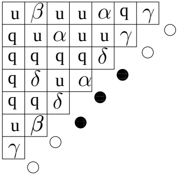


Define its *type* to be the word in $\{\bullet, \circ\}^N$ obtained by reading the southeast border and assigning a $\bullet$ to an α or δ and a $\circ$ to a β or γ.

Tableaux formulas for probabilities in the general case.

Assign q's to some empty boxes (according to deterministic local RULE).

Define *weight* $\text{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

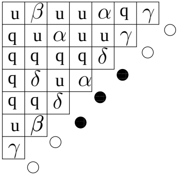
Consider the ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general.
The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \text{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities in the general case.

Assign q's to some empty boxes (according to deterministic local RULE).
Define *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

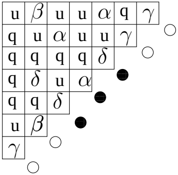
Consider the ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general.
The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities in the general case.

Assign q's to some empty boxes (according to deterministic local RULE).
Define *weight* $\text{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

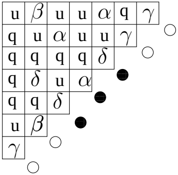
Consider the ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general.
The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \text{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities in the general case.

Assign q's to some empty boxes (according to deterministic local RULE).
Define *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

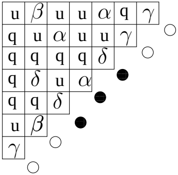
Consider the ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general.
The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities in the general case.

Assign q's to some empty boxes (according to deterministic local RULE).
Define *weight* $\text{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

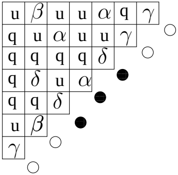
Consider the ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general.
The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \text{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

Tableaux formulas for probabilities in the general case.

Assign q's to some empty boxes (according to deterministic local RULE).
Define *weight* $\mathrm{wt}(\mathcal{T})$ of tableau $\mathcal{T}$ as product of all boxes.



Let $Z_N = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, summing over all tableaux of size N.

Theorem (Corteel–W.)

Consider the ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general.
The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all tableaux $\mathcal{T}$ of type σ.

How and why did we come up with staircase tableaux?

- 2005 (W.): Enumerated cells according to dimension in the *positive Grassmannian* $Gr^+_{K,N}$, using Postnikov's *Le-diagrams*. Obtained: $\hat{E}_{K,N}(q) = q^{K-K^2} \sum_{i=0}^{K-1} (-1)^i [K-i]_q{}^N q^{Ki-K} \left(\binom{N}{i} q^{K-i} + \binom{N}{i-1} \right)$.
- 2007 (Corteel): Proved that if $\alpha = \beta = 1$, $\gamma = \delta = 0$, the steady state probability that the ASEP on N sites is in configuration with exactly K particles is $\frac{\hat{E}_{K+1,N+1}(q)}{Z_N}$.
- 2007 (Corteel-W): When α, β, q general and $\gamma = \delta = 0$, gave formulas for all steady state probabilities of the ASEP using Le-diagrams.

How and why did we come up with staircase tableaux?

- 2005 (W.): Enumerated cells according to dimension in the *positive Grassmannian* $Gr^{+}_{K,N}$, using Postnikov's *Le-diagrams*. Obtained: $\hat{E}_{K,N}(q) = q^{K-K^2} \sum_{i=0}^{K-1} (-1)^i [K-i]_q{}^N q^{Ki-K} \left(\binom{N}{i} q^{K-i} + \binom{N}{i-1} \right)$.
- 2007 (Corteel): Proved that if $\alpha = \beta = 1$, $\gamma = \delta = 0$, the steady state probability that the ASEP on N sites is in configuration with exactly K particles is $\frac{\hat{E}_{K+1,N+1}(q)}{Z_N}$.
- 2007 (Corteel-W): When α, β, q general and $\gamma = \delta = 0$, gave formulas for all steady state probabilities of the ASEP using Le-diagrams.

How and why did we come up with staircase tableaux?

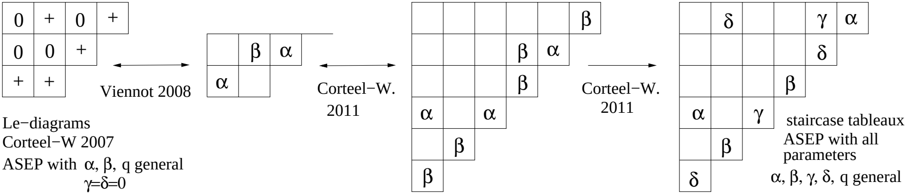
- 2005 (W.): Enumerated cells according to dimension in the *positive Grassmannian* $Gr^+_{K,N}$, using Postnikov's *Le-diagrams*. Obtained: $\hat{E}_{K,N}(q) = q^{K-K^2} \sum_{i=0}^{K-1} (-1)^i [K-i]_q{}^N q^{Ki-K} \left(\binom{N}{i} q^{K-i} + \binom{N}{i-1} \right)$.
- 2007 (Corteel): Proved that if $\alpha = \beta = 1$, $\gamma = \delta = 0$, the steady state probability that the ASEP on N sites is in configuration with exactly K particles is $\frac{\hat{E}_{K+1,N+1}(q)}{Z_N}$.
- 2007 (Corteel-W): When α, β, q general and $\gamma = \delta = 0$, gave formulas for all steady state probabilities of the ASEP using Le-diagrams.

How and why did we come up with staircase tableaux?

- 2005 (W.): Enumerated cells according to dimension in the *positive Grassmannian* $Gr^+_{K,N}$, using Postnikov's *Le-diagrams*. Obtained: $\hat{E}_{K,N}(q) = q^{K-K^2} \sum_{i=0}^{K-1} (-1)^i [K-i]_q{}^N q^{Ki-K} \left(\binom{N}{i} q^{K-i} + \binom{N}{i-1} \right)$.
- 2007 (Corteel): Proved that if $\alpha = \beta = 1$, $\gamma = \delta = 0$, the steady state probability that the ASEP on N sites is in configuration with exactly K particles is $\frac{\hat{E}_{K+1,N+1}(q)}{Z_N}$.
- 2007 (Corteel-W): When α, β, q general and $\gamma = \delta = 0$, gave formulas for all steady state probabilities of the ASEP using Le-diagrams.

How and why did we come up with staircase tableaux?

- 2005 (W.): Enumerated cells according to dimension in the *positive Grassmannian* $Gr^+_{K,N}$, using Postnikov's *Le-diagrams*. Obtained: $\hat{E}_{K,N}(q) = q^{K-K^2} \sum_{i=0}^{K-1} (-1)^i [K-i]_q{}^N q^{Ki-K} \left(\binom{N}{i} q^{K-i} + \binom{N}{i-1} \right)$.
- 2007 (Corteel): Proved that if $\alpha = \beta = 1$, $\gamma = \delta = 0$, the steady state probability that the ASEP on N sites is in configuration with exactly K particles is $\frac{\hat{E}_{K+1,N+1}(q)}{Z_N}$.
- 2007 (Corteel-W): When α, β, q general and $\gamma = \delta = 0$, gave formulas for all steady state probabilities of the ASEP using Le-diagrams.



Staircase tableaux are nice objects

Let $Z_n(\alpha,\beta,\gamma,\delta;q) = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})$, where the sum is over all staircase tableaux of size n.

α	β	γ	δ	q	$Z_n(\alpha,\beta,\gamma,\delta;q)$
1	1	1	1	1	$4^n n! = 4n!!!!$
1	1	1	0	1	$(2n+1)!!$
1	1	0	0	1	$(n+1)!$
1	1	0	0	0	$C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$
α	β	γ	δ	1	$\prod_{j=0}^{n-1}(\alpha+\beta+\gamma+\delta+j(\alpha+\gamma)(\beta+\delta))$
α	β	γ	$-\beta$	q	$\prod_{j=0}^{n-1}(\alpha+q^j\gamma)$

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\Pr_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \quad (1)$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \quad (2)$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \quad (3)$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\Pr_N(\sigma_1, \dots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\Pr_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\mathrm{Pr}_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \quad (1)$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \quad (2)$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \quad (3)$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \dots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N (\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\mathrm{Pr}_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\mathrm{Pr}_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \quad (1)$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \quad (2)$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \quad (3)$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \dots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\mathrm{Pr}_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\mathrm{Pr}_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \quad (1)$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \quad (2)$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \quad (3)$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \dots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\mathrm{Pr}_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\Pr_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \tag{1}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{2}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{3}$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\Pr_N(\sigma_1, \ldots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N (\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\Pr_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\mathrm{Pr}_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \quad (1)$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \quad (2)$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \quad (3)$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1,\ldots,\sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\mathrm{Pr}_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\mathrm{Pr}_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \tag{1}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{2}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{3}$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1,\ldots,\sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\mathrm{Pr}_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\mathrm{Pr}_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \quad (1)$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \quad (2)$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \quad (3)$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \dots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\mathrm{Pr}_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\mathrm{Pr}_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \tag{1}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{2}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{3}$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \dots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\mathrm{Pr}_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

How to relate tableaux to ASEP: the Matrix Ansatz

Let $\Pr_N(\sigma)$ be the steady state prob. of configuration $\sigma \in \{0,1\}^N$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \tag{1}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{2}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{3}$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\Pr_N(\sigma_1,\ldots,\sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Ex. $\Pr_5(1,1,0,0,1) = \frac{\langle W|DDEED|V\rangle}{Z_5}$.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \quad (4)$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \quad (5)$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \quad (6)$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \ldots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^{N}(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Idea of proving the tableaux formulas for steady state probabilities: find matrices and vectors $D, E, \langle W|, |V\rangle$ which are *transfer matrices* for staircase tableaux ... such that e.g. $\langle W|DDEED|V\rangle$ enumerates the weights of all tableaux of type • • ○ ○ •.

Then show that the matrices/vectors satisfy the relations above.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \tag{4}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{5}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{6}$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \ldots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Idea of proving the tableaux formulas for steady state probabilities:

find matrices and vectors $D, E, \langle W|, |V\rangle$ which are *transfer matrices* for staircase tableaux ... such that e.g. $\langle W|DDEED|V\rangle$ enumerates the weights of all tableaux of type $\bullet \bullet \circ \circ \bullet$.

Then show that the matrices/vectors satisfy the relations above.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \tag{4}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{5}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{6}$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \ldots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N(\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Idea of proving the tableaux formulas for steady state probabilities: find matrices and vectors $D, E, \langle W|, |V\rangle$ which are *transfer matrices* for staircase tableaux ... such that e.g. $\langle W|DDEED|V\rangle$ enumerates the weights of all tableaux of type $\bullet \bullet \circ \circ \bullet$.

Then show that the matrices/vectors satisfy the relations above.

Theorem (Derrida, Evans, Hakim, Pasquier 1993):

Suppose there exists matrices D, E, and vectors $\langle W|$ and $|V\rangle$, such that:

$$DE - qED = D + E \tag{4}$$

$$(\beta D - \delta E)|V\rangle = |V\rangle \tag{5}$$

$$\langle W|(\alpha E - \gamma D) = \langle W| \tag{6}$$

Let $Z_N := \langle W|(D+E)^N|V\rangle$ (the *partition function*). Then

$$\mathrm{Pr}_N(\sigma_1, \ldots, \sigma_N) = \frac{\langle W|(\prod_{i=1}^N (\sigma_i D + (1-\sigma_i)E))|V\rangle}{Z_N}.$$

Idea of proving the tableaux formulas for steady state probabilities: find matrices and vectors $D, E, \langle W|, |V\rangle$ which are *transfer matrices* for staircase tableaux ... such that e.g. $\langle W|DDEED|V\rangle$ enumerates the weights of all tableaux of type $\bullet \bullet \circ \circ \bullet$.

Then show that the matrices/vectors satisfy the relations above.

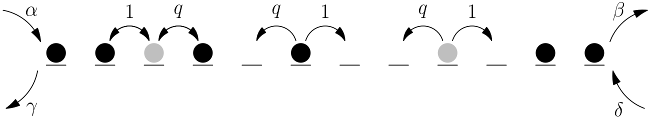
The two-species ASEP

Same as the ASEP, but with two kinds of particles, *heavy* and *light*.

Sometimes represent these particles by 2 and 1, and a hole by a 0.

Fix a $1D$ lattice of N sites. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$.

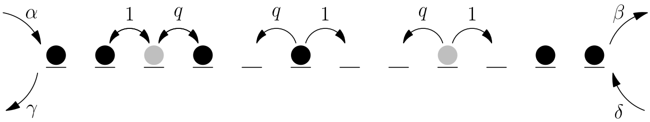
(Usually $u = 1$.)



- New *heavy* particles can enter and exit the lattice (swapping places with a hole) from the left at rates α, γ, and *heavy* particles can exit and enter from the right (swapping places with a hole) at rates β, δ.
- Light particles cannot leave the lattice, so their number is conserved.
- For two adjacent sites on the lattice, we have $21 \to 12$ and $20 \to 02$ and $10 \to 01$ with rate u. The reverse transitions happen with rate q.

The two-species ASEP

Same as the ASEP, but with two kinds of particles, *heavy* and *light*.
Sometimes represent these particles by 2 and 1, and a hole by a 0.
Fix a $1D$ lattice of N sites. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$.
(Usually $u = 1$.)



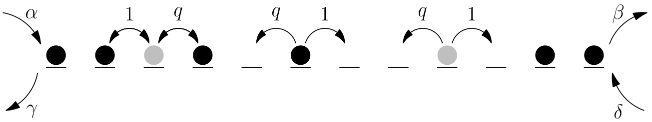
- New *heavy* particles can enter and exit the lattice (swapping places with a hole) from the left at rates α, γ, and *heavy* particles can exit and enter from the right (swapping places with a hole) at rates β, δ.

- Light particles cannot leave the lattice, so their number is conserved.

- For two adjacent sites on the lattice, we have $21 \to 12$ and $20 \to 02$ and $10 \to 01$ with rate u. The reverse transitions happen with rate q.

The two-species ASEP

Same as the ASEP, but with two kinds of particles, *heavy* and *light*.
Sometimes represent these particles by 2 and 1, and a hole by a 0.
Fix a $1D$ lattice of N sites. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$.
(Usually $u = 1$.)



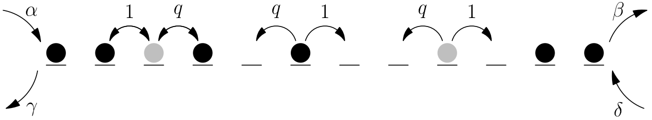
- New *heavy* particles can enter and exit the lattice (swapping places with a hole) from the left at rates α, γ, and *heavy* particles can exit and enter from the right (swapping places with a hole) at rates β, δ.

- Light particles cannot leave the lattice, so their number is conserved.

- For two adjacent sites on the lattice, we have $21 \to 12$ and $20 \to 02$ and $10 \to 01$ with rate u. The reverse transitions happen with rate q.

The two-species ASEP

Same as the ASEP, but with two kinds of particles, *heavy* and *light*. Sometimes represent these particles by 2 and 1, and a hole by a 0. Fix a $1D$ lattice of N sites. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$. (Usually $u = 1$.)



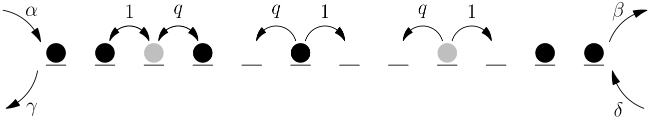
• New *heavy* particles can enter and exit the lattice (swapping places with a hole) from the left at rates α, γ, and *heavy* particles can exit and enter from the right (swapping places with a hole) at rates β, δ.

• Light particles cannot leave the lattice, so their number is conserved.

• For two adjacent sites on the lattice, we have $21 \to 12$ and $20 \to 02$ and $10 \to 01$ with rate u. The reverse transitions happen with rate q.

The two-species ASEP

Same as the ASEP, but with two kinds of particles, *heavy* and *light*.
Sometimes represent these particles by 2 and 1, and a hole by a 0.
Fix a $1D$ lattice of N sites. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$.
(Usually $u = 1$.)



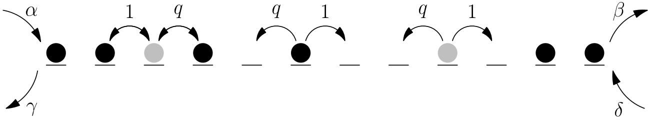
• New *heavy* particles can enter and exit the lattice (swapping places with a hole) from the left at rates α, γ, and *heavy* particles can exit and enter from the right (swapping places with a hole) at rates β, δ.

• Light particles cannot leave the lattice, so their number is conserved.

• For two adjacent sites on the lattice, we have $21 \to 12$ and $20 \to 02$ and $10 \to 01$ with rate u. The reverse transitions happen with rate q.

The two-species ASEP

Same as the ASEP, but with two kinds of particles, *heavy* and *light*.
Sometimes represent these particles by 2 and 1, and a hole by a 0.
Fix a $1D$ lattice of N sites. Choose parameters $q, u, \alpha, \beta, \gamma, \delta$.
(Usually $u = 1$.)



• New *heavy* particles can enter and exit the lattice (swapping places with a hole) from the left at rates α, γ, and *heavy* particles can exit and enter from the right (swapping places with a hole) at rates β, δ.

• Light particles cannot leave the lattice, so their number is conserved.

• For two adjacent sites on the lattice, we have $21 \to 12$ and $20 \to 02$ and $10 \to 01$ with rate u. The reverse transitions happen with rate q.

Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.
New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.
The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:

The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:

Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.
New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.
The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:

The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:

Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.

New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.

The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:

The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:

Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.
New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.
The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:

The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:

Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

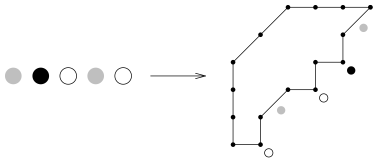
When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.
New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.
The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:

The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:

Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.
New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.
The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:

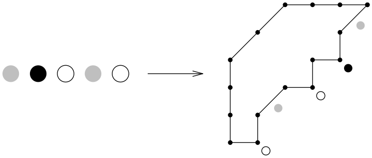


The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:

Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.
New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.
The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:

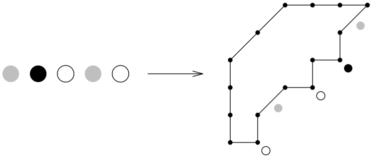


The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:

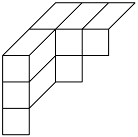
Theorem (Corteel-Mandelshtam-W.):

There is an explicit combinatorial formula for all steady state probabilities of the two-species ASEP using *rhombic staircase tableaux*.

When $\gamma = \delta = 0$, analogous formula first proved by Mandelshtam-Viennot.
New idea: one should use "generalized" Young diagrams containing diagonal edges as well as horizontal and vertical ones.
The diagram $\Gamma(\sigma)$ associated to a state σ is obtained as follows:



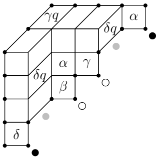
The *maximum rhombic tiling* of $\Gamma(\sigma)$ is the following:



Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

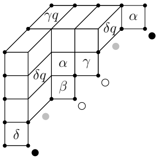


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* $\mathrm{wt}(\mathcal{T})$ of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

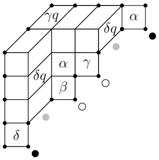


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* $\mathrm{wt}(\mathcal{T})$ of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

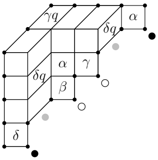


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* $\mathrm{wt}(\mathcal{T})$ of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

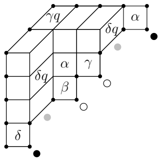


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* wt($\mathcal{T}$) of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

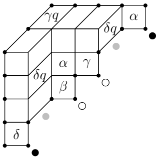


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* wt($\mathcal{T}$) of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

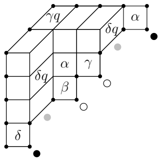


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* $\mathrm{wt}(\mathcal{T})$ of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

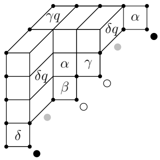


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* wt($\mathcal{T}$) of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or $\delta \boldsymbol{q}$
- Short rhombi are empty or contain α or $\gamma \boldsymbol{q}$
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.

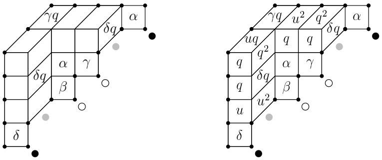


Assign a monomial to each empty tile according to deterministic local RULE. The *weight* $\mathrm{wt}(\mathcal{T})$ of a tableau is the product of all tiles.

Tableaux formulas for probabilities.

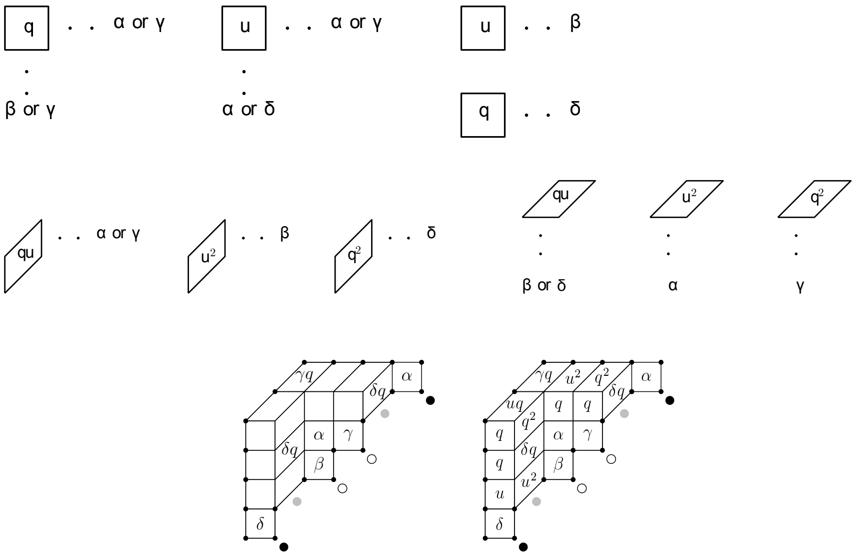
Def. (Corteel-Mandelshtam-W.) A *rhombic staircase tableau* of type $\Gamma(\sigma)$ is a filling of its maximum rhombic tiling such that:

- Squares are empty or filled with $\alpha, \beta, \gamma, \delta$
- Tall rhombi are empty or contain β or δq
- Short rhombi are empty or contain α or γq
- The lowest square in each vertical strip must be filled according to σ
- Each tile "above" an α or γ is empty
- Each tile "left of" a β or δ is empty.



Assign a monomial to each empty tile according to deterministic local RULE. The *weight* $\mathrm{wt}(\mathcal{T})$ of a tableau is the product of all tiles.

RULE for filling in blank tiles and determining the weight



Tableaux formulas for probabilities.

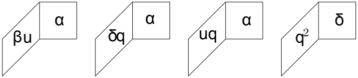
Theorem (Corteel–Mandelshtam – W.)

Consider the 2-species ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all rhombic tableaux $\mathcal{T}$ of type σ.

Example: Consider the two-species ASEP on a lattice of 2 sites with precisely 1 light particle. Let $\sigma = (2, 1)$. The tableaux of type σ are:



So the probability of state σ is equal to $\frac{\alpha\beta u + \alpha\delta q + \alpha u q + \delta q^2}{\mathbf{Z}_{2,1}}$.

Tableaux formulas for probabilities.

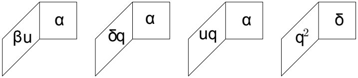
Theorem (Corteel–Mandelshtam – W.)

Consider the 2-species ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all rhombic tableaux $\mathcal{T}$ of type σ.

Example: Consider the two-species ASEP on a lattice of 2 sites with precisely 1 light particle. Let $\sigma = (2, 1)$. The tableaux of type σ are:



So the probability of state σ is equal to $\frac{\alpha\beta u + \alpha\delta q + \alpha u q + \delta q^2}{Z_{2,1}}$.

Tableaux formulas for probabilities.

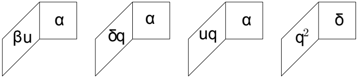
Theorem (Corteel–Mandelshtam – W.)

Consider the 2-species ASEP with parameters $\alpha, \beta, \gamma, \delta, q$ general. The steady state probability that the ASEP is in configuration σ is

$$\frac{\sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})}{Z_N},$$

where sum is over all rhombic tableaux $\mathcal{T}$ of type σ.

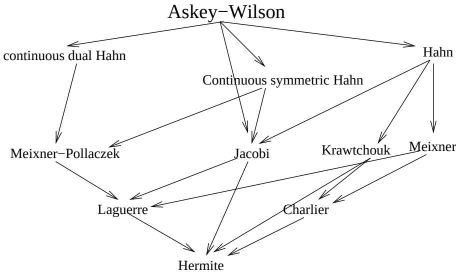
Example: Consider the two-species ASEP on a lattice of 2 sites with precisely 1 light particle. Let $\sigma = (2, 1)$. The tableaux of type σ are:



So the probability of state σ is equal to $\frac{\alpha\beta u + \alpha\delta q + \alpha u q + \delta q^2}{\mathbf{Z}_{2,1}}$.

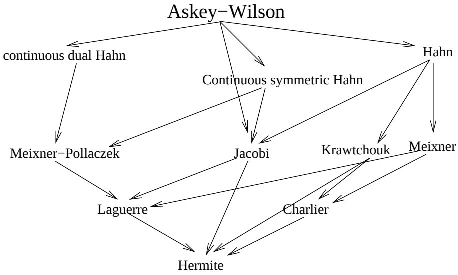
Applications to Orthogonal polynomials

- Choose a measure μ. We say that $\{P_k(x)\}_{k\geq 0}$ is a family of *orthogonal polynomials* if $\int P_j(x)P_k(x)d\mu(x) = 0$ for $j \neq k$.
- Given such a measure (or family of orthogonal polynomials), we define the Nth moment μ_N to be $\mu_N = \int x^N d\mu$.
- Note: if one understands the moments, then by linearity, one can integrate any polynomial with respect to the measure.
- *Askey-Wilson polynomials* $P_n(x, a, b, c, d|q)$ are at the top of the hierarchy of classical orthogonal polynomials in one variable.



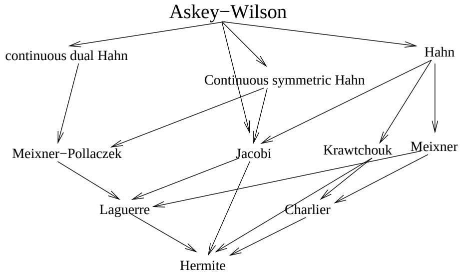
Applications to Orthogonal polynomials

- Choose a measure μ. We say that $\{P_k(x)\}_{k\geq 0}$ is a family of *orthogonal polynomials* if $\int P_j(x)P_k(x)d\mu(x) = 0$ for $j \neq k$.
- Given such a measure (or family of orthogonal polynomials), we define the Nth moment μ_N to be $\mu_N = \int x^N d\mu$.
- Note: if one understands the moments, then by linearity, one can integrate any polynomial with respect to the measure.
- *Askey-Wilson polynomials* $P_n(x, a, b, c, d|q)$ are at the top of the hierarchy of classical orthogonal polynomials in one variable.



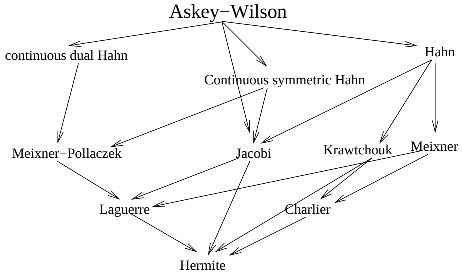
Applications to Orthogonal polynomials

- Choose a measure μ. We say that $\{P_k(x)\}_{k\geq 0}$ is a family of *orthogonal polynomials* if $\int P_j(x)P_k(x)d\mu(x) = 0$ for $j \neq k$.
- Given such a measure (or family of orthogonal polynomials), we define the Nth moment μ_N to be $\mu_N = \int x^N d\mu$.
- Note: if one understands the moments, then by linearity, one can integrate any polynomial with respect to the measure.
- *Askey-Wilson polynomials* $P_n(x, a, b, c, d|q)$ are at the top of the hierarchy of classical orthogonal polynomials in one variable.



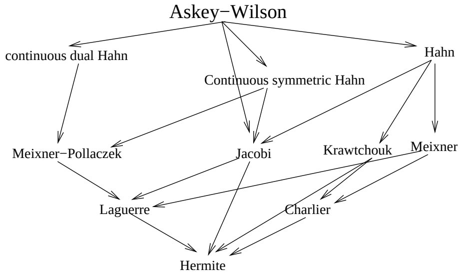
Applications to Orthogonal polynomials

- Choose a measure μ. We say that $\{P_k(x)\}_{k\geq 0}$ is a family of *orthogonal polynomials* if $\int P_j(x)P_k(x)d\mu(x) = 0$ for $j \neq k$.
- Given such a measure (or family of orthogonal polynomials), we define the Nth moment μ_N to be $\mu_N = \int x^N d\mu$.
- Note: if one understands the moments, then by linearity, one can integrate any polynomial with respect to the measure.
- *Askey-Wilson polynomials* $P_n(x, a, b, c, d|q)$ are at the top of the hierarchy of classical orthogonal polynomials in one variable.



Applications to Orthogonal polynomials

- Choose a measure μ. We say that $\{P_k(x)\}_{k\geq 0}$ is a family of *orthogonal polynomials* if $\int P_j(x)P_k(x)d\mu(x) = 0$ for $j \neq k$.
- Given such a measure (or family of orthogonal polynomials), we define the Nth moment μ_N to be $\mu_N = \int x^N d\mu$.
- Note: if one understands the moments, then by linearity, one can integrate any polynomial with respect to the measure.
- *Askey-Wilson polynomials* $P_n(x, a, b, c, d|q)$ are at the top of the hierarchy of classical orthogonal polynomials in one variable.



Combinatorics of (one-variable) orthogonal polynomials

- Since the 1970's, there was a lot of work developing a combinatorial theory of orthogonal polynomials (Viennot, Flajolet, Foata, Stanton, Ismail, . . .), but no results for Askey-Wilson polynomials.
- 2005: Uchiyama-Sasamoto-Wadati discovered a close link between the ASEP and the Askey-Wilson moments. Tridiagonal solution to Matrix Ansatz ...
- 2011: Using USW and our work on the ASEP, we gave a formula for Askey-Wilson moments in terms of staircase tableaux (Corteel-W.)

Combinatorics of (one-variable) orthogonal polynomials

- Since the 1970's, there was a lot of work developing a combinatorial theory of orthogonal polynomials (Viennot, Flajolet, Foata, Stanton, Ismail, ...), but no results for Askey-Wilson polynomials.
- 2005: Uchiyama-Sasamoto-Wadati discovered a close link between the ASEP and the Askey-Wilson moments. Tridiagonal solution to Matrix Ansatz ...
- 2011: Using USW and our work on the ASEP, we gave a formula for Askey-Wilson moments in terms of staircase tableaux (Corteel-W.)

Combinatorics of (one-variable) orthogonal polynomials

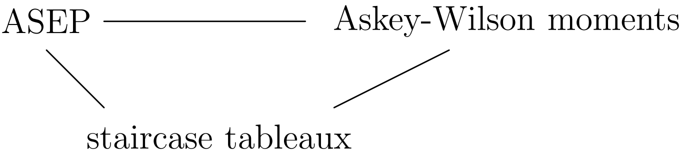
- Since the 1970's, there was a lot of work developing a combinatorial theory of orthogonal polynomials (Viennot, Flajolet, Foata, Stanton, Ismail, . . .), but no results for Askey-Wilson polynomials.
- 2005: Uchiyama-Sasamoto-Wadati discovered a close link between the ASEP and the Askey-Wilson moments. Tridiagonal solution to Matrix Ansatz ...
- 2011: Using USW and our work on the ASEP, we gave a formula for Askey-Wilson moments in terms of staircase tableaux (Corteel-W.)

Combinatorics of (one-variable) orthogonal polynomials

- Since the 1970's, there was a lot of work developing a combinatorial theory of orthogonal polynomials (Viennot, Flajolet, Foata, Stanton, Ismail, . . .), but no results for Askey-Wilson polynomials.
- 2005: Uchiyama-Sasamoto-Wadati discovered a close link between the ASEP and the Askey-Wilson moments. Tridiagonal solution to Matrix Ansatz ...
- 2011: Using USW and our work on the ASEP, we gave a formula for Askey-Wilson moments in terms of staircase tableaux (Corteel-W.)

Combinatorics of (one-variable) orthogonal polynomials

- Since the 1970's, there was a lot of work developing a combinatorial theory of orthogonal polynomials (Viennot, Flajolet, Foata, Stanton, Ismail, . . .), but no results for Askey-Wilson polynomials.
- 2005: Uchiyama-Sasamoto-Wadati discovered a close link between the ASEP and the Askey-Wilson moments. Tridiagonal solution to Matrix Ansatz ...
- 2011: Using USW and our work on the ASEP, we gave a formula for Askey-Wilson moments in terms of staircase tableaux (Corteel-W.)



(Macdonald-)Koornwinder polynomials

- Let $x = (x_1, \dots, x_m)$, $\lambda = (\lambda_1, \dots, \lambda_m)$ be a partition, and a, b, c, d, q, t be generic complex parameters.
- The *Koornwinder polynomials* $P_\lambda(x; a, b, c, d|q, t)$ are multivariate orthogonal polynomials which are the type BC-case of Macdonald polynomials. Include **Askey-Wilson polynomials** as a limiting case.
- Macdonald polynomials have deep relationship with affine Hecke algebras and Hilbert schemes. In type A, lots of amazing combinatorics: Haiman (n! conjecture), Haglund-Haiman-Loehr explicit formula, etc.
- So far not much combinatorics of Koornwinder polynomials. But understanding these would be very desirable – the Macdonald polynomials associated to any classical root system can be expressed as limits or special cases of Koornwinder polynomials (Van Diejen).

(Macdonald-)Koornwinder polynomials

- Let $x = (x_1, \ldots, x_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, and a, b, c, d, q, t be generic complex parameters.
- The *Koornwinder polynomials* $P_\lambda(x; a, b, c, d|q, t)$ are multivariate orthogonal polynomials which are the type BC-case of Macdonald polynomials. Include **Askey-Wilson polynomials** as a limiting case.
- Macdonald polynomials have deep relationship with affine Hecke algebras and Hilbert schemes. In type A, lots of amazing combinatorics: Haiman (n! conjecture), Haglund-Haiman-Loehr explicit formula, etc.
- So far not much combinatorics of Koornwinder polynomials. But understanding these would be very desirable – the Macdonald polynomials associated to any classical root system can be expressed as limits or special cases of Koornwinder polynomials (Van Diejen).

(Macdonald-)Koornwinder polynomials

- Let $x = (x_1, \ldots, x_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, and a, b, c, d, q, t be generic complex parameters.
- The *Koornwinder polynomials* $P_\lambda(x; a, b, c, d|q, t)$ are multivariate orthogonal polynomials which are the type BC-case of Macdonald polynomials. Include **Askey-Wilson polynomials** as a limiting case.
- Macdonald polynomials have deep relationship with affine Hecke algebras and Hilbert schemes. In type A, lots of amazing combinatorics: Haiman (n! conjecture), Haglund-Haiman-Loehr explicit formula, etc.
- So far not much combinatorics of Koornwinder polynomials. But understanding these would be very desirable – the Macdonald polynomials associated to any classical root system can be expressed as limits or special cases of Koornwinder polynomials (Van Diejen).

(Macdonald-)Koornwinder polynomials

- Let $x = (x_1, \ldots, x_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, and a, b, c, d, q, t be generic complex parameters.
- The *Koornwinder polynomials* $P_\lambda(x; a, b, c, d|q, t)$ are multivariate orthogonal polynomials which are the type BC-case of Macdonald polynomials. Include **Askey-Wilson polynomials** as a limiting case.
- Macdonald polynomials have deep relationship with affine Hecke algebras and Hilbert schemes. In type A, lots of amazing combinatorics: Haiman (n! conjecture), Haglund-Haiman-Loehr explicit formula, etc.
- So far not much combinatorics of Koornwinder polynomials. But understanding these would be very desirable – the Macdonald polynomials associated to any classical root system can be expressed as limits or special cases of Koornwinder polynomials (Van Diejen).

(Macdonald-)Koornwinder polynomials

- Let $x = (x_1, \ldots, x_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, and a, b, c, d, q, t be generic complex parameters.
- The *Koornwinder polynomials* $P_\lambda(x; a, b, c, d|q, t)$ are multivariate orthogonal polynomials which are the type BC-case of Macdonald polynomials. Include **Askey-Wilson polynomials** as a limiting case.
- Macdonald polynomials have deep relationship with affine Hecke algebras and Hilbert schemes. In type A, lots of amazing combinatorics: Haiman (n! conjecture), Haglund-Haiman-Loehr explicit formula, etc.
- So far not much combinatorics of Koornwinder polynomials. But understanding these would be very desirable – the Macdonald polynomials associated to any classical root system can be expressed as limits or special cases of Koornwinder polynomials (Van Diejen).

(Macdonald-)Koornwinder polynomials

- Let $x = (x_1, \ldots, x_m)$, $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition, and a, b, c, d, q, t be generic complex parameters.
- The *Koornwinder polynomials* $P_\lambda(x; a, b, c, d|q, t)$ are multivariate orthogonal polynomials which are the type BC-case of Macdonald polynomials. Include **Askey-Wilson polynomials** as a limiting case.
- Macdonald polynomials have deep relationship with affine Hecke algebras and Hilbert schemes. In type A, lots of amazing combinatorics: Haiman (n! conjecture), Haglund-Haiman-Loehr explicit formula, etc.
- So far not much combinatorics of Koornwinder polynomials. But understanding these would be very desirable – the Macdonald polynomials associated to any classical root system can be expressed as limits or special cases of Koornwinder polynomials (Van Diejen).

Koornwinder polynomials and Koornwinder moments

- Question (Haiman): Can one generalize the relationships

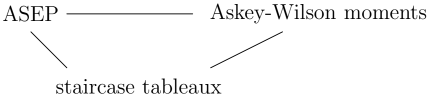
  replacing "Askey-Wilson" by Koornwinder?
- First need to define *Koornwinder moments*. How do we generalize $\int x^n d\mu$ in a multivariable setting? One option: replace x^n by the degree n homogeneous symmetric polynomial in m variables.
- At $q = t$, Koornwinder polynomials are equal to

$$P_\lambda(x; a, b, c, d|q, q) = const \cdot \frac{\det(p_{m-j+\lambda_j}(x_i; a, b, c, d|q))_{i,j=1}^m}{\det(p_{m-j}(x_i; a, b, c, d|q))_{i,j=1}^m},$$

where the p_i's are Askey-Wilson polynomials.

Koornwinder polynomials and Koornwinder moments

- Question (Haiman): Can one generalize the relationships



replacing "Askey-Wilson" by Koornwinder?

- First need to define *Koornwinder moments.* How do we generalize $\int x^n d\mu$ in a multivariable setting? One option: replace x^n by the degree n homogeneous symmetric polynomial in m variables.

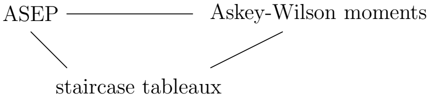
- At $q = t$, Koornwinder polynomials are equal to

$$P_\lambda(x; a, b, c, d|q, q) = const \cdot \frac{\det(p_{m-j+\lambda_j}(x_i; a, b, c, d|q))_{i,j=1}^m}{\det(p_{m-j}(x_i; a, b, c, d|q))_{i,j=1}^m},$$

where the p_i's are Askey-Wilson polynomials.

Koornwinder polynomials and Koornwinder moments

- Question (Haiman): Can one generalize the relationships



replacing "Askey-Wilson" by Koornwinder?

- First need to define *Koornwinder moments*. How do we generalize $\int x^n d\mu$ in a multivariable setting? One option: replace x^n by the degree n homogeneous symmetric polynomial in m variables.

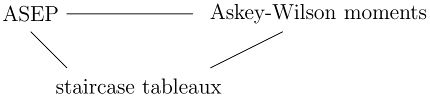
- At $q = t$, Koornwinder polynomials are equal to

$$P_\lambda(x; a, b, c, d|q, q) = const \cdot \frac{\det(p_{m-j+\lambda_j}(x_i; a, b, c, d|q))_{i,j=1}^m}{\det(p_{m-j}(x_i; a, b, c, d|q))_{i,j=1}^m},$$

where the p_i's are Askey-Wilson polynomials.

Koornwinder polynomials and Koornwinder moments

- Question (Haiman): Can one generalize the relationships



replacing "Askey-Wilson" by Koornwinder?

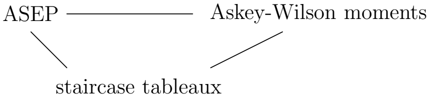
- First need to define *Koornwinder moments*. How do we generalize $\int x^n d\mu$ in a multivariable setting? One option: replace x^n by the degree n homogeneous symmetric polynomial in m variables.
- At $q = t$, Koornwinder polynomials are equal to

$$P_\lambda(x; a, b, c, d|q, q) = const \cdot \frac{\det(p_{m-j+\lambda_j}(x_i; a, b, c, d|q))_{i,j=1}^m}{\det(p_{m-j}(x_i; a, b, c, d|q))_{i,j=1}^m},$$

where the p_i's are Askey-Wilson polynomials.

Koornwinder polynomials and Koornwinder moments

- Question (Haiman): Can one generalize the relationships



replacing "Askey-Wilson" by Koornwinder?

- First need to define *Koornwinder moments*. How do we generalize $\int x^n d\mu$ in a multivariable setting? One option: replace x^n by the degree n homogeneous symmetric polynomial in m variables.

- At $q = t$, Koornwinder polynomials are equal to

$$P_\lambda(x; a, b, c, d|q, q) = const \cdot \frac{\det(p_{m-j+\lambda_j}(x_i; a, b, c, d|q))_{i,j=1}^m}{\det(p_{m-j}(x_i; a, b, c, d|q))_{i,j=1}^m},$$

where the p_i's are Askey-Wilson polynomials.

Koornwinder polynomials and Koornwinder moments

- Eric Rains defined the following "Koornwinder moments at $q = t$": for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, set

$$M_\lambda = I_K(s_\lambda(x_1, \ldots, x_m); a, b, c, d; q, q),$$

  where s_λ is a Schur polynomial, and I_K means integrate with respect to the Koornwinder density.

- One can express M_λ as a ratio of two determinants in the Askey-Wilson moments.

- When $\lambda = (\lambda_1, 0, 0, \ldots, 0)$, we are integrating a homogeneous symmetric polynomial of degree λ_1 in $x_1, \ldots, x_m$, which is a very natural analogue of $\int x^n d\mu$.

Koornwinder polynomials and Koornwinder moments

- Eric Rains defined the following "Koornwinder moments at $q = t$": for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, set

$$M_\lambda = I_K(s_\lambda(x_1, \dots, x_m); a, b, c, d; q, q),$$

where s_λ is a Schur polynomial, and I_K means integrate with respect to the Koornwinder density.

- One can express M_λ as a ratio of two determinants in the Askey-Wilson moments.

- When $\lambda = (\lambda_1, 0, 0, \dots, 0)$, we are integrating a homogeneous symmetric polynomial of degree λ_1 in $x_1, \dots, x_m$, which is a very natural analogue of $\int x^n d\mu$.

Koornwinder polynomials and Koornwinder moments

- Eric Rains defined the following "Koornwinder moments at $q = t$": for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, set

$$M_\lambda = I_K(s_\lambda(x_1, \ldots, x_m); a, b, c, d; q, q),$$

  where s_λ is a Schur polynomial, and I_K means integrate with respect to the Koornwinder density.

- One can express M_λ as a ratio of two determinants in the Askey-Wilson moments.

- When $\lambda = (\lambda_1, 0, 0, \ldots, 0)$, we are integrating a homogeneous symmetric polynomial of degree λ_1 in $x_1, \ldots, x_m$, which is a very natural analogue of $\int x^n d\mu$.

Koornwinder polynomials and Koornwinder moments

- Eric Rains defined the following "Koornwinder moments at $q = t$": for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, set

$$M_\lambda = I_K(s_\lambda(x_1, \ldots, x_m); a, b, c, d; q, q),$$

  where s_λ is a Schur polynomial, and I_K means integrate with respect to the Koornwinder density.

- One can express M_λ as a ratio of two determinants in the Askey-Wilson moments.

- When $\lambda = (\lambda_1, 0, 0, \ldots, 0)$, we are integrating a homogeneous symmetric polynomial of degree λ_1 in $x_1, \ldots, x_m$, which is a very natural analogue of $\int x^n d\mu$.

Koornwinder polynomials and Koornwinder moments

Theorem (Corteel-W.)

Let $Z_{N,r}$ be the partition function for the two-species ASEP on a lattice of N sites with precisely r light particles. We have

$$Z_{N,r} = \text{const. } M_{(N-r,0^r)},$$

where $M_{(N-r,0^r)}$ is the homogeneous Koornwinder moment (and there is a particular change of variable between $\alpha, \beta, \gamma, \delta$ and a, b, c, d).

Theorem (Corteel-Mandelshtam-W.):

$M_{(N-r,0^r)} = \text{const.} \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, where the sum is over all rhombic staircase tableaux of size N with precisely r diagonal steps on the border.

Koornwinder polynomials and Koornwinder moments

Theorem (Corteel-W.)

Let $Z_{N,r}$ be the partition function for the two-species ASEP on a lattice of N sites with precisely r light particles. We have

$$Z_{N,r} = \text{const. } M_{(N-r,0^r)},$$

where $M_{(N-r,0^r)}$ is the homogeneous Koornwinder moment (and there is a particular change of variable between $\alpha, \beta, \gamma, \delta$ and a, b, c, d).

Theorem (Corteel-Mandelshtam-W.):

$M_{(N-r,0^r)} = \text{const.} \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, where the sum is over all rhombic staircase tableaux of size N with precisely r diagonal steps on the border.

Koornwinder polynomials and Koornwinder moments

Theorem (Corteel-W.)

Let $Z_{N,r}$ be the partition function for the two-species ASEP on a lattice of N sites with precisely r light particles. We have

$$Z_{N,r} = \text{const. } M_{(N-r,0^r)},$$

where $M_{(N-r,0^r)}$ is the homogeneous Koornwinder moment (and there is a particular change of variable between $\alpha, \beta, \gamma, \delta$ and a, b, c, d).

Theorem (Corteel-Mandelshtam-W.):

$M_{(N-r,0^r)} = \text{const.} \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, where the sum is over all rhombic staircase tableaux of size N with precisely r diagonal steps on the border.

Koornwinder polynomials and Koornwinder moments

Theorem (Corteel-W.)

Let $Z_{N,r}$ be the partition function for the two-species ASEP on a lattice of N sites with precisely r light particles. We have

$$Z_{N,r} = \text{const. } M_{(N-r,0^r)},$$

where $M_{(N-r,0^r)}$ is the homogeneous Koornwinder moment (and there is a particular change of variable between $\alpha, \beta, \gamma, \delta$ and a, b, c, d).

Theorem (Corteel-Mandelshtam-W.):

$M_{(N-r,0^r)} = \text{const.} \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, where the sum is over all rhombic staircase tableaux of size N with precisely r diagonal steps on the border.

Rk: $M_{(N-r,0^r)}$ is the evaluation of a Koornwinder polynomial at $x_i = 1 \forall i$.

Koornwinder polynomials and Koornwinder moments

Theorem (Corteel-W.)

Let $Z_{N,r}$ be the partition function for the two-species ASEP on a lattice of N sites with precisely r light particles. We have

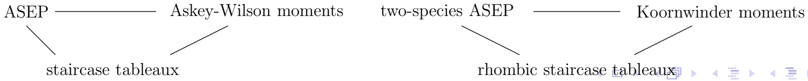
$$Z_{N,r} = \text{const. } M_{(N-r,0^r)},$$

where $M_{(N-r,0^r)}$ is the homogeneous Koornwinder moment (and there is a particular change of variable between $\alpha, \beta, \gamma, \delta$ and a, b, c, d).

Theorem (Corteel-Mandelshtam-W.):

$M_{(N-r,0^r)} = \text{const.} \sum_{\mathcal{T}} \text{wt}(\mathcal{T})$, where the sum is over all rhombic staircase tableaux of size N with precisely r diagonal steps on the border.

Rk: $M_{(N-r,0^r)}$ is the evaluation of a Koornwinder polynomial at $x_i = 1 \forall i$.



Next steps

- Positivity conjecture: M_λ can be written as a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients. (We proved it for partitions with one non-zero part.)
- Use rhombic staircase tableaux to give formulas for Koornwinder moments and polynomials? (Cantini: our Koornwinder moments are specializations of certain Koornwinder polynomials)
- Particle model interpretation of general moments M_λ?
- Relate our work to that of Cantini, and Cantini-de Gier-Wheeler, which also relates various Macdonald polynomials to multispecies exclusion process?
- Relate our work to that of Borodin-Corwin on Macdonald processes?
- Is there a dynamics on the tableaux themselves?

Next steps

- Positivity conjecture: M_λ can be written as a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients. (We proved it for partitions with one non-zero part.)
- Use rhombic staircase tableaux to give formulas for Koornwinder moments and polynomials? (Cantini: our Koornwinder moments are specializations of certain Koornwinder polynomials)
- Particle model interpretation of general moments M_λ?
- Relate our work to that of Cantini, and Cantini-de Gier-Wheeler, which also relates various Macdonald polynomials to multispecies exclusion process?
- Relate our work to that of Borodin-Corwin on Macdonald processes?
- Is there a dynamics on the tableaux themselves?

Next steps

- Positivity conjecture: M_λ can be written as a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients. (We proved it for partitions with one non-zero part.)
- Use rhombic staircase tableaux to give formulas for Koornwinder moments and polynomials? (Cantini: our Koornwinder moments are specializations of certain Koornwinder polynomials)
- Particle model interpretation of general moments M_λ?
- Relate our work to that of Cantini, and Cantini-de Gier-Wheeler, which also relates various Macdonald polynomials to multispecies exclusion process?
- Relate our work to that of Borodin-Corwin on Macdonald processes?
- Is there a dynamics on the tableaux themselves?

Next steps

- Positivity conjecture: M_λ can be written as a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients. (We proved it for partitions with one non-zero part.)
- Use rhombic staircase tableaux to give formulas for Koornwinder moments and polynomials? (Cantini: our Koornwinder moments are specializations of certain Koornwinder polynomials)
- Particle model interpretation of general moments M_λ?
- Relate our work to that of Cantini, and Cantini-de Gier-Wheeler, which also relates various Macdonald polynomials to multispecies exclusion process?
- Relate our work to that of Borodin-Corwin on Macdonald processes?
- Is there a dynamics on the tableaux themselves?

Next steps

- Positivity conjecture: M_λ can be written as a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients. (We proved it for partitions with one non-zero part.)
- Use rhombic staircase tableaux to give formulas for Koornwinder moments and polynomials? (Cantini: our Koornwinder moments are specializations of certain Koornwinder polynomials)
- Particle model interpretation of general moments M_λ?
- Relate our work to that of Cantini, and Cantini-de Gier-Wheeler, which also relates various Macdonald polynomials to multispecies exclusion process?
- Relate our work to that of Borodin-Corwin on Macdonald processes?
- Is there a dynamics on the tableaux themselves?

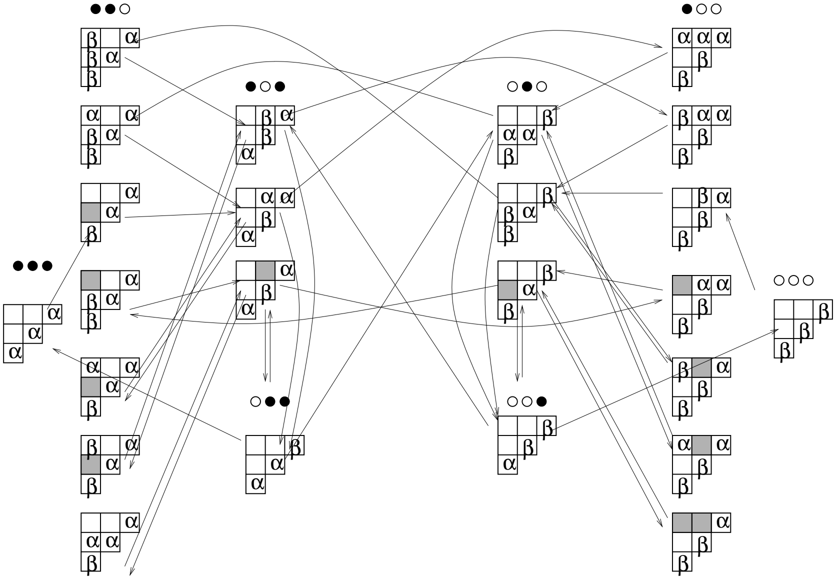
Next steps

- Positivity conjecture: M_λ can be written as a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients. (We proved it for partitions with one non-zero part.)
- Use rhombic staircase tableaux to give formulas for Koornwinder moments and polynomials? (Cantini: our Koornwinder moments are specializations of certain Koornwinder polynomials)
- Particle model interpretation of general moments M_λ?
- Relate our work to that of Cantini, and Cantini-de Gier-Wheeler, which also relates various Macdonald polynomials to multispecies exclusion process?
- Relate our work to that of Borodin-Corwin on Macdonald processes?
- Is there a dynamics on the tableaux themselves?

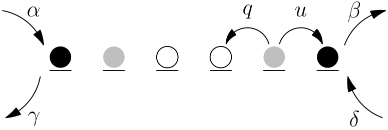
Next steps

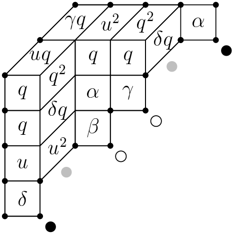
- Positivity conjecture: M_λ can be written as a polynomial in $\alpha, \beta, \gamma, \delta, q$ with positive coefficients. (We proved it for partitions with one non-zero part.)
- Use rhombic staircase tableaux to give formulas for Koornwinder moments and polynomials? (Cantini: our Koornwinder moments are specializations of certain Koornwinder polynomials)
- Particle model interpretation of general moments M_λ?
- Relate our work to that of Cantini, and Cantini-de Gier-Wheeler, which also relates various Macdonald polynomials to multispecies exclusion process?
- Relate our work to that of Borodin-Corwin on Macdonald processes?
- Is there a dynamics on the tableaux themselves?

Dynamics on staircase tableaux when $\gamma = \delta = 0$



Thank you for listening!





- Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials (with Corteel), *Duke Math.*, 2011.
- Macdonald-Koornwinder moments and the two-species exclusion process (with Corteel), `arXiv:1505.00843`.
- Combinatorics of the two-species ASEP and Koornwinder moments (with Corteel and Mandelshtam), `arXiv:1510.05023`.

ASEP —— Askey-Wilson moments   two-species ASEP —— Koornwinder moments

staircase tableaux   rhombic staircase tableaux

Relationship between ASEP and Askey-Wilson moments

Let $Z_N(\xi;\alpha,\beta,\gamma,\delta;q) = \sum_{\mathcal{T}} \mathrm{wt}(\mathcal{T})\xi^{b(\mathcal{T})}$, where $b(\mathcal{T})$ is the number of black particles in the type of $\mathcal{T}$. This is the *fugacity partition function*.

Theorem (Corteel-Stanton-Stanley-W.)

The N^{th} Askey-Wilson moment is equal to

$$\mu_N(a,b,c,d|q) = \frac{(1-q)^N}{2^N i^N} Z_N(-1;\alpha,\beta,\gamma,\delta;q),$$

where $i^2 = -1$ and

$$\alpha = \frac{1-q}{1-ac+ai+ci}, \qquad \beta = \frac{1-q}{1-bd-bi-di},$$
$$\gamma = \frac{(1-q)ac}{1-ac+ai+ci}, \qquad \delta = \frac{(1-q)bd}{1-bd-bi-di}.$$