The Atlas model, in and out of equilibrium

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Jointly with Tsai (in equilibrium)
and
with Cabezas, Sarantsev, Sidoravicius (out of equilibrium)
\( X(t) = (X_i(t), i \geq 0) \) is \( \mathbb{R}^N \)-valued stochastic process.

Markov process: for \( t \geq s \geq 0 \) and suitable \( f(\cdot) \in S \):
\[
\mathbb{E}[f(X(t))|\mathcal{F}_s^X] = \int p_{t-s}(X(s), dy)f(y) =: (p_{t-s}f)(X(s)) .
\]

\( \{p_u(\cdot, \cdot)\} \) transition probabilities semi-group:

(a) \( p_u(x, \cdot) \) probability measure on \( \mathbb{R}^N \), per \( u, x \).
(b) \( p_u(\cdot, A) \) Borel function, per \( u, A \subset \mathbb{R}^N \) Borel.
(c) Semi-group: \( p_{u+v}(x, A) = \int p_u(x, dy)p_v(y, A) \), for \( u, v \geq 0 \).

Brownian Motion: \( t \mapsto W_i(t) \) continuous, Markov process
\[
p_t(x, A) = \int_A p_t(x - y)dy, \text{ heat kernel } p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} \text{ (here } N = 1).\]
\[
u(t, x) = (p_t f)(x) \text{ solves HE: } u_t = \frac{1}{2} u_{xx}.
\]

Brownian scaling: \( W_i^b(t) = bW_i(t/b^2) \overset{(d)}{=} W_i(t) \) for any \( b > 0 \).

\((W_i(t), t \geq 0), i \geq 0 \) independent BM-s \( \iff \) product measures.

If \( f(\underline{x}) = \prod_i g_i(x_i) \) then \( (p_t f)(\underline{x}) = \prod_i (p_t g_i)(x_i) \).
$Z_k \sim \text{Exp}(\lambda) \iff P(Z_k \geq z) = e^{-\lambda z}, \ z \geq 0, \ \lambda > 0.$

Independent Exponentials $Z^{(\lambda)} := (Z_k, k \geq 1) \sim \rho_\lambda = \bigotimes_{k=1}^\infty \text{Exp}(\lambda)$.

Poisson process has points at $Y = (Y_k, k \geq 1)$:
$(Y_k, k \geq 1) \sim \text{PPP}_+(\lambda) \iff Y_1 = 0, \ Y_{k+1} = Y_k + Z_k, \ k \geq 1$

Continuous $\mathbb{R}$-valued $t \mapsto M(t)$ is $L^2$-MG
$\iff \mathbb{E}[M_t | \mathcal{F}_s^M] = M_s \quad \& \quad \mathbb{E}[M_t^2] < \infty \ \forall t \geq s \geq 0.$
$\implies M_t^2 - [M]_t$ is MG (for quadratic variation $[M]_t$), by Doob-Meyer.

For $f \in C_b^{1,2}(\mathbb{R})$ let $\mathcal{L}f = f_t + \frac{1}{2}f_{xx}$.

Ito’s lemma: $M_t^f := f(t, W(t)) - f(0, W(0)) - \int_0^t (\mathcal{L}f)(s, W(s))ds$ is $L^2$-MG
$[M^f]_t = \int_0^t f_x^2(s, W(s))ds.$
Interacting particles: Markov process $R(t)$ with interaction.

**SSEP:** $R(t) \in \{0, 1\}^\mathbb{Z}$.

Jumps: $\Delta_k(i) \in \{-1, +1\}$ i.i.d. $P(\Delta_k(i) = +1) = \frac{1}{2}$ independent of i.i.d. PPP$_+$ $(1)$ 'clock' processes $\{\tau_k(i)\}$ for $i \in \mathbb{Z}$.

Order $\{\tau_k(i)\}$, $i \in \mathbb{Z}$ and $k \geq 2$.
Sequentially, if $R_i(\tau_k(i)) = 1$ and $R_{i+\Delta_k(i)}(\tau_k(i)) = 0$ exchange these values. Otherwise, do nothing (exclusion).

Hydrodynamics: $b \sum_{i=0}^{x/b} R_i(t/b^2) \rightarrow Q_*(t, x)$ as $b \rightarrow 0$.
$Q_*$ non-random solves some PDE (for suitable $R(0)$).
\[ X_i(t) = X_i(0) + W_i(t) + \int_0^t 1_{\{X_i(s) = X(0)(s)\}} \, ds, \quad i \geq 0. \]

\((W_i(t), t \geq 0), i \geq 0\) independent BM-s.

\[ X(0) = (X_i(0), i \geq 0) \sim \text{PPP}_+(\lambda), \quad \lambda \in (0, \infty), \]

\[ \iff Z(0) = Z^{(\lambda)} \sim \rho_\lambda = \bigotimes_{k=1}^{\infty} \text{Exp}(\lambda). \]

\[ X(0)(t) = \min_i \{X_i(t)\} \quad \text{left-most particle.} \]

Ranked process \(Y\) and spacings process \(Z\):

\[ Z_k(t) := Y_{k+1}(t) - Y_k(t) := X_k(t) - X_{(k-1)}(t), \quad k \geq 1 \]

\((Y_k(\cdot) \text{ and } Z_k(\cdot) \text{ are } k-\text{th ranked particle and } k-\text{th spacing}, \text{ resp.}).\)

[Ichiba-Karatzas-Shkolnikov 13, Pal-Pitman 08] \(\exists\) unique, rankable weak sol. \(X\).
RBM representation for $Z(t)$ based on

$$Y_k(t) - Y_k(0) = t \mathbf{1}_{\{k=1\}} + B_k(t) + L_{k-1}(t) - L_k(t)$$

($B(t)$) independent BM-s

$L_0(t) = 0$, $L_k(t)$ local time at $\{Z_k(s) = 0\}$, $k \geq 1$ (collisions).
[Pal-Pitman 08] \( \lambda = 2 \Rightarrow \) Spacings equilibrium \( (Z(t) \overset{d}{=} Z(0)) \).

(利用 [Williams 87] 工作在 RBM-s on polyhedra).

[Conj. 2]: Unique invariant measure (Open).

[Conj. 3]: (resolved in [D-Tsai 15]).

\[
t^{-1/4} X_{(0)}(t) \overset{d}{\to} N(0, c), \quad t \to \infty, \quad \text{some } c \in (0, \infty).
\]

(标记粒子的 Harris 系统 [Harris 65, Dürr-Goldstein-Lebowitz 85], 和 SSEP [Arratia 83, Rost-Vares 85, Landim-Volchan 00, De Masi-Ferrari 02])

By spacing equilibrium, [D-Tsai 15] resolve [Conj. 3, PP08] by showing that asymptotic fluctuation at scale \( b^{-1/2} \) follows ASHE with Neumann BC at 0.

**Question**: Out of equilibrium? Expects

\[
X_{(0)}(s) \to \pm \infty, \quad \text{according to } \quad \text{sgn}(2 - \lambda).
\]
Asymptotics $b \downarrow 0$ of point processes on $\mathbb{R}_+ \times \mathbb{R}$

$$Q^b(t, \cdot) := b \sum_{i=0}^{\infty} \delta_{t, X^b_i(t)}, \quad X^b_i(t) = bX_i(t/b^2), \quad i \geq 0.$$ 

$Q^b(t, \cdot) \in M_*(\mathbb{R}) = \{ \text{all Borel } \mu \geq 0 \text{ with } \mu(( - \infty, r]) \text{ finite } \forall r \}$,

$C_* := \{ f \in C_b(\mathbb{R}) \text{ eventually zero}\}$-topology, metrizable by $d_*$.

$Q^b(\cdot, \cdot) \in \mathcal{C} = \{ \text{all continuous } t \mapsto \mu(t, \cdot) : \mathbb{R}_+ \to (M_*(\mathbb{R}), d_*) \}$,

with topology of uniform convergence on compacts in $\mathbb{R}_+$. 
Theorem (CDSS 15)

For $\text{ATLAS}_\infty(\lambda)$ as $b \to 0$ we have $Q^b(\cdot, \cdot) \to Q_*(\cdot, \cdot)$ in $\mathcal{C}$.

The $Q_*$-density with respect to Lebesgue

$$u_*(t, x) := \left[ c_1 + c_2 \Phi(x/\sqrt{t}) \right] 1_{\{x > y_*(t)\}} , \quad y_*(t) := \kappa \sqrt{t} , \quad \forall t > 0$$

$\Phi(\cdot)$ CDF of $N(0,1)$ and

$$c_1 := \frac{2 - \lambda \Phi(\kappa)}{1 - \Phi(\kappa)} , \quad c_2 := \frac{\lambda - 2}{1 - \Phi(\kappa)} .$$

$\text{sgn}(\kappa) = \text{sgn}(2 - \lambda)$ for $\kappa$ unique such that

$$\frac{\kappa(1 - \Phi(\kappa))}{\Phi'(\kappa)} \equiv 1 - \frac{\lambda}{2} .$$

Left-most particle $X^b_{(0)}(t) \to y_*(t)$ as $b \to 0$ (uniformly on compacts).
Stefan problem for $\text{ATLAS}_\infty(\lambda)$

$y_*(t) = \inf\{x : u_*(t, x) > 0\}$ differentiable and $u_*(t, x)$ unique, uniformly bounded and uniformly positive on $x > y(t)$, solution of 1-sided Stefan problem:

$$u_t(t, x) = \frac{1}{2} u_{xx}(t, x), \quad \forall x > y(t). \quad \text{HE}$$

$$\lim_{t \downarrow 0} u(t, x) = \lambda 1_{x > 0}, \quad \forall x \neq 0. \quad \text{IC}$$

$$u(t, y(t)^+) := \lim_{x \downarrow y(t)} u(t, x) = 2, \quad \forall t > 0. \quad \text{EQ-LBV}$$

$$u(t, y(t)^+) \frac{dy}{dt}(t) + \frac{1}{2} u_x(t, y(t)^+) = 0, \forall t > 0. \quad \text{FLX-BD}$$
The flux condition: consequences

\[
\frac{dy}{dt} = -\frac{1}{4} u_x(t, y(t)^+) , \forall t > 0 . \quad \text{FLX-BD}
\]

\[
\lambda - 2 > 0 \implies \kappa < 0 \quad \text{(expanding),}
\]

\[
\lambda - 2 < 0 \implies \kappa > 0 \quad \text{(contracting)}.
\]

Non-random rate of expansion/contraction

\[
\lim_{s \to \infty} \frac{Y_1(s)}{\sqrt{s}} = \kappa .
\]

\[
u_*(1, \cdot) \text{ as limiting particle density profile:}
\]

\[
\lim_{s \to \infty} Q^{1/\sqrt{s}}(1, x + [-\epsilon, \epsilon]) = \int_{-\epsilon}^{\epsilon} u_*(1, x + r)dr , \quad \epsilon > 0 .
\]

\# of particles at time \( s \gg 1 \) near \( \sqrt{s}x \) has density \( u_*(1, x) \).
Stochastic monotonicity and spacing at the edge

**Definition**

\[ \xi \preccurlyeq \xi' \iff P(\xi \geq y) \leq P(\xi' \geq y), \quad \forall y \in \mathbb{R}^N. \]

**Theorem (CDSS 15)**

\[ Z(0) = Z^{(\lambda)} \sim \rho_{\lambda}. \]

\[ \lambda < 2 \quad \Rightarrow \quad Z^{(2)} \preccurlyeq Z(t) \preccurlyeq Z(s) \preccurlyeq Z^{(\lambda)}, \quad \forall t \geq s \geq 0, \]

and \( Z(t) \to Z^{(2)} \) (convergence of f.d.d.).

\[ \lambda > 2 \quad \Rightarrow \quad Z^{(\lambda)} \preccurlyeq Z(s) \preccurlyeq Z(t) \preccurlyeq Z^{(2)}, \quad \forall t \geq s \geq 0. \]
[Landim-Olla-Volchan 98] get same Stefan problem for effect of tagged asymmetric particle on (truly) doubly-infinite SSEP, by [Arratia 85] transform of spacings in -SEP to constant rate zero-range process.

Here purely one-sided system. Stochastic monotonicity (RBM theory) plus LD for i.i.d. BM-s and for $\text{PPP}_+(\lambda)$ give pre-compactness/regularity of $\{Q^b, b > 0\}$ ($\mathcal{C}$-limit-points $Q^0$ as $b \to 0$, with bounded $Q^0$-density and $X^b_{(0)}(t) \to y_{Q^0}(t)$).

By Ito’s lemma (diminishing martingale term as $b \to 0$), all limit points satisfy same weak (distributional) form of our Stefan problem. A-priori regularity and standard PDE tools [Ishii 81] give uniqueness of solution.
Space-time particle fluctuations at $\lambda = 2$: Setting

Asymptotics $b \downarrow 0$ of re-scaled point processes on $\mathbb{R}_+ \times \mathbb{R}$

$$\hat{Q}^b(t, \cdot) := \sqrt{b/2} \left[ \sum_{i=0}^{\infty} \delta_{X^b_i(t)} - (2/b) \text{Leb}(\mathbb{R}_+) \right], \quad X^b_i(t) = bX_i(t/b^2), \quad i \geq 0.$$ 

Heat kernel $p_t(x) = \Phi'_t(x)$ for $\Phi_t(x) = \Phi(x/\sqrt{t}) - 1$.

Neumann kernel $p^N_t(y, x) = \partial_y \Psi_t(y, x)$ for

$$\Psi_t(y, x) := \Phi_t(y - x) + \Phi_t(y + x).$$

$B(\cdot)$ Brownian motion, $\mathcal{W}(t, x)$ standard white noise are independent.

$$\hat{\mathcal{W}}_t(x) := \int_0^\infty \Psi_t(y, x) dB(y),$$

$$\hat{\mathcal{M}}_t(x) := \int_0^t \int_0^\infty p^N_{t-s}(y, x) d\mathcal{W}(y, s).$$

$C(\mathbb{R}_+^2; \mathbb{R})$-valued Gaussian process $\hat{U}^0(t, x) = \hat{\mathcal{W}}_t(x) + \hat{\mathcal{M}}_t(x)$, solves the ASHE

$$(\partial_t - \frac{1}{2} \partial_{xx}) \hat{U}^0(t, x) = \mathcal{W}(t, x), \quad \hat{U}^0(0, x) = B(x).$$
Equip $D(\mathbb{R}^2_+)$ with uniform convergence on compacts and let

$$\hat{U}^b(t, x) := \sqrt{b/2} \left( 2X_{\lfloor x/(2b) \rfloor}(t/b^2) - \lfloor x/(2b) \rfloor \right).$$

**Theorem (D-Tsai 15)**

For ATLAS$_\infty(2)$ as $b \to 0$,

$$\hat{U}^b(\cdot, \cdot) \Rightarrow \hat{U}^0(\cdot, \cdot).$$

In particular, $b^{-1/2}X(0)(t/b^2) \Rightarrow (2/\pi)^{1/4}V(t)$ a 1/4-FBM.

$$\hat{U}^b(t, x) \approx F^{b, r}(t, x) := \langle \hat{Q}^b(t, \cdot), \Psi_{b^{1+r}}(\cdot, x + b^r) \rangle$$

(some $0 < r < 1/2$).

Ito's lemma for $F^{b, r}(t, x)$:

- martingale contribution goes to $\hat{M}_t(x)$,
- IC contribution goes in law to $\hat{W}_t(x)$,
- HE and choice of $\Psi$ eliminate $\mathcal{L}F$ part.
Thank you!