Time evolution of out-of-equilibrium integrable models

Jacopo De Nardis

In collaboration with:

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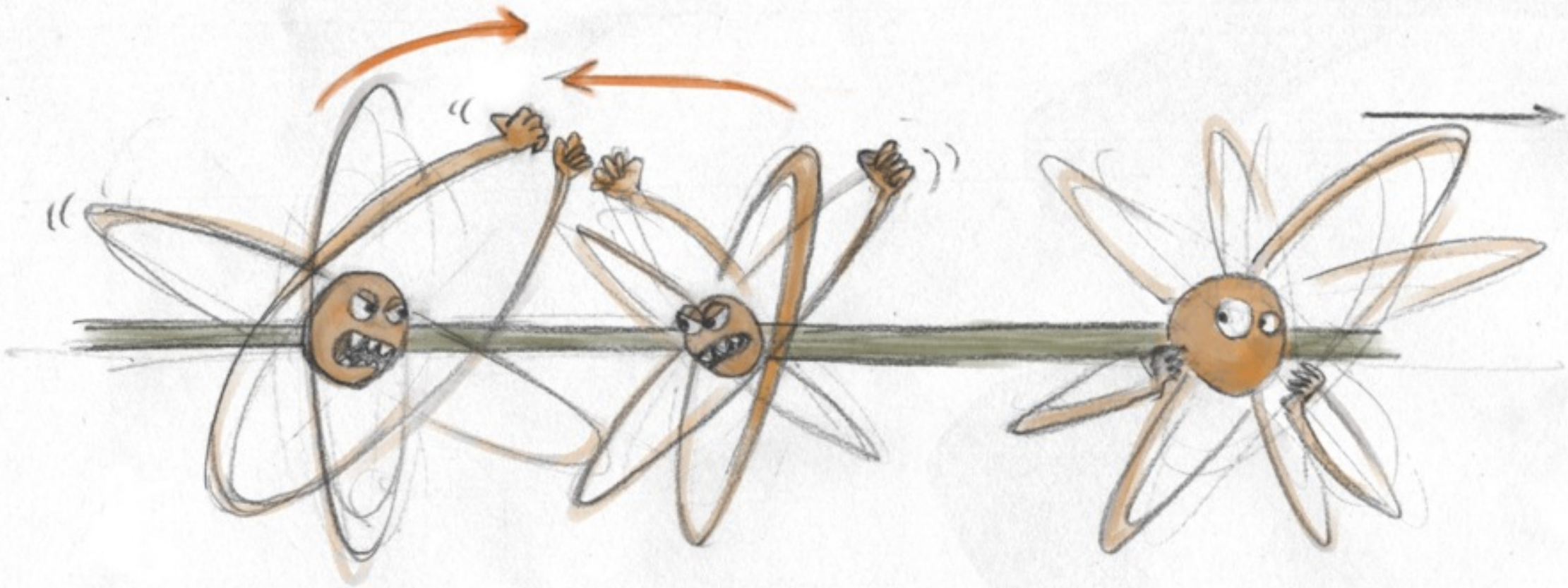
Michael Brockmann	MPI Dresden
Lorenzo Piroli	SISSA
Tomasz Prosen	Ljubljana
Fabian Essler	Oxford

FOM

ENS ÉCOLE NORMALE SUPÉRIEURE

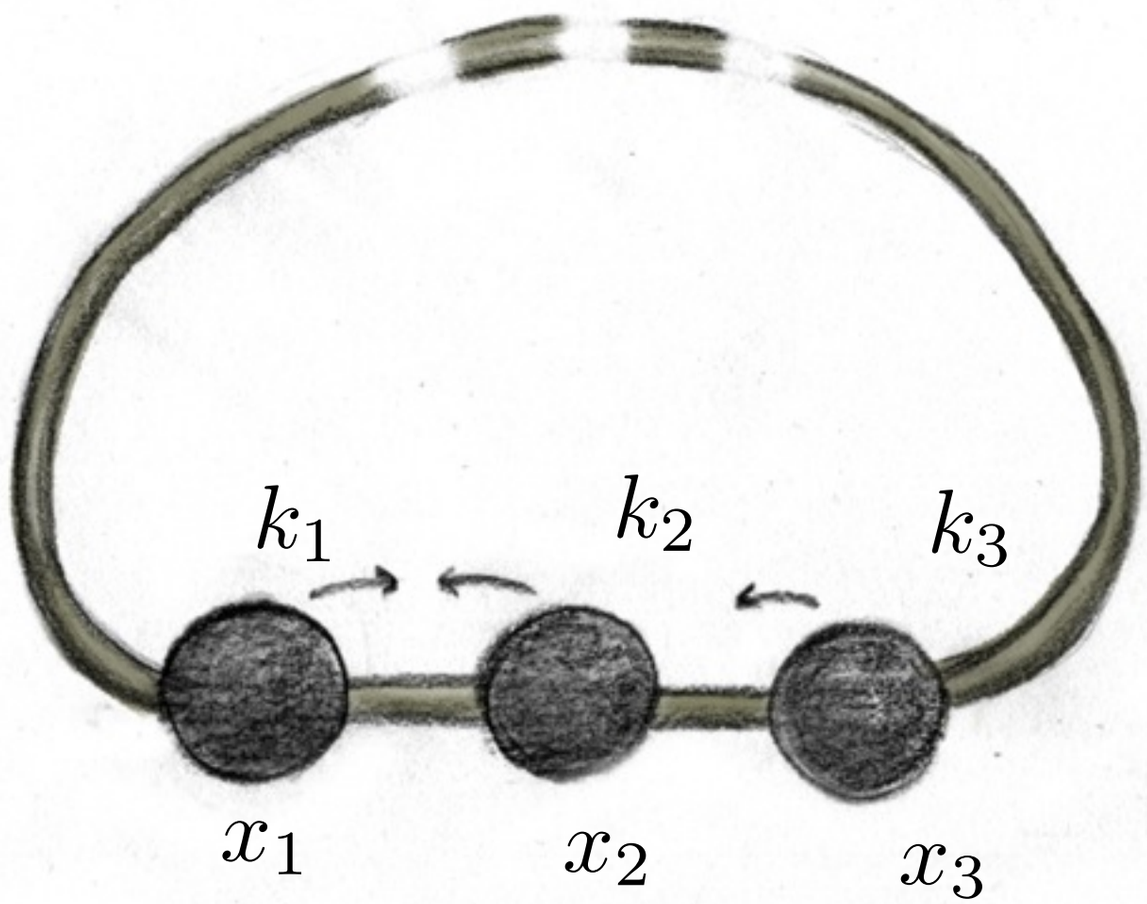
UNIVERSITEIT VAN AMSTERDAM

Many-body interacting systems in 1D



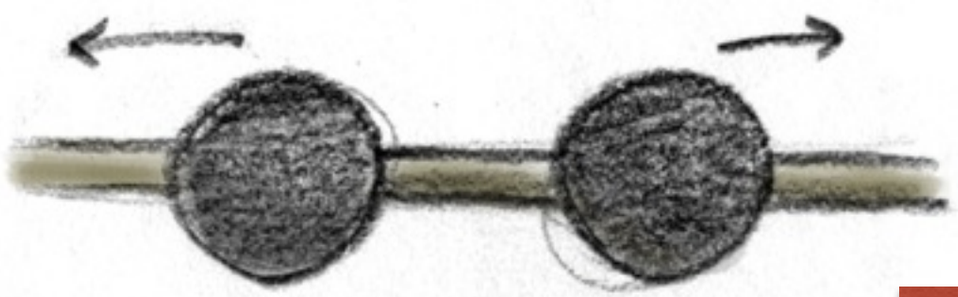
Many-body wave function

$$\Psi(\{x_j\}_{j=1}^N|\{k_j\}_{j=1}^N)$$



$$E = \epsilon_0(k_1) + \epsilon_0(k_2) = \epsilon_0(k_1') + \epsilon_0(k_2')$$

$$P = p_0(k_1) + p_0(k_2) = p_0(k_1') + p_0(k_2')$$



Phase-shift due to interaction

$$\Psi \to e^{ix_1k_1 + ix_2k_2} + e^{i\phi(k_1, k_2)} e^{ix_1k_2 + ix_2k_1}$$

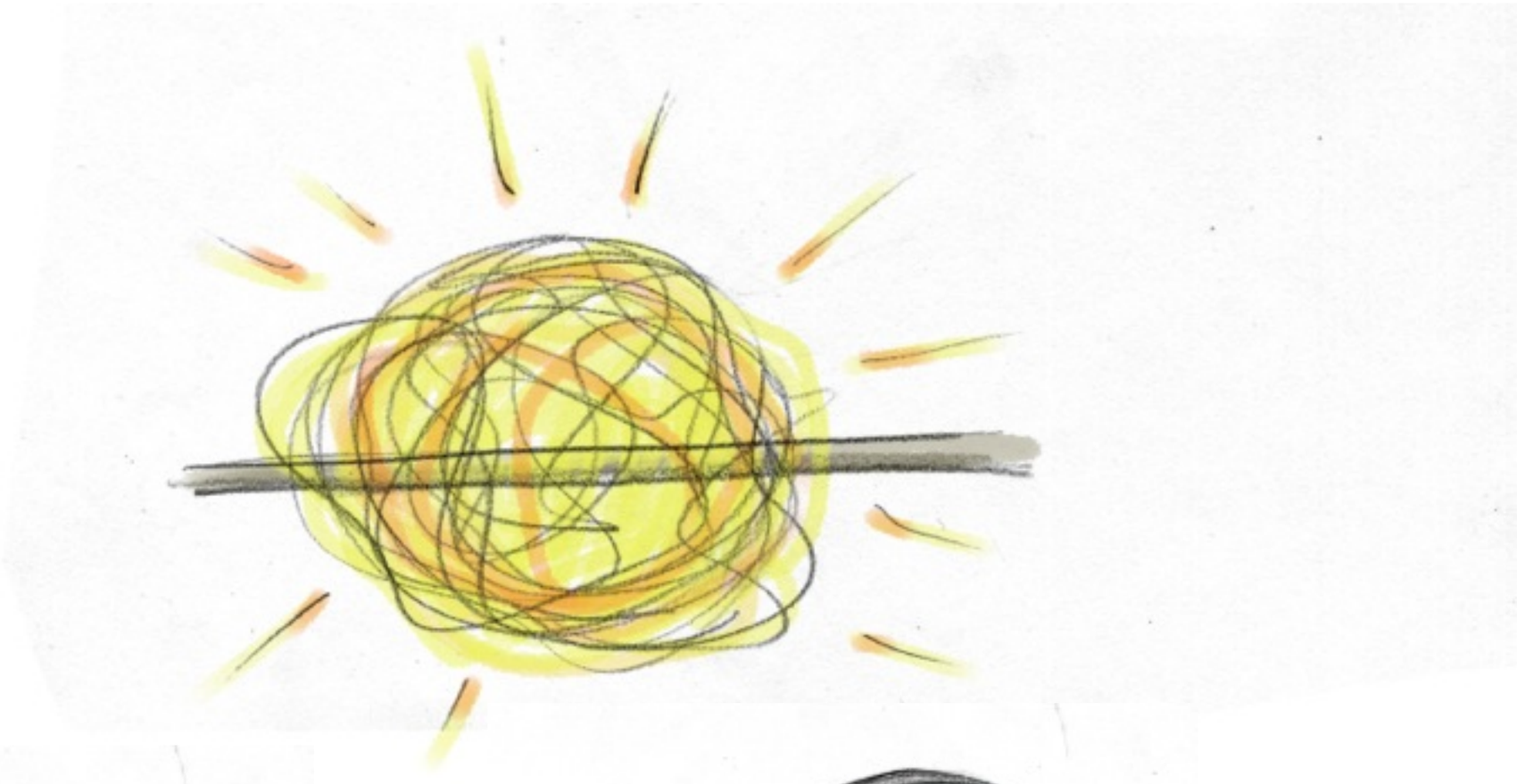
$$P = p_0(k_1) + p_0(k_2) + p_0(k_3) = p_0(k_1') + p_0(k_2') + p_0(k_3')$$

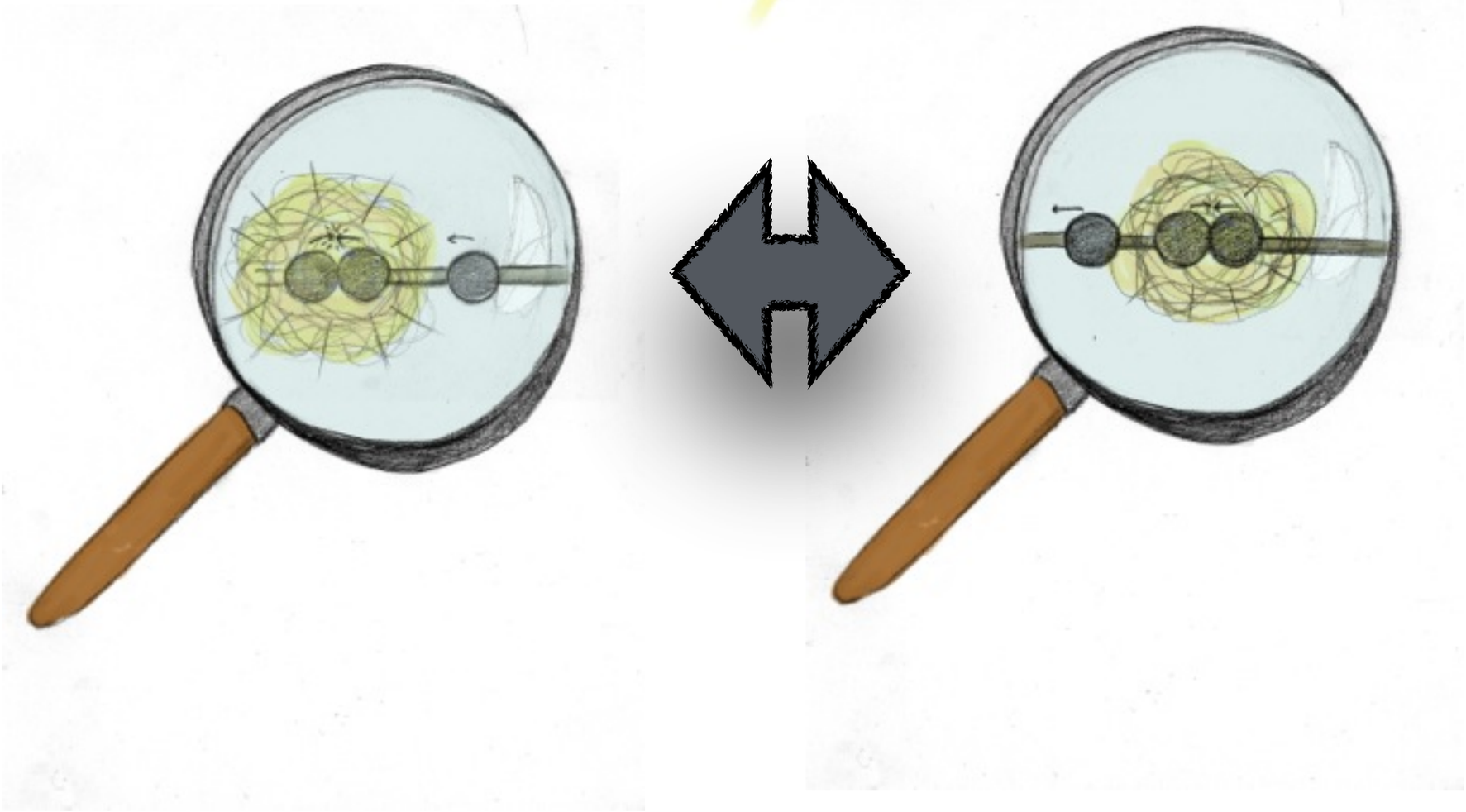
$$E = \epsilon_0(k_1) + \epsilon_0(k_2) + \epsilon_0(k_3) = \epsilon_0(k_1') + \epsilon_0(k_2') + \epsilon_0(k_3')$$

$$\Psi \to \sum_P \Phi(P)\, e^{ix_1 k_{P_1} + ix_2 k_{P_2} + ix_3 k_{P_3}} + \iiint_{k_1' < k_2' < k_3'} dk_1' dk_2' dk_3'\, S[k_1', k_2', k_3']\, e^{ix_1 k_1' + ix_2 k_2' + ix_3 k_3'}$$

2-particle scatterings

Creation of all possible momenta

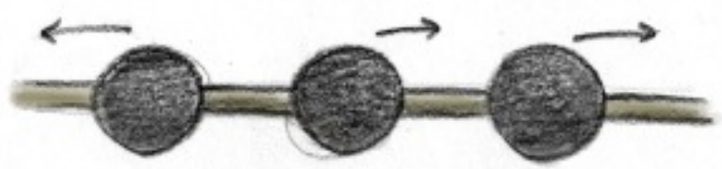




$$P = p_0(k_1) + p_0(k_2) + p_0(k_3) = p_0(k_1') + p_0(k_2') + p_0(k_3')$$

$$E = \epsilon_0(k_1) + \epsilon_0(k_2) + \epsilon_0(k_3) = \epsilon_0(k_1') + \epsilon_0(k_2') + \epsilon_0(k_3')$$

$$Q_3 = q_0^{(3)}(k_1) + q^{(3)}(k_2) + q^{(3)}(k_3) = q^{(3)}(k_1') + q^{(3)}(k_2') + q^{(3)}(k_3')$$

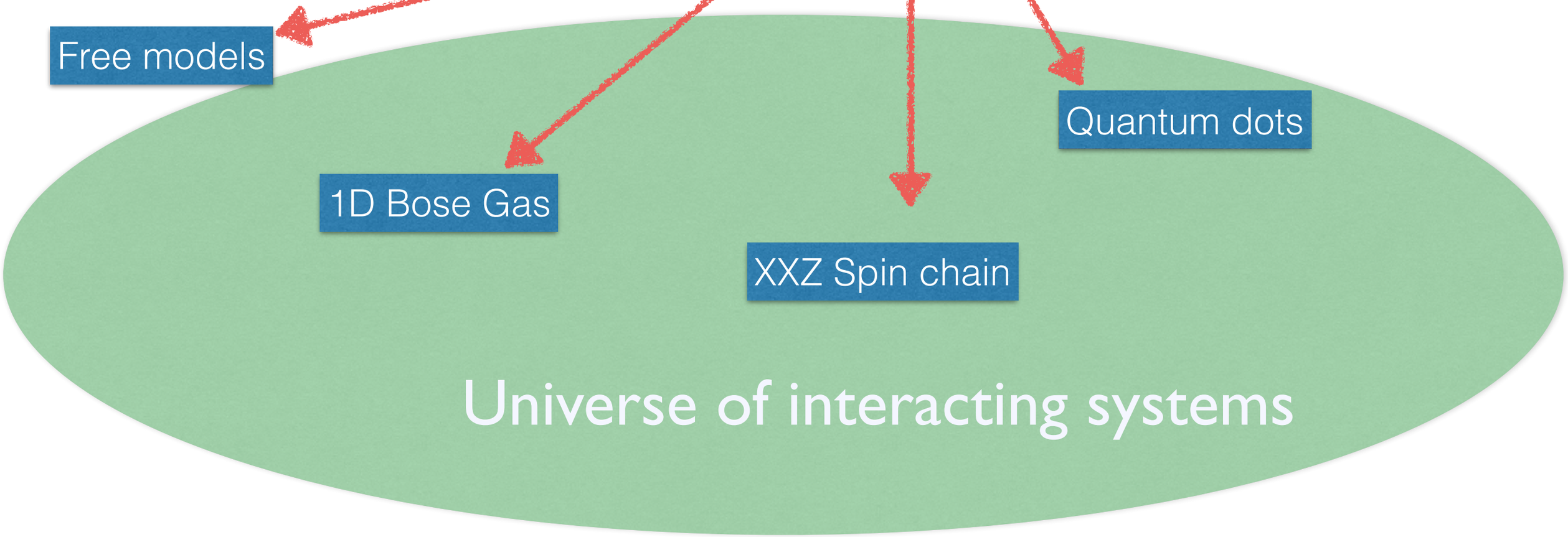


$$\Psi \to \sum_P \Phi(P)\, e^{i x_1 k_{P_1} + i x_2 k_{P_2} + i x_3 k_{P_3}} + \cancel{\iiint_{k_1' < k_2' < k_3'} dk_1' dk_2' dk_3'\, S[k_1', k_2', k_3']\, e^{i x_1 k_1' + i x_2 k_2' + i x_3 k_3'}}$$

The Bethe wave function

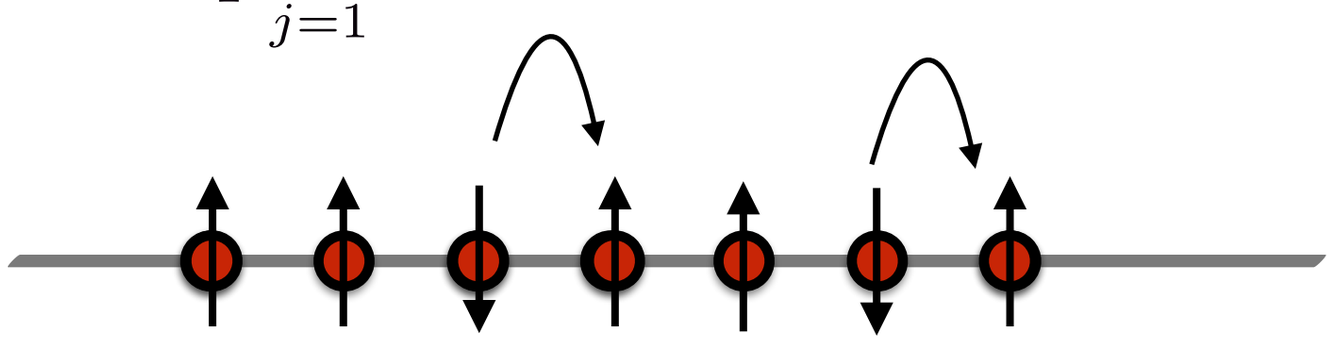
$$\Psi_B(\{x_j\}_{j=1}^N|\{k_j\}_{j=1}^N)$$

$$= \mathcal{N}^{-1} \sum_{P\in\mathcal{S}_N} e^{-\frac{i}{2}\sum_{i<j=1}^N \theta(k_{P_i}-k_{P_j})} \prod_{j=1}^N e^{ix_j k_{P_j}}$$



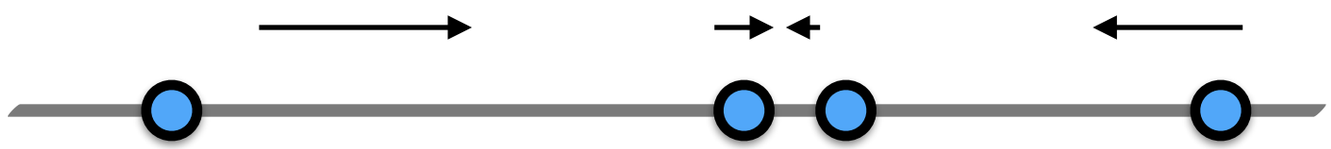
XXZ Spin chain

$$H = \frac{J}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)\right)$$



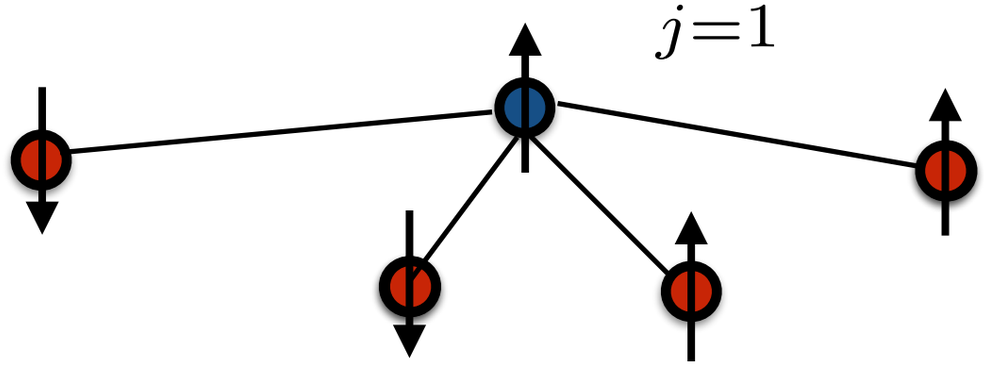
1D Bose Gas

$$H = -\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2} + 2c\sum_{j>k}\delta(x_j - x_k)$$



Quantum dots

$$H = hS_0^z + \sum_{j=1}^{N} A_j \boldsymbol{S}_0 \cdot \boldsymbol{I}_j$$



$$H\Psi_B(\{x_j\}_{j=1}^N|\{\lambda_j\}_{j=1}^N) =$$

$$E(\{\lambda_j\}_{j=1}^N)\Psi_B(\{x_j\}_{j=1}^N|\{\lambda_j\}_{j=1}^N)$$

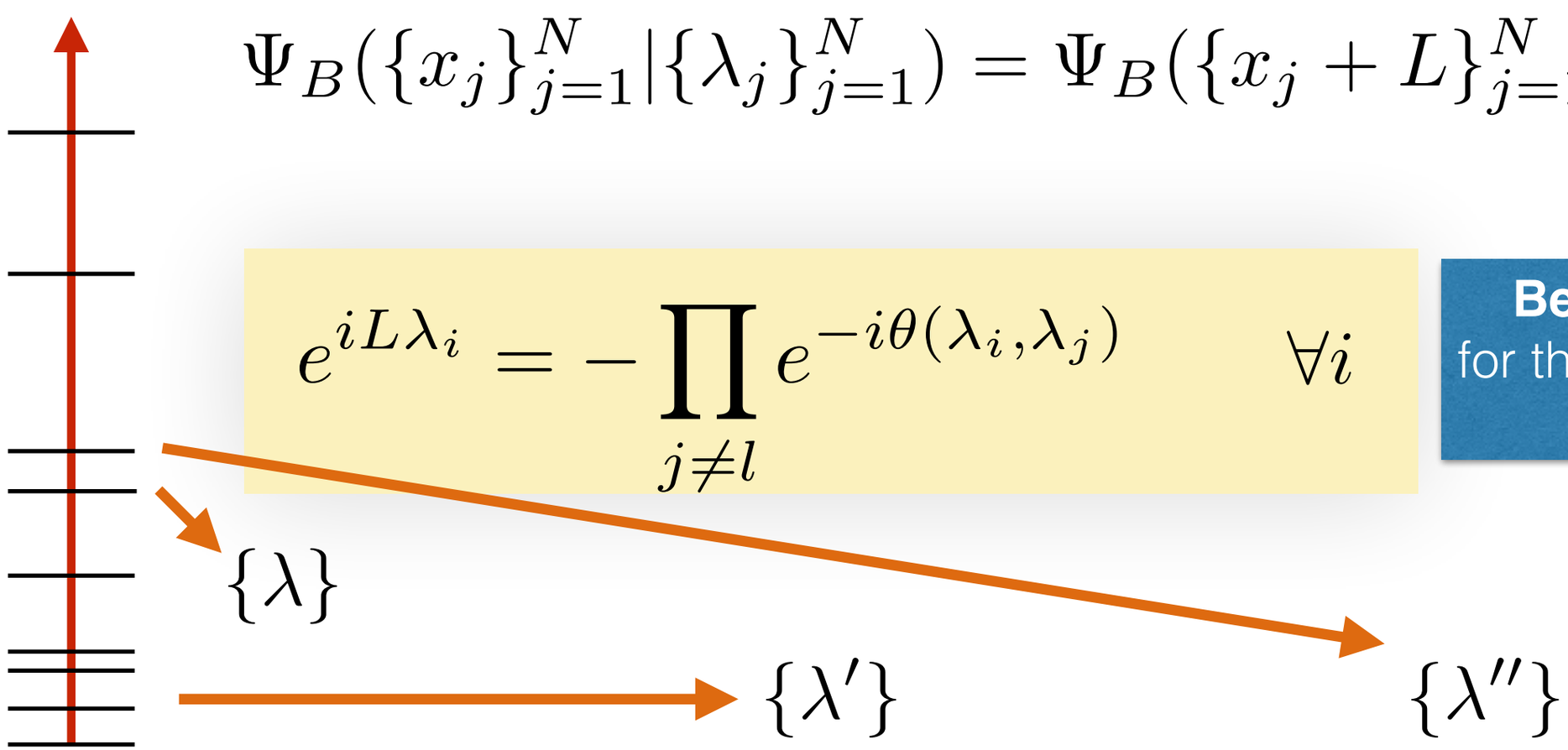
$$E = \sum_{j=1}^N \epsilon_0(\lambda_j)$$

Quantization = Fixing the boundary conditions

$$\Psi_B(\{x_j\}_{j=1}^N|\{\lambda_j\}_{j=1}^N) = \Psi_B(\{x_j + L\}_{j=1}^N|\{\lambda_j\}_{j=1}^N)$$

$$e^{iL\lambda_i} = -\prod_{j\neq l} e^{-i\theta(\lambda_i,\lambda_j)} \qquad \forall i$$

Bethe Equations
for the quasi-momenta
(**rapidities**)



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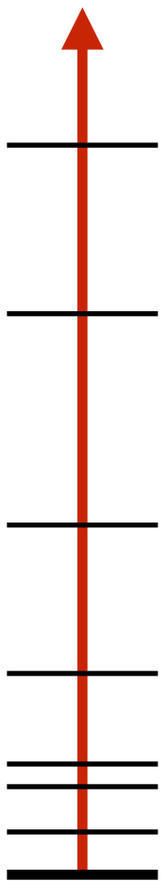
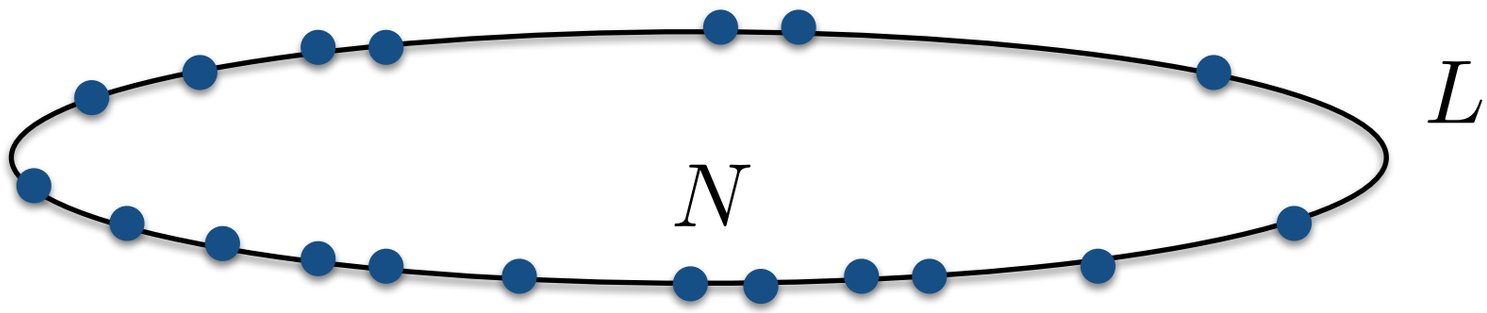




$$c \longrightarrow +\infty$$

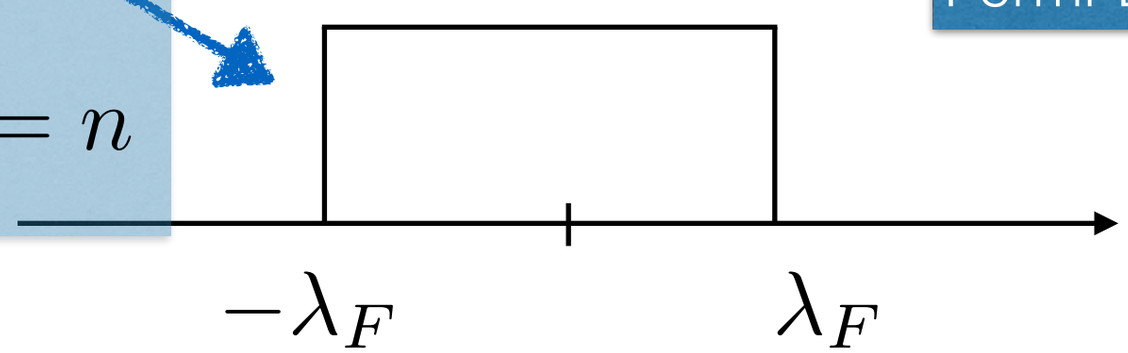
$$\lambda_j = \frac{2\pi I_j}{L}$$

Hard-core bosons

Free fermions quantization

$$L \to \infty \qquad N/L = n$$
$$N \to \infty$$

Fermi-Dirac distribution



1D Bose Gas

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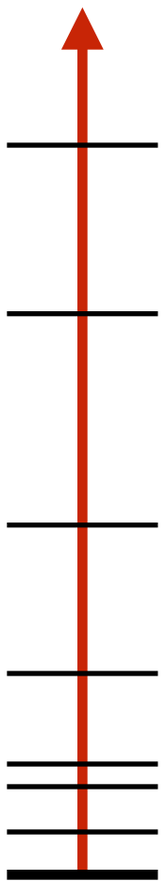
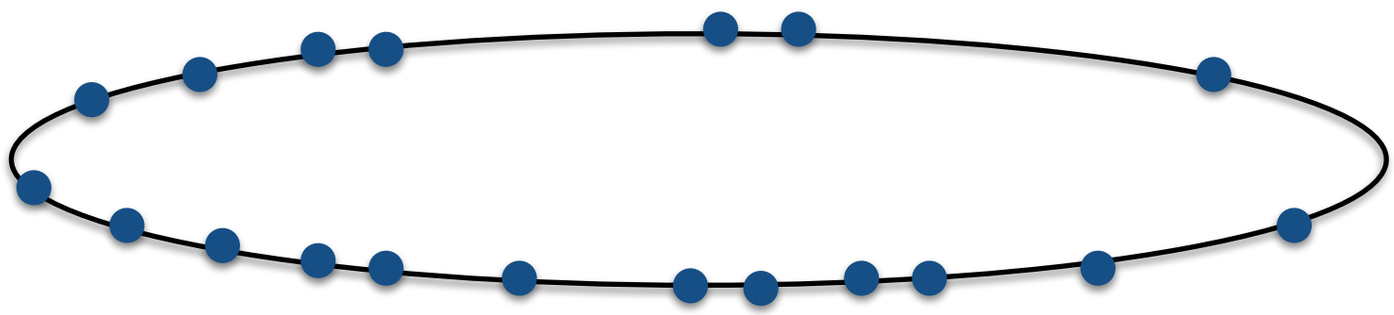
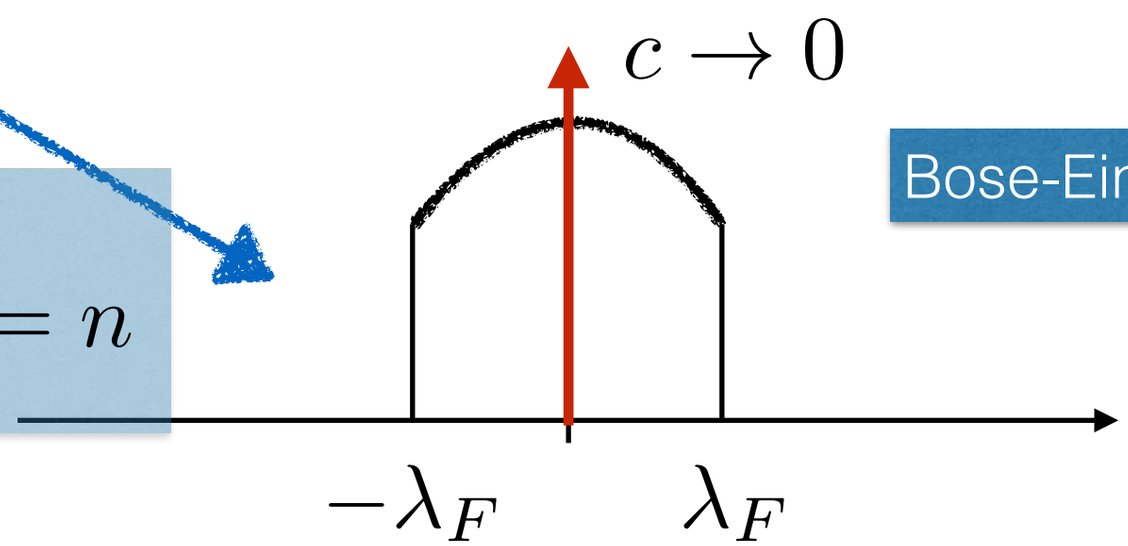




Finite coupling c

$$\lambda_j = \frac{2\pi I_j}{L} - \frac{2}{L} \sum_{k=1}^{N} \arctan\left(\frac{\lambda_j - \lambda_k}{c}\right)$$

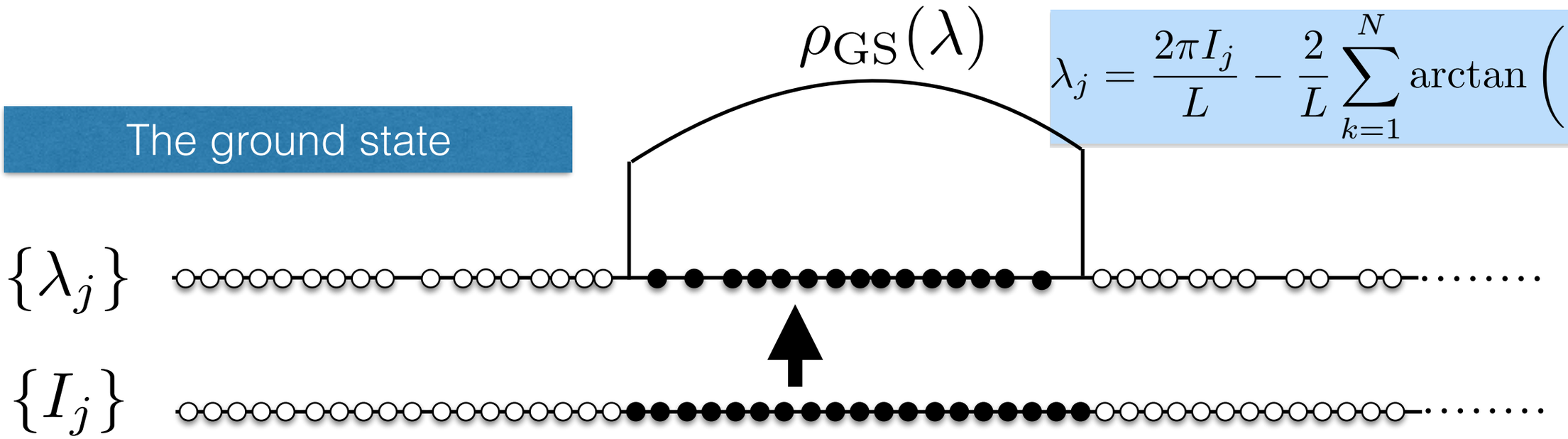
$$L \to \infty \qquad N/L = n$$
$$N \to \infty$$



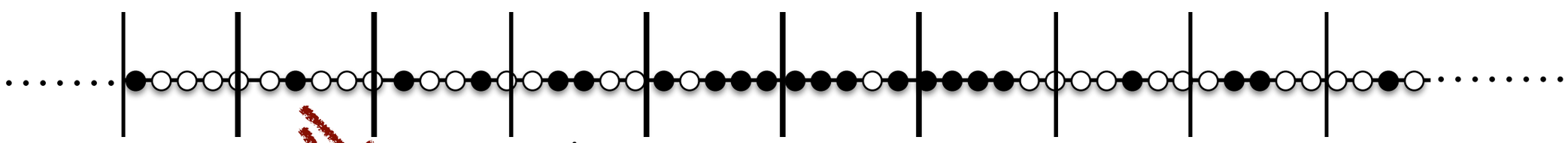
Bose-Einstein Condensate

$$\lambda_j = \frac{2\pi I_j}{L} - \frac{2}{L}\sum_{k=1}^{N} \arctan\left(\frac{\lambda_j - \lambda_k}{c}\right)$$

The ground state



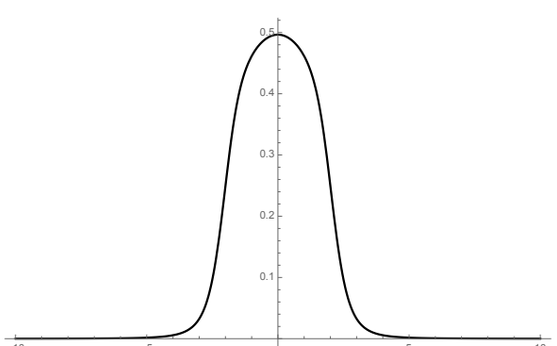
Non-zero entropy states

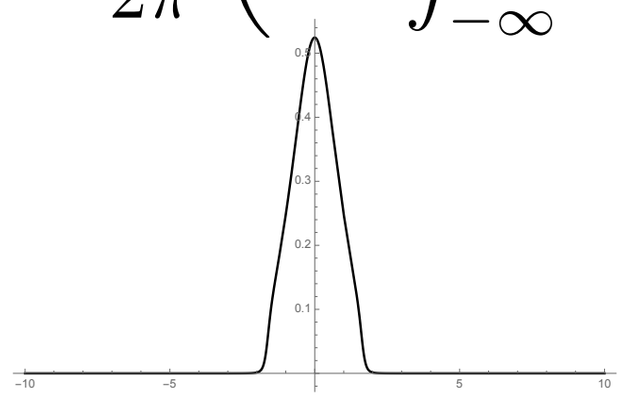


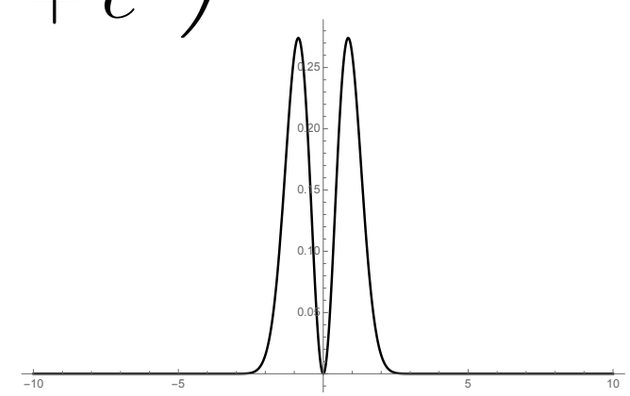
$$n_p^j = L\rho(\lambda)d\lambda + O(1)$$

$$n_h^j = L\rho^h(\lambda)d\lambda + O(1)$$

$$\rho(\lambda) + \rho^h(\lambda) = \frac{1}{2\pi}\left(1 + \int_{-\infty}^{\infty} d\mu \frac{2c\,\rho(\mu)}{(\lambda-\mu)^2 + c^2}\right)$$



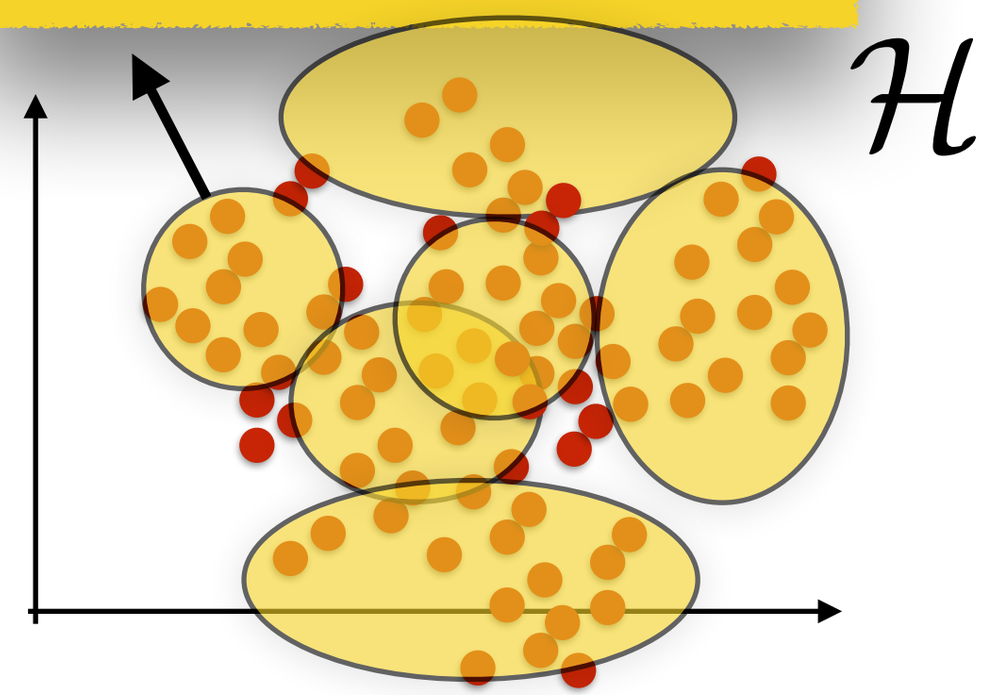




Thermodynamic Bethe states

$$L \to \infty \qquad N/L = n$$
$$N \to \infty$$

$$\rho(\lambda) \qquad \int_{-\infty}^{\infty} d\lambda \rho(\lambda) = n$$



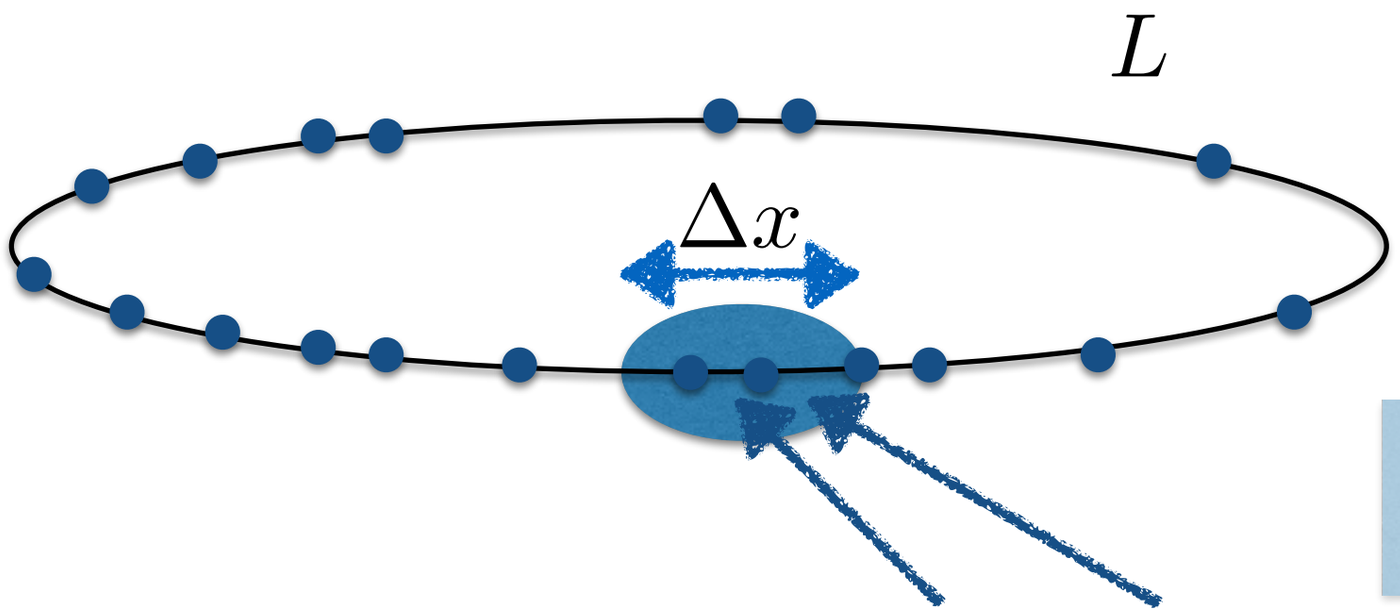
$$E = L \int_{-\infty}^{\infty} d\lambda \epsilon_0(\lambda)\rho(\lambda) + O(1)$$

$$P = L \int_{-\infty}^{\infty} d\lambda p_0(\lambda)\rho(\lambda) + O(1)$$

$$S_{YY}[\rho] = L \int_{-\infty}^{\infty} d\lambda \big((\rho + \rho^h) \ln(\rho + \rho^h) - \rho \ln \rho - \rho^h \ln \rho^h \big)$$

Yang C N and Yang C P 1969 J. Math. Phys.

Correlation functions of local operators



Density operator

$$\hat{\rho}(x) = \Psi^\dagger(x)\Psi(x)$$

$$\langle GS|\hat{\rho}(x)\hat{\rho}(0)|GS\rangle$$

$$= \left(\prod_{j=2}^{N}\int_0^L dx_j\right) \Psi^*_{\mathrm{GS}}(\{x_j\}_{j=1}^M|\{k_j\}_{j=1}^M)\Big|_{x_1=x} \Psi_{\mathrm{GS}}(\{x_j\}_{j=1}^M|\{k_j\}_{j=1}^M)\Big|_{x_1=0}$$

$$= \sum_{\{\mu\}} \langle GS|\hat{\rho}(x)|\{\mu\}\rangle\langle\{\mu\}|\hat{\rho}(0)|GS\rangle$$

Form factors

$$\langle\{\lambda\}|\hat{\rho}|\{\mu\}\rangle$$

N. A. Slavnov, Theor. Math. Phys. 79, 502 (1989)

Form factors in interacting models

$$\langle\{\mu\}|\hat{\rho}(0)|\{\lambda\}\rangle = \left(\sum_{j=1}^{N}(\mu_j-\lambda_j)\right)\prod_{j=1}^{N}\left(V_j^+ - V_j^-\right)\prod_{j,k}^{N}\left(\frac{\lambda_j-\lambda_k+ic}{\mu_j-\lambda_k}\right)\frac{\det_N\left(\delta_{jk}+U_{jk}\right)}{V_p^+ - V_p^-}\boxed{\frac{1}{|\{\lambda\}|}\frac{1}{|\{\mu\}|}}$$

$$V_j^{\pm} = \prod_{k=1}^{N}\frac{\mu_k-\lambda_j\pm ic}{\lambda_k-\lambda_j\pm ic}$$

$$U_{jk} = i\frac{\mu_j-\lambda_j}{V_j^+ - V_j^-}\prod_{m\neq j}^{N}\left(\frac{\mu_m-\lambda_j}{\lambda_m-\lambda_j}\right)\Big(K\left(\lambda_j-\lambda_k\right)-K\left(\lambda_p-\lambda_k\right)\Big)$$

$$|\{\lambda\}|^2 = c^N\prod_{j\neq k}^{N}\frac{\lambda_j-\lambda_k+ic}{\lambda_j-\lambda_k}\det_N\mathcal{G}$$

Norm of a Bethe state

$$\mathcal{G}_{jk} = \delta_{jk}\left(L+\sum_{m=1}^{N}K\left(\lambda_j-\lambda_m\right)\right)-K\left(\lambda_j-\lambda_k\right),$$

$$K(\lambda) = \frac{2c}{\lambda^2+c^2}$$

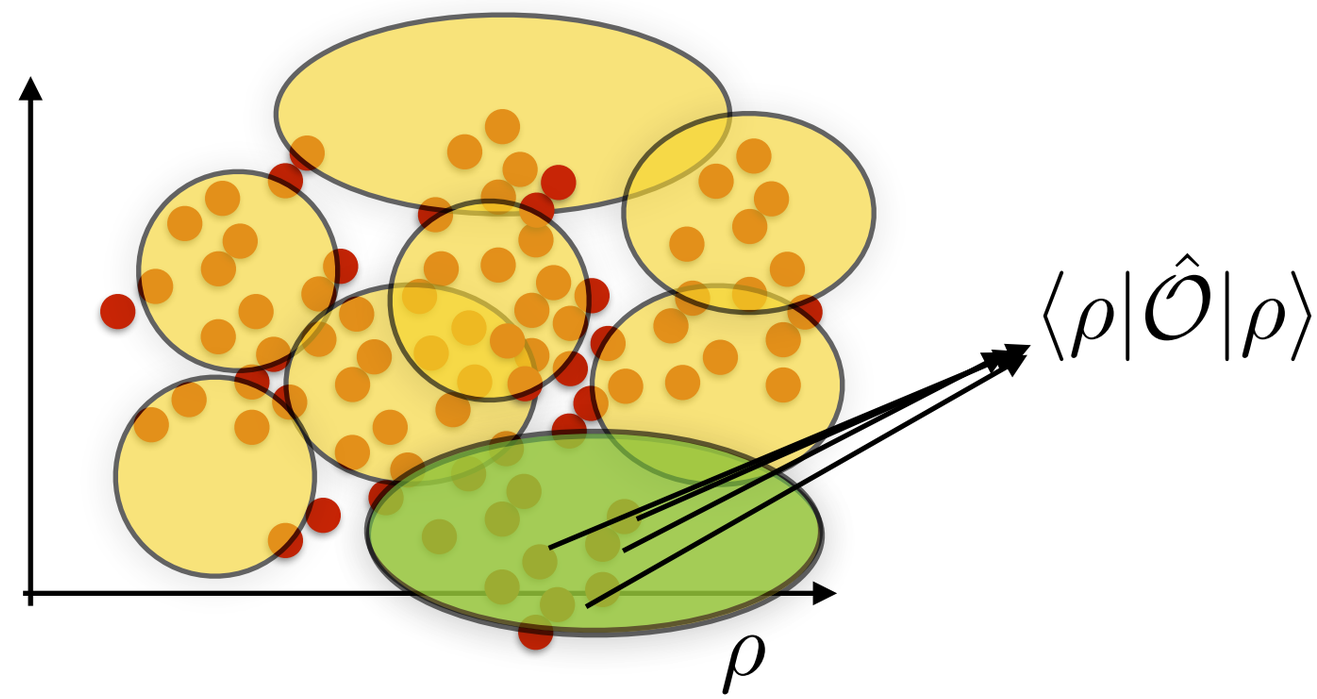
Gaudin Matrix

Gaudin M 1971 J. Math. Phys

Properties of form factors of local operators

$$\langle\{\lambda\}|\hat{\mathcal{O}}|\{\lambda\}\rangle \to$$
$$\langle\rho|\mathcal{O}|\rho\rangle + O(L^{-1})$$

Smooth



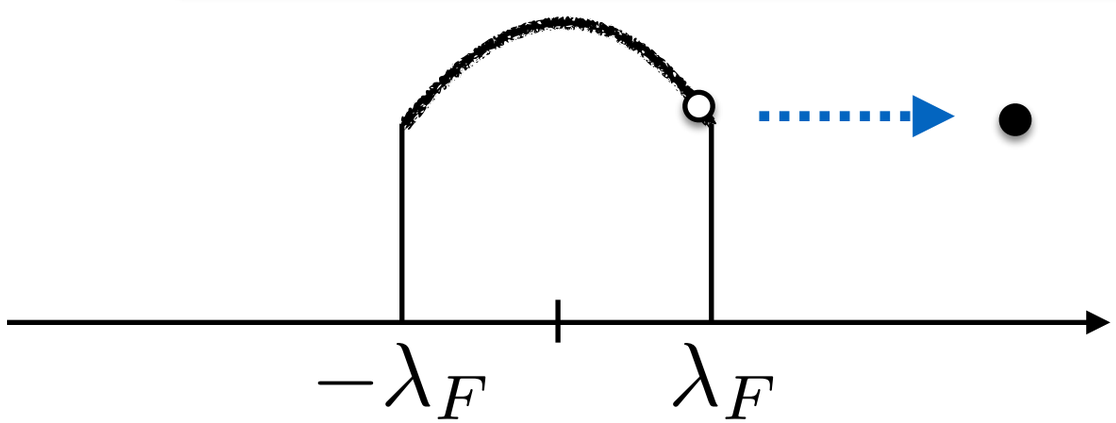
$$\langle\{\lambda_{\rm ref}\}|\mathcal{O}(x)\mathcal{O}(0)|\{\lambda_{\rm ref}\}\rangle = \sum_{\{\mu\}} e^{-ixP_\mu}\left|\langle\{\lambda\}_{\rm ref}|\mathcal{O}|\{\mu\}\rangle\right|^2$$

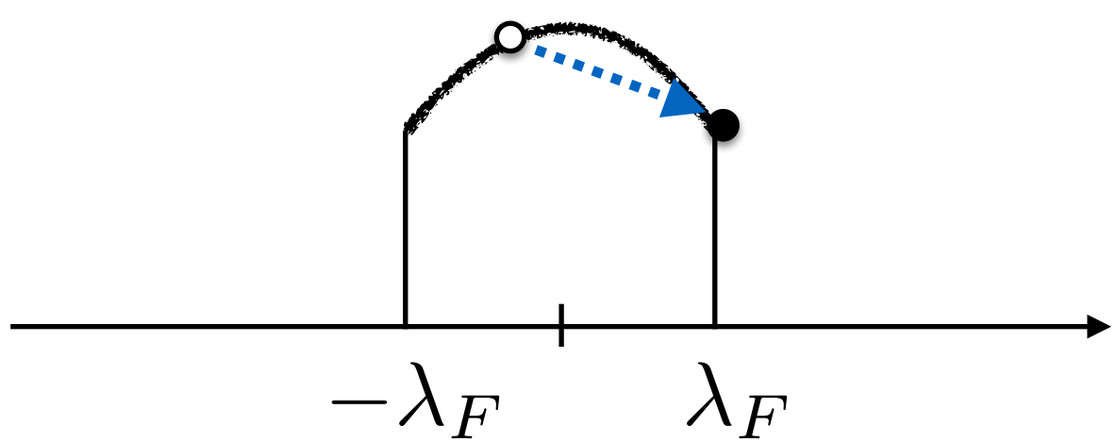
$$\to \sum_{\{\mu\}} e^{-ixP_\mu}\left|\langle\rho_{\rm ref}|\mathcal{O}|\{\mu\}\rangle\right|^2$$

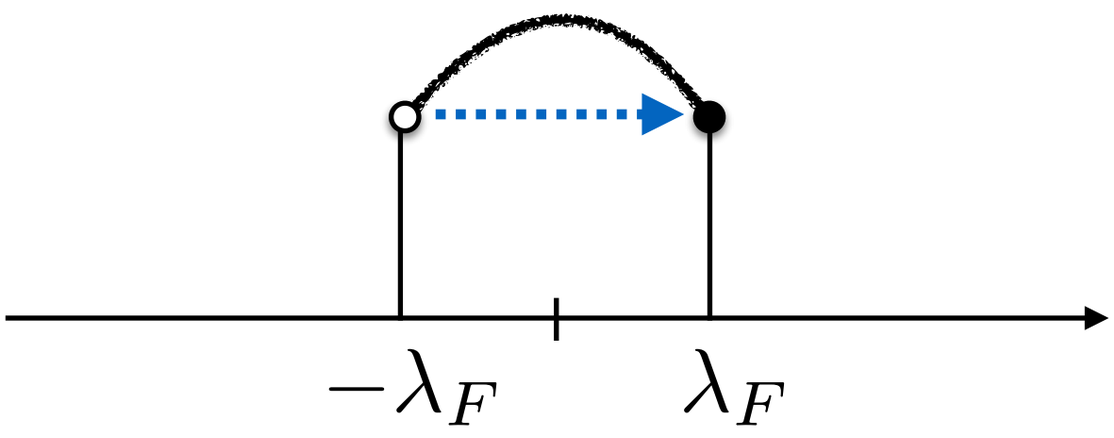
Local in the Hilbert space

Dynamical correlations of the ground state

$$S(k,\omega) = \sum_{\{\mu\}} \left| \langle \mathrm{GS} | \hat{\rho} | \{\mu\} \rangle \right|^2 \delta_{P_\mu,k} \delta_{E_\mu,\omega}$$







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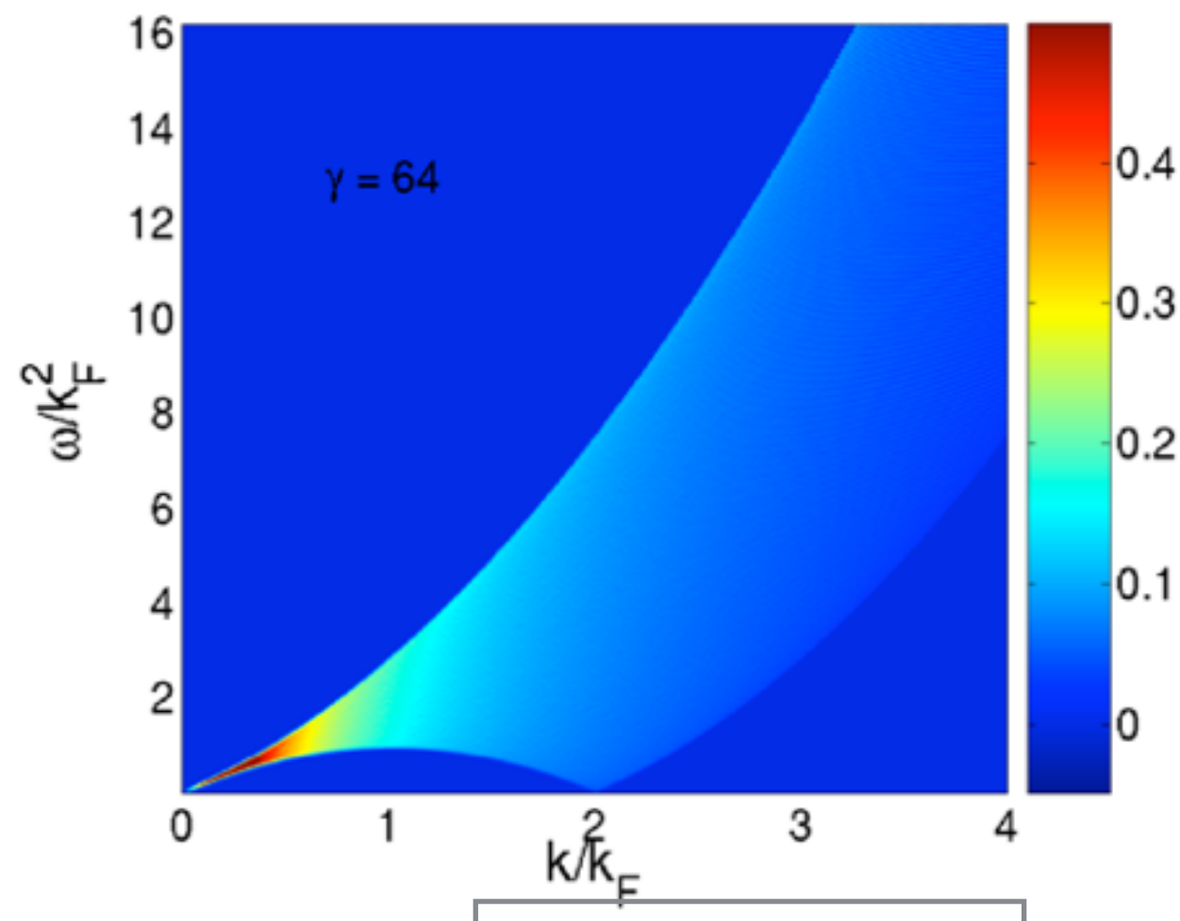
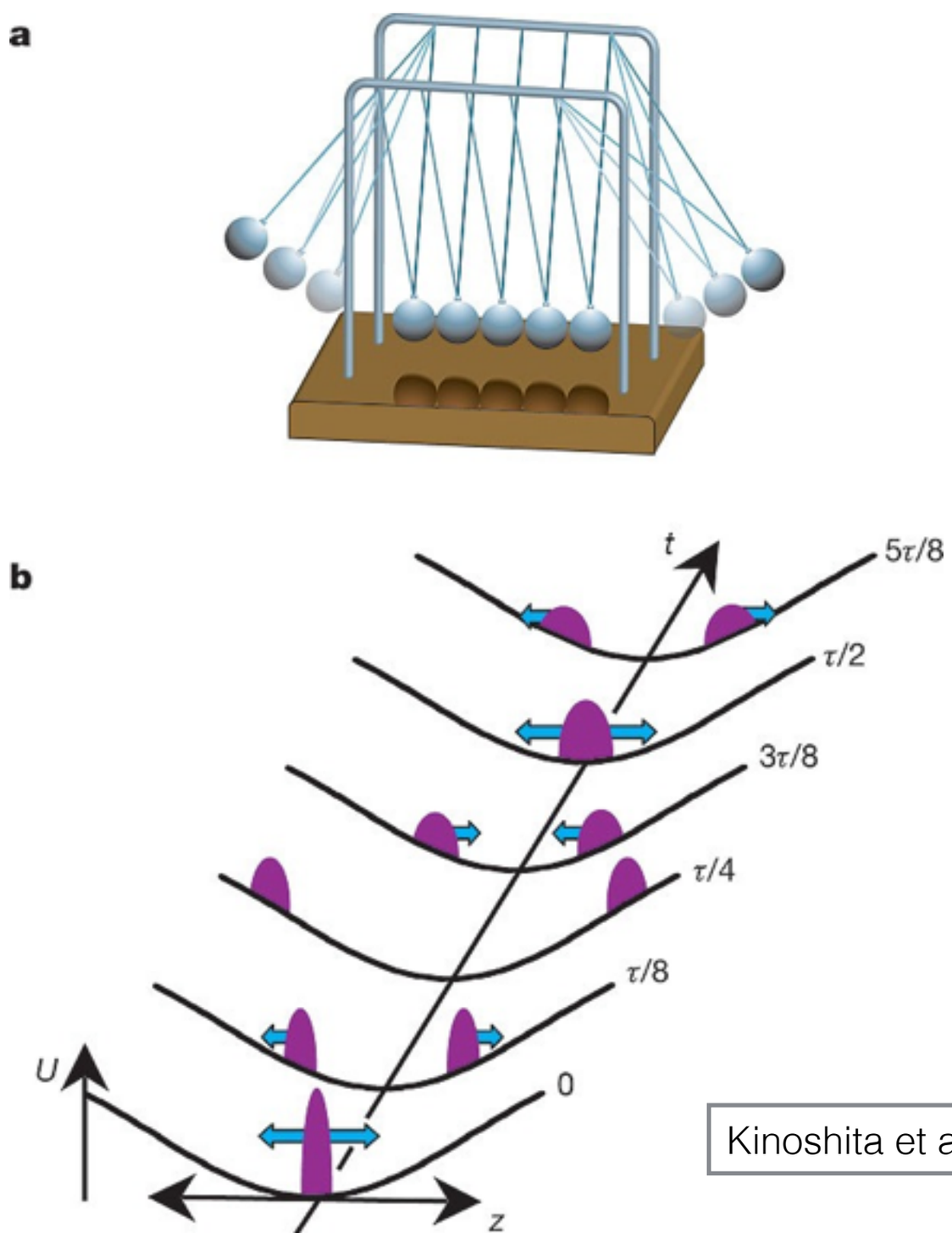


Caux&Calabrese, PRA 74

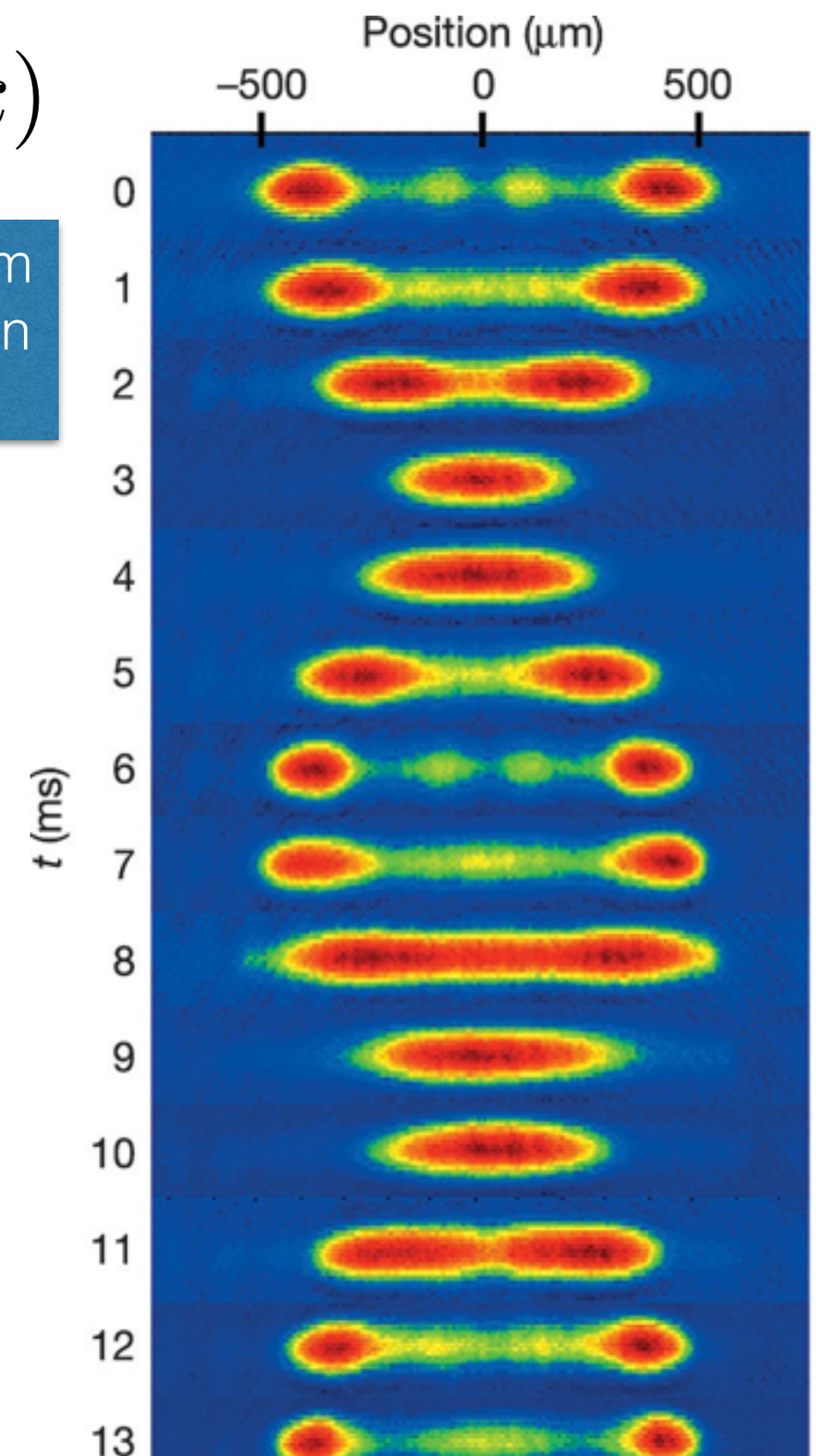
Out-of-equilibrium many-body physics

Quantum Newton's Cradle



$n_t(k)$

Momentum distribution function

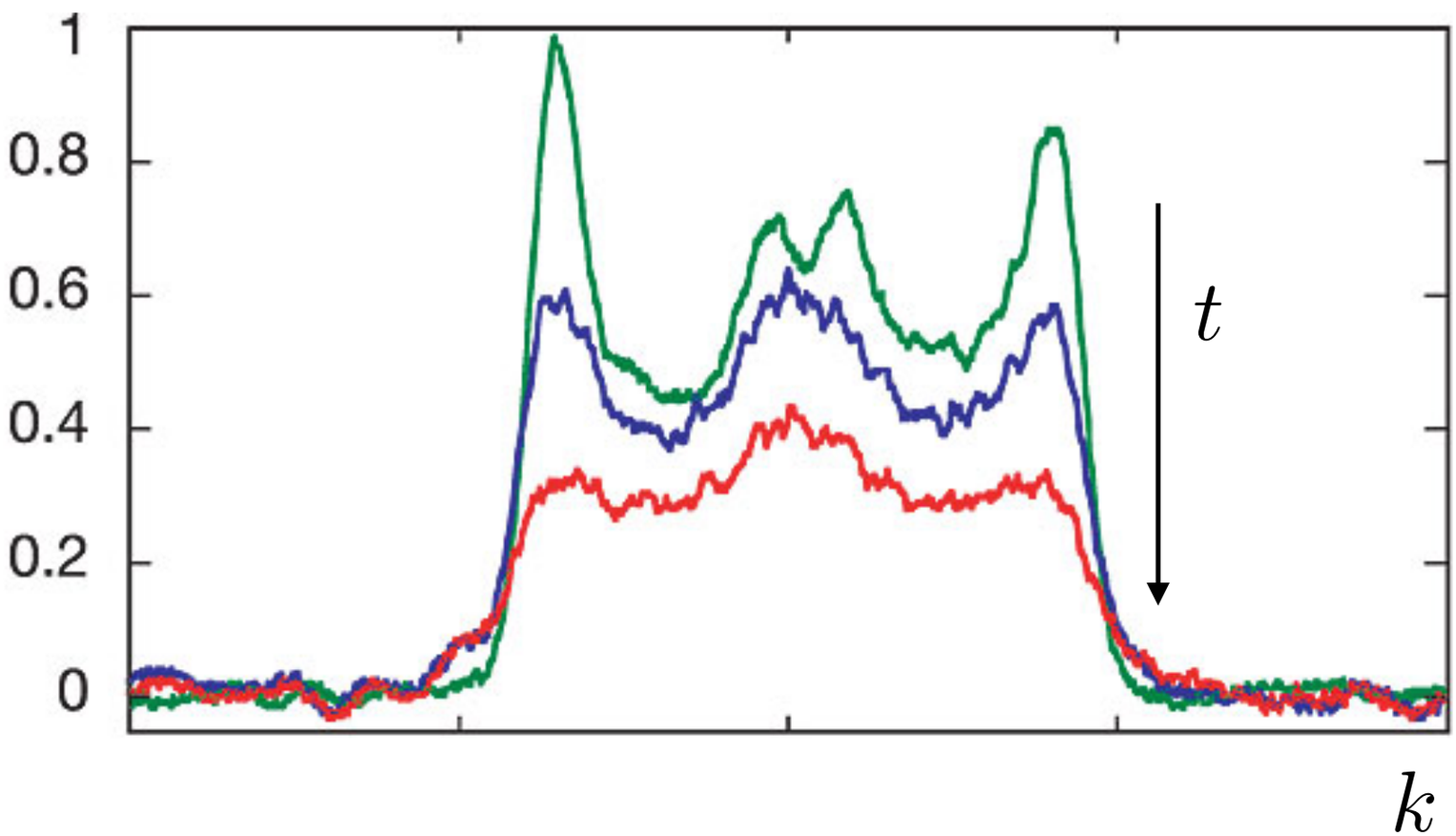


Kinoshita et al, Nature 440

$$\langle n(k) \rangle_\infty = \frac{1}{T} \int_0^T dt\, n_t(k)$$

$$T \to \infty$$

Memory of the initial state



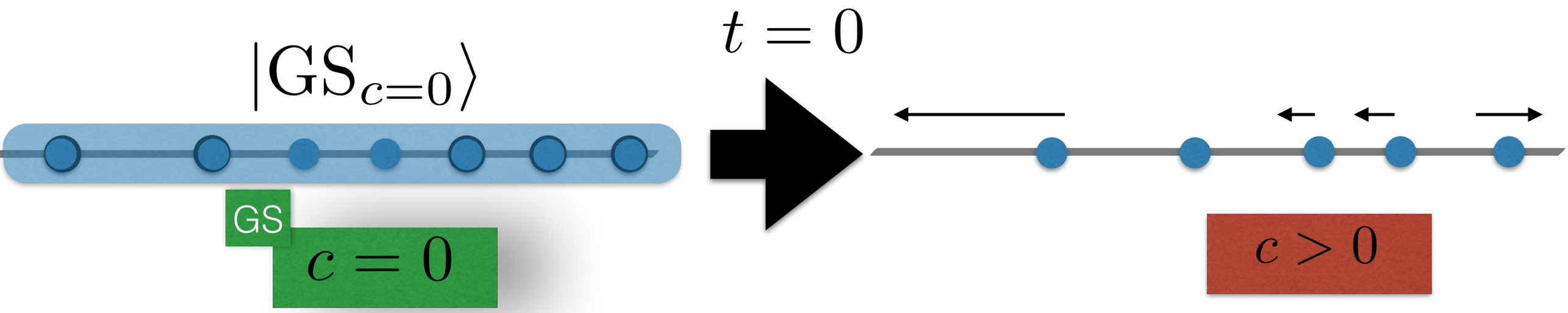
Kinoshita et al, Nature 440

If not thermal then what?
Can i construct in this way **new steady states**?

Quantum quenches

1D Bose Gas

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j>k} \delta(x_j - x_k)$$



Calabrese P and Cardy J 2006 Phys. Rev. Lett. 96

Bloch I, Dalibard J and Zwerger W 2008 Rev. Mod. Phys. 80

Calabrese P, Essler F H L and Fagotti M 2011 Phys. Rev. Lett. 106

Rigol M, Dunjko V and Olshanii M 2008 Nature 452

Gring M, Kuhnert M, Langen T, Kitagawa T, Rauer B, Schreitl M, Mazets I, Smith D A, Demler E and Schmiedmayer J 2012 Science 337

1D Bose Gas

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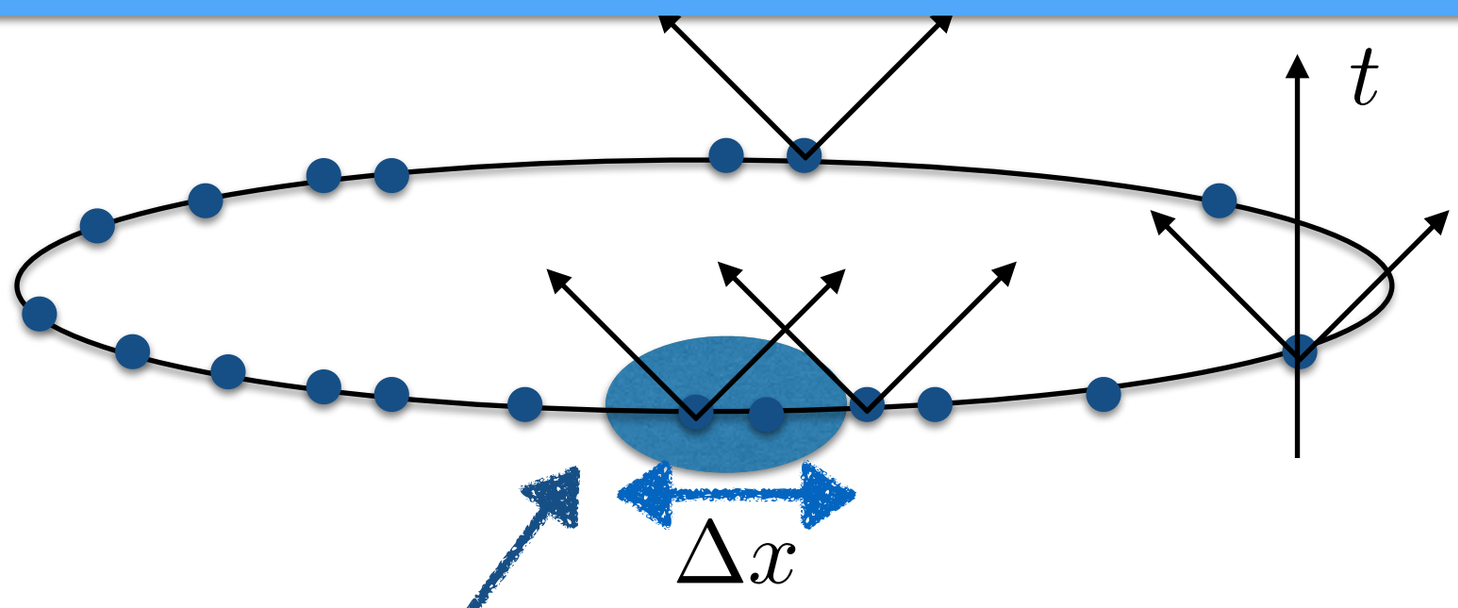
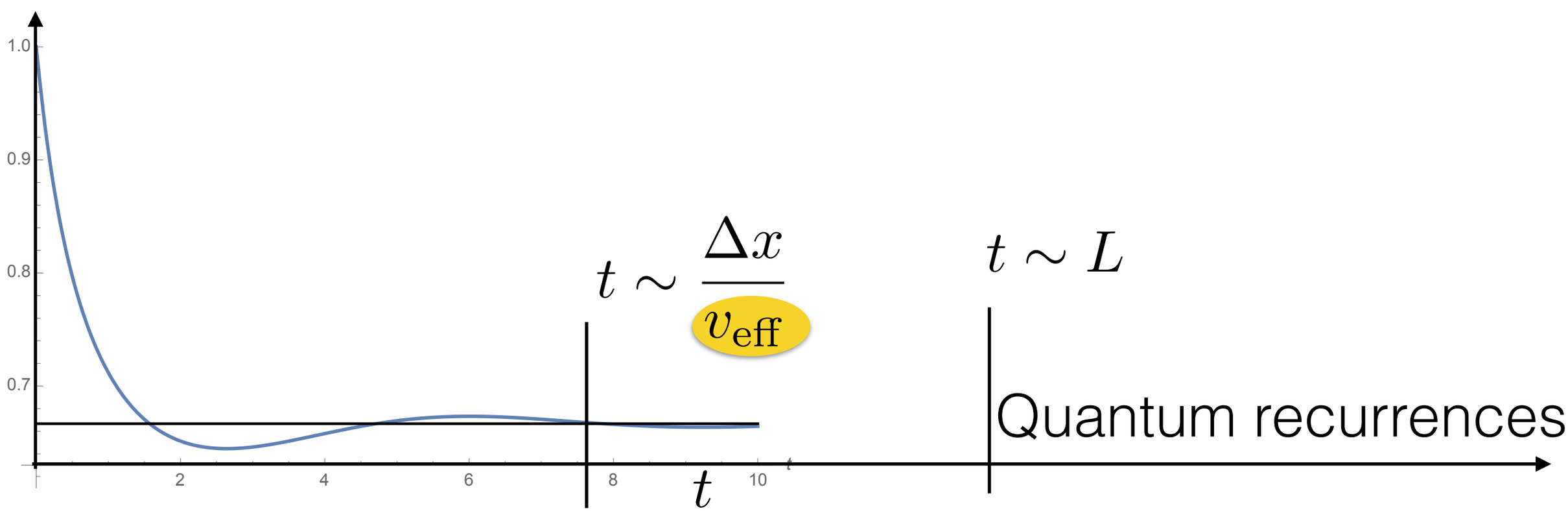
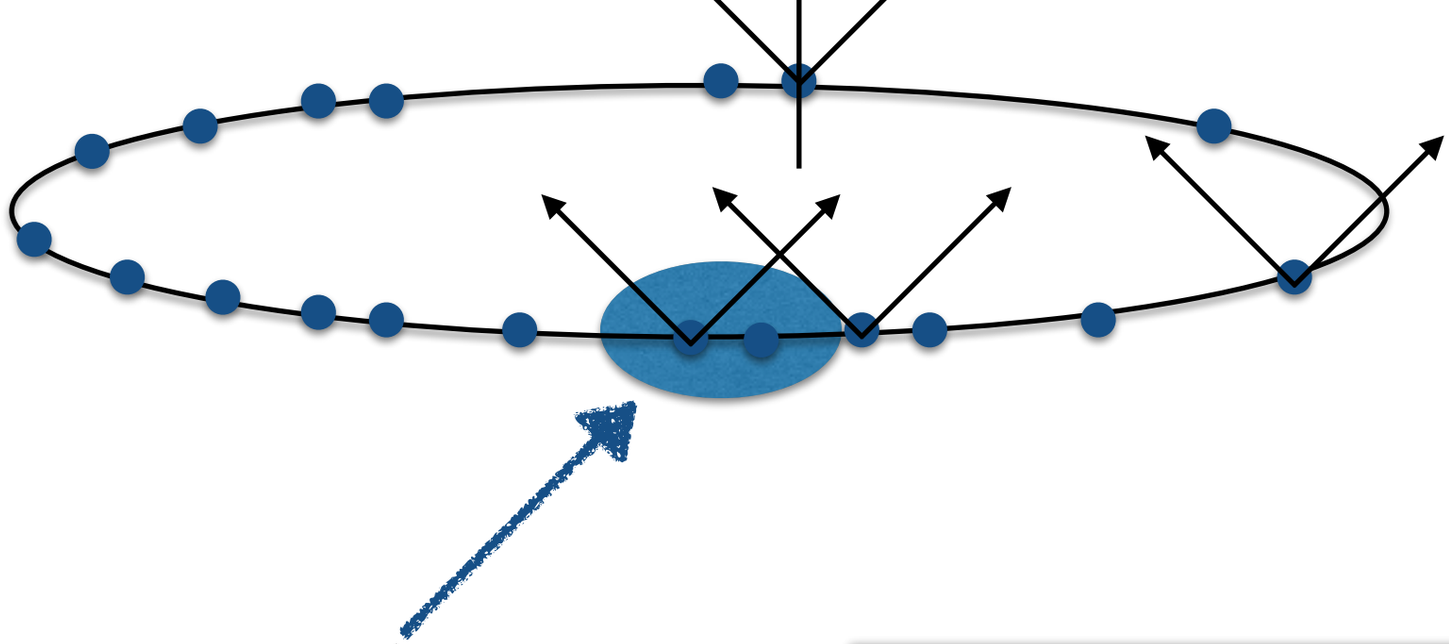


Calabrese P and Cardy J 2006 Phys. Rev. Lett. 96

$$\langle \Psi_0(t) | \hat{\mathcal{O}} | \Psi_0(t) \rangle$$





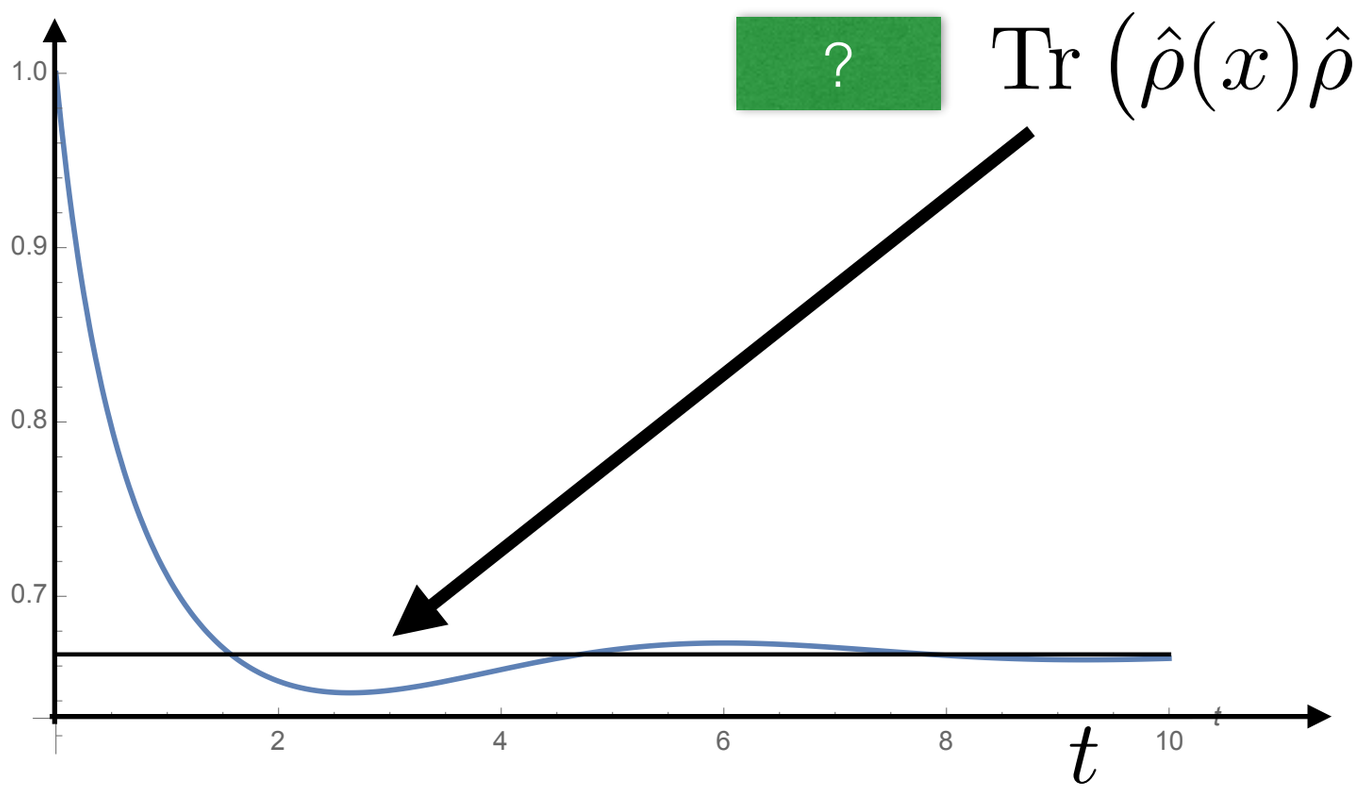
$$\langle \Psi_0(t)|\hat{\mathcal{O}}|\Psi_0(t)\rangle$$

overlaps with the initial state

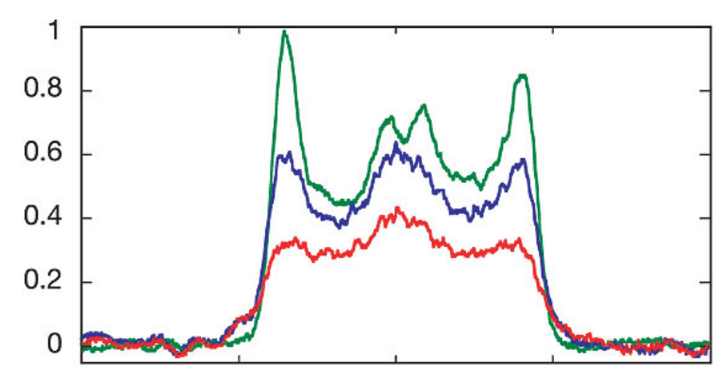
$$= \lim_{L\to\infty} \sum_{\{\mu\}} \sum_{\{\nu\}} \langle \Psi_0|\{\mu\}\rangle \langle \Psi_0|\{\nu\}\rangle^* \langle \{\mu\}|\hat{\mathcal{O}}|\{\nu\}\rangle e^{-it(E_\mu - E_\nu)}$$

?

$$\mathrm{Tr}\left(\hat{\rho}(x)\hat{\rho}(x+\Delta x)e^{-\beta_{\mathrm{eff}}H}\right)$$



?



Overlaps of Bethe states with BEC state

$$|\Psi_0\rangle = |\mathrm{GS}_{c=0}\rangle = |\mathrm{BEC}\rangle$$

M. Brockmann, J. De Nardis, B. Wouters, and J.-S. Caux, J. Phys. A 47

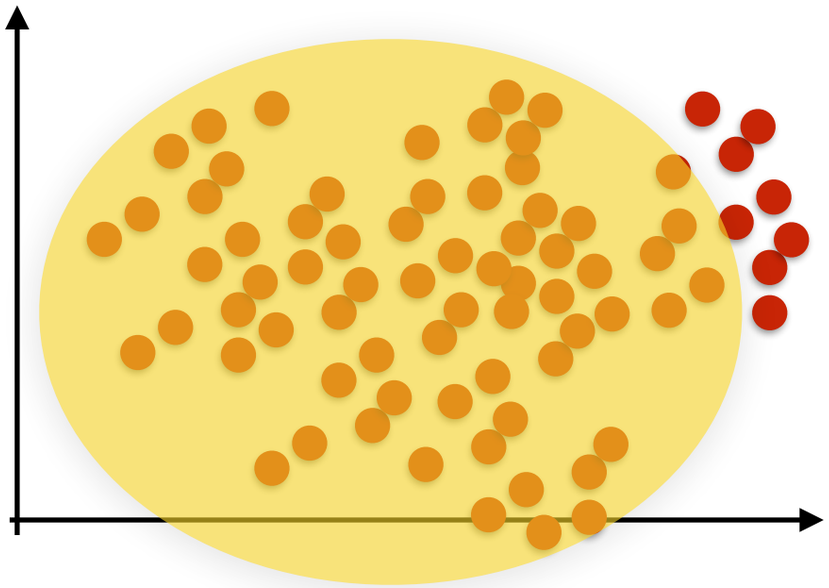
$$\langle\{\lambda\},\{-\lambda\}|\mathrm{BEC}\rangle = \sqrt{\frac{(cL)^{-N}N!}{\det_{j,k=1}^{N} G_{jk}}}\,\frac{\det_{j,k=1}^{N/2} G^{Q}_{jk}}{\prod_{j=1}^{N/2}\frac{\lambda_j}{c}\sqrt{\frac{\lambda_j^2}{c^2}+\frac{1}{4}}}$$

$$G^{Q}_{jk} = \delta_{jk}\Big(L + \sum_{l=1}^{N/2} K^{Q}(\lambda_j,\lambda_l)\Big) - K^{Q}(\lambda_j,\lambda_k)$$

$$K^{Q}(\lambda,\mu) = K(\lambda-\mu) + K(\lambda+\mu)$$

Parity invariant Bethe states

$\langle\mathrm{BEC}|$



Thermodynamic limit - quench action approach

$$L \to \infty \qquad N/L = n$$
$$N \to \infty$$

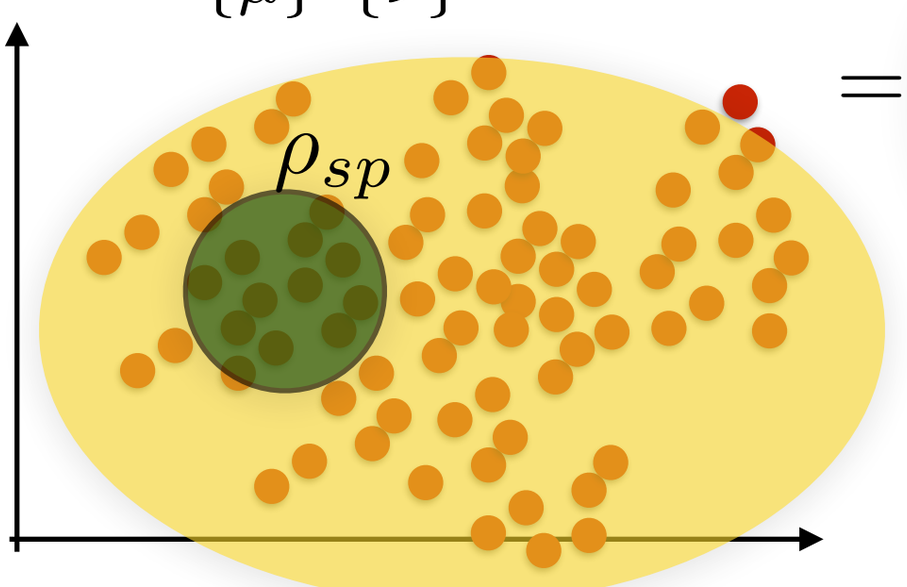
J.-S. Caux and
F. H. L. Essler, PRL 110

$$\frac{\det_{j,k=1}^{N/2} G^Q_{jk}}{\sqrt{\det_{j,k=1}^{N} G_{jk}}} \to 1$$

$$\left(\prod_{j=1}^{N/2} \frac{\lambda_j}{c}\sqrt{\frac{\lambda_j^2}{c^2}+\frac{1}{4}}\right)^{-1} \to \exp\left(-L\int_0^\infty d\lambda\ \rho(\lambda)\log\left[\frac{\lambda}{c}\sqrt{\frac{\lambda^2}{c^2}+\frac{1}{4}}\right]+O(1)\right)$$

$$\lim_{L\to\infty}\langle\Psi_0(t)|\hat{\mathcal{O}}|\Psi_0(t)\rangle =$$

$$\lim_{L\to\infty}\sum_{\{\mu\}}\sum_{\{\nu\}}\langle\Psi_0|\{\mu\}\rangle\langle\{\nu\}|\Psi_0\rangle e^{-it(E_\mu-E_\nu)}\langle\{\mu\}|\hat{\mathcal{O}}|\{\nu\}\rangle$$



$$= \sum_{\{\mu\}} e^{-\delta s_\mu - it(E_\mu - E_{sp})}\langle\rho_{sp}|\hat{\mathcal{O}}|\{\mu\}\rangle$$

$$\left.\frac{\delta\left[-\int_0^\infty d\lambda\ \rho(\lambda)\log\left[\frac{\lambda}{c}\sqrt{\frac{\lambda^2}{c^2}+\frac{1}{4}}\right]+S_{\mathrm{YY}}[\rho]\right]}{\delta\rho}\right|_{\rho=\rho_{sp}} = 0$$

Post-quench saddle point distribution of quasi-momenta

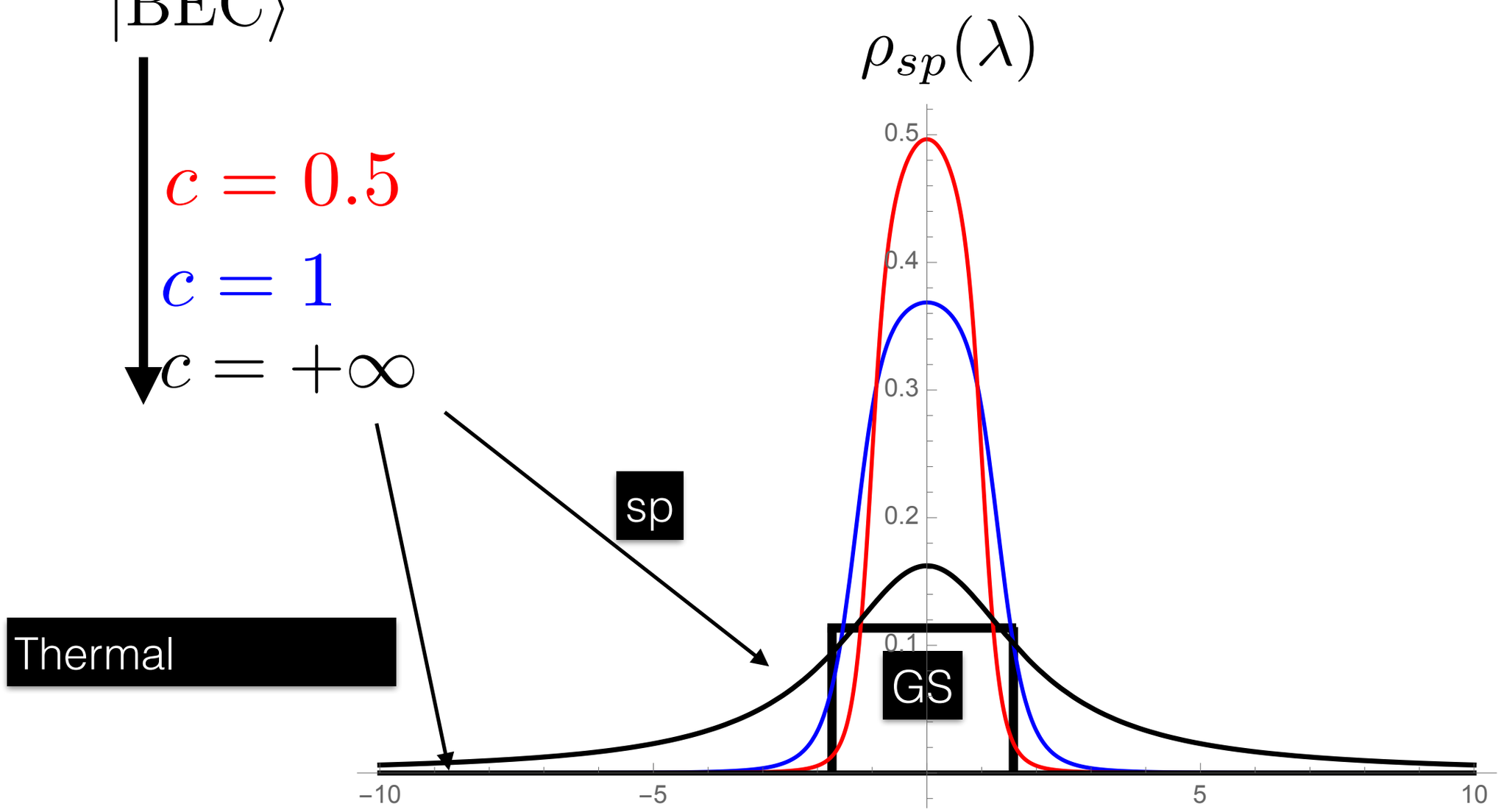
$$\frac{\delta\left[-\int_0^\infty d\lambda\ \rho(\lambda)\log\left[\frac{\lambda}{c}\sqrt{\frac{\lambda^2}{c^2}+\frac{1}{4}}\right]+S_{\mathrm{YY}}[\rho]\right]}{\delta\rho}\Bigg|_{\rho=\rho_{sp}}=0$$

$|\mathrm{BEC}\rangle$

$c=0.5$

$c=1$

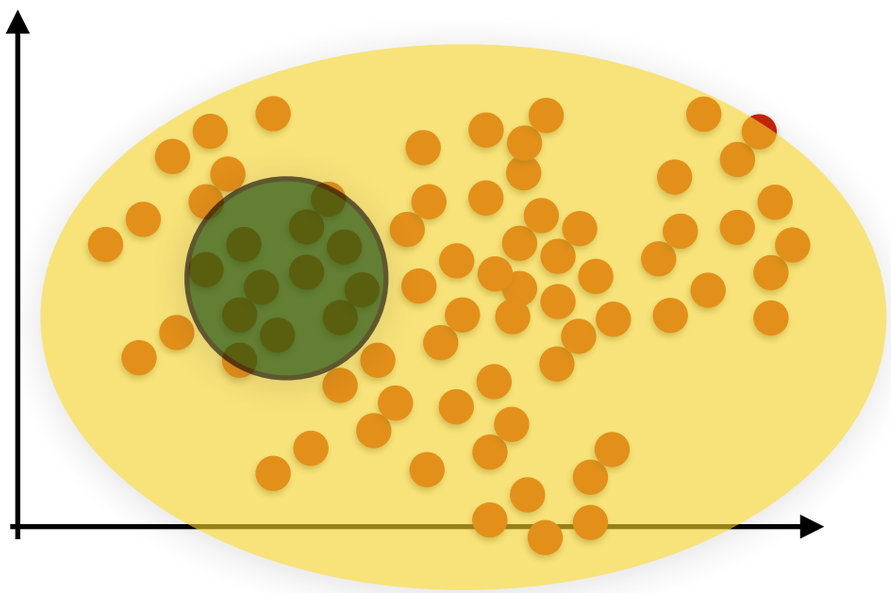
$c=+\infty$



De Nardis et al, PRA 89

$$\lim_{L\to\infty}\langle\Psi_0(t)|\hat{\mathcal{O}}|\Psi_0(t)\rangle \xrightarrow{t\to\infty} \langle\rho_{sp}|\hat{\mathcal{O}}|\rho_{sp}\rangle$$

Time evolution towards equilibrium



$$\lim_{L\to\infty} \langle \Psi_0(t)|\hat{\mathcal{O}}|\Psi_0(t)\rangle =$$

$$\sum_{\{\mu\}} e^{-\delta s_\mu - it(E_\mu - E_{sp})} \langle \rho_{sp}|\hat{\mathcal{O}}|\{\mu\}\rangle$$

$$\hat{\mathcal{O}} = \mathbf{\Psi}^+(x)\mathbf{\Psi}(0)$$

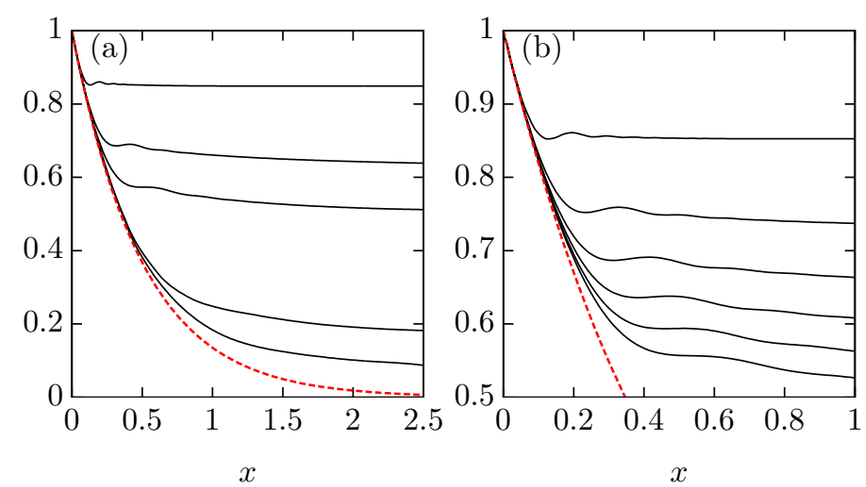
$$c = 0 \to c = +\infty$$

$$\lim_{L\to\infty} \langle \mathrm{BEC}(t)|\mathbf{\Psi}^+(x)\mathbf{\Psi}(0)|\mathrm{BEC}(t)\rangle =$$

$$\sqrt{\mathrm{Det}\begin{pmatrix} 1+K'\rho_{sp} & -K'\varphi_+^{(t)} \\ K'^*\varphi_-^{(t)} & 1+K'^*\rho_{sp}\end{pmatrix}} - \sqrt{\mathrm{Det}\begin{pmatrix} 1+K\rho_{sp} & -K\varphi_+^{(t)} \\ K\varphi_-^{(t)} & 1+K\rho_{sp}\end{pmatrix}}$$

$$K'(u,v) = K(u,v) + e^{-\frac{x}{2}i(u+v)}$$

$$K(u,v) = -4\frac{\sin(\frac{x}{2}(u-v))}{u-v}$$

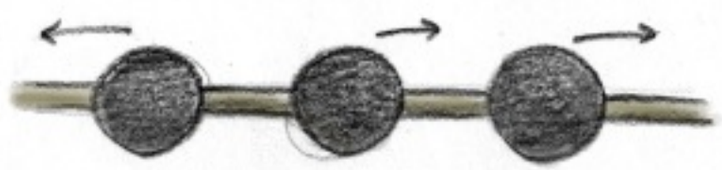


De Nardis J and Caux J S J. Stat. Mech. (2014)

$$p(k_1) + p(k_2) + p(k_3) = p(k_1') + p(k_2') + p(k_3')$$

$$\epsilon(k_1) + \epsilon(k_2) + \epsilon(k_3) = \epsilon(k_1') + \epsilon(k_2') + \epsilon(k_3')$$

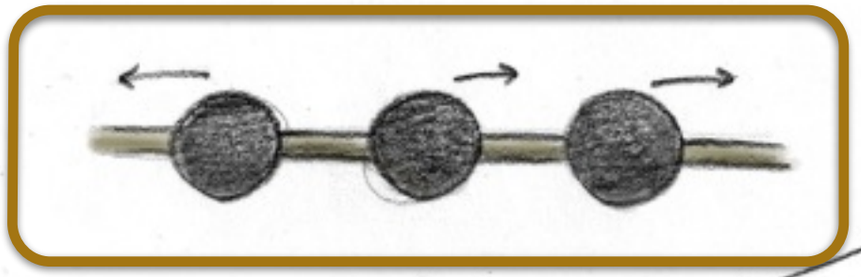
$$q_3(k_1) + q_3(k_2) + q_3(k_3) = q_3(k_1') + q_3(k_2') + q_3(k_3')$$



$$\Psi \to \sum_P \Phi(P)\, e^{i x_1 k_{P_1} + i x_2 k_{P_2} + i x_3 k_{P_3}} + \cancel{\iiint_{k_1' < k_2' < k_3'} dk_1' dk_2' dk_3'\, S[k_1', k_2', k_3']\, e^{i x_1 k_1' + i x_2 k_2' + i x_3 k_3'}}$$

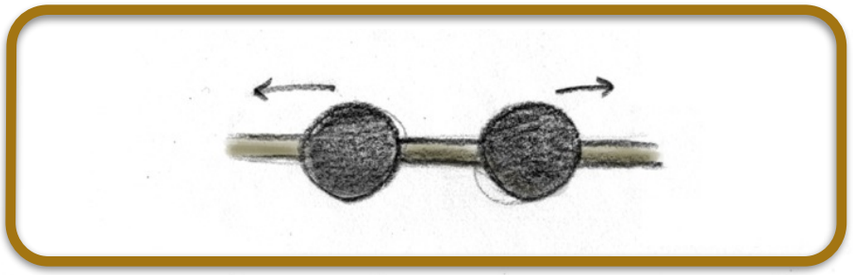
Local conserved quantities

Q_3



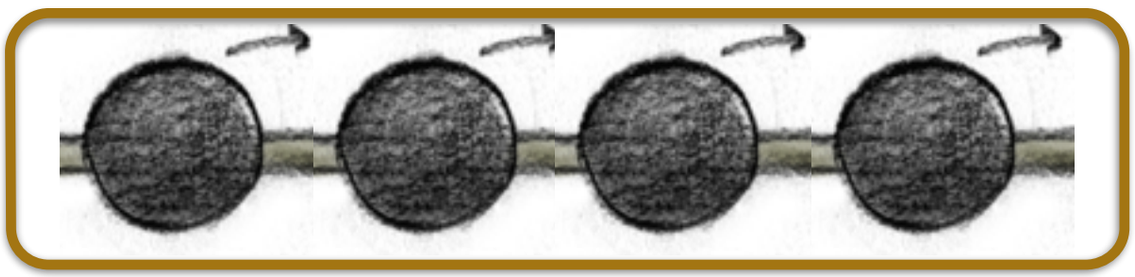
$$\Psi \to \sum_P \Phi(P)\, e^{ix_1 k_{P_1} + ix_2 k_{P_2} + ix_3 k_{P_3}} + \cancel{\iiint_{k'_1 < k'_2 < k'_3} dk'_1\, dk'_2\, dk'_3\, S[k'_1, k'_2, k'_3]\, e^{ix_1 k'_1 + ix_2 k'_2 + ix_3 k'_3}}$$

$Q_2 \sim H$

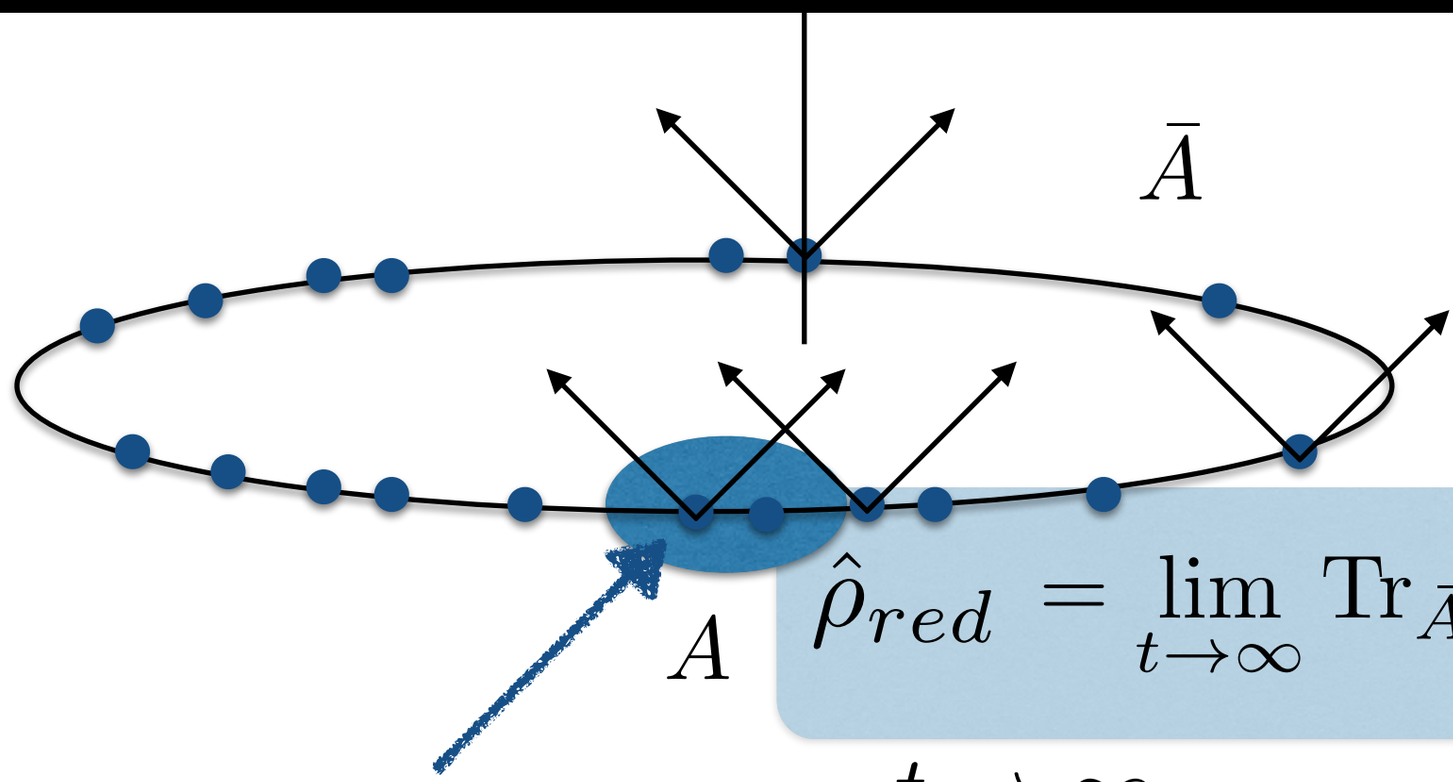


$$\Psi \to e^{ix_1 k_1 + ix_2 k_2} + e^{i\phi(k_1, k_2)}\, e^{ix_1 k_2 + ix_2 k_1}$$

Q_4



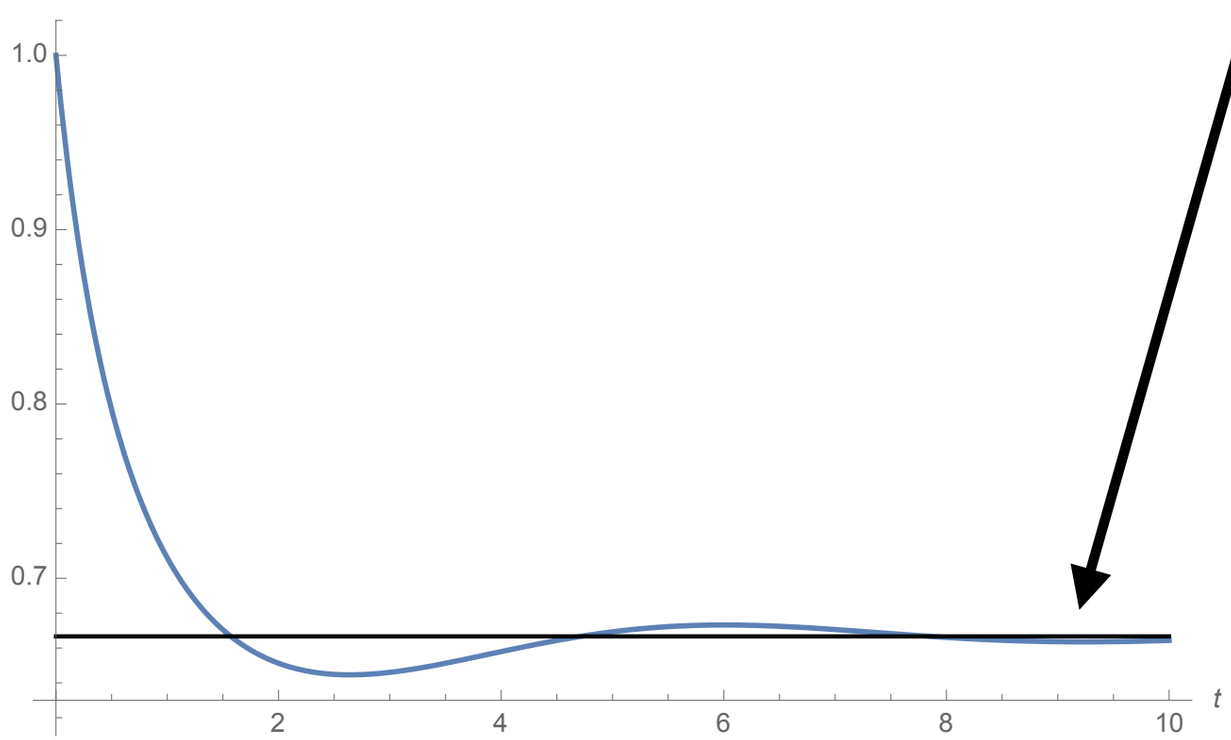
Generalized Gibbs Ensemble (GGE)



$$\hat{\rho}_{red} = \lim_{t\to\infty} \mathrm{Tr}_{\bar{A}} \left(|\Psi_0(t)\rangle\langle\Psi_0(t)| \right)$$

$$\langle\Psi_0(t)|\hat{\mathcal{O}}|\Psi_0(t)\rangle \xrightarrow{t\to\infty} \mathrm{Tr}\left(\hat{\mathcal{O}}\hat{\rho}_{\mathrm{red}}\right)$$

$$\hat{\rho}_{\mathrm{red}} = \exp\left(-\sum_n \beta_n Q_n\right)$$



Rigol et al, PRL 110

Generalized Gibbs Ensemble

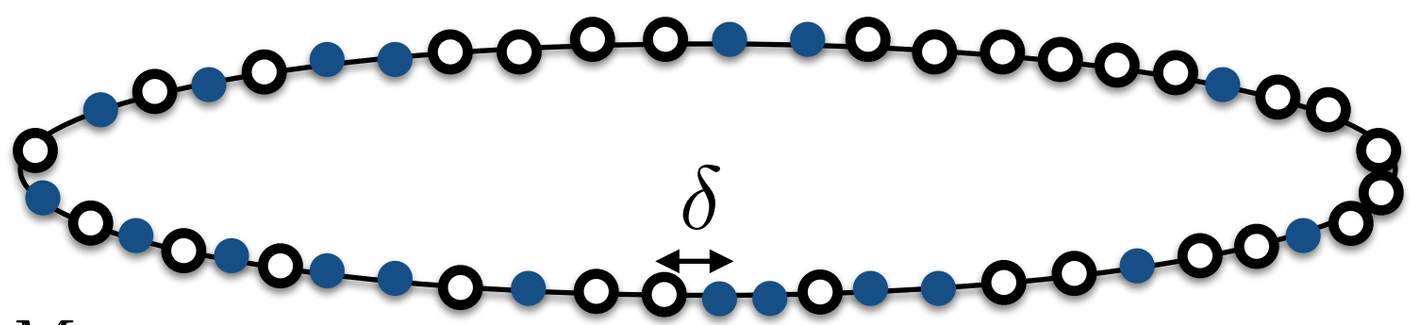
Divergence of the local charges

$$\lim_{L\to\infty} \frac{\langle \mathrm{BEC}|H|\mathrm{BEC}\rangle}{L} = c\left(\frac{N}{L}\right)^2$$

$$\frac{\langle \mathrm{BEC}|Q_4|\mathrm{BEC}\rangle}{L} \sim \delta(0)^2$$

$$\frac{\langle \mathrm{BEC}|Q_{2n}|\mathrm{BEC}\rangle}{L} \sim \delta(0)^{2n-2}$$

Lattice q-bosons regularization



$$H_q = -\frac{1}{\delta^2}\sum_{j=1}^{M}(B_j^\dagger B_{j+1} + B_{j+1}^\dagger B_j - 2N_j)$$

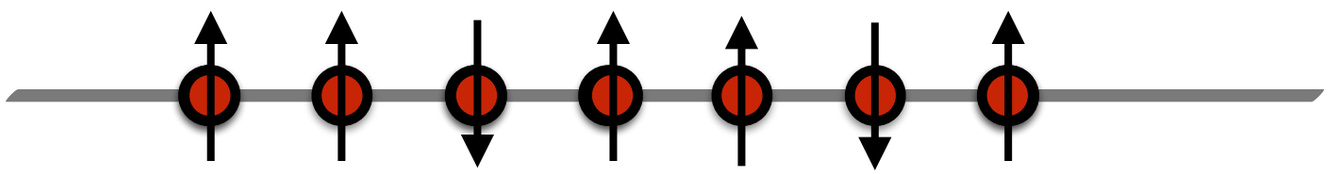
$$B_j B_j^\dagger - q^{-2} B_j^\dagger B_j = 1\,, \quad q > 1$$

$$\frac{\langle \mathrm{BEC}|Q_{2n}|\mathrm{BEC}\rangle}{L} \sim \delta^{-(2n-2)}$$

M. Kormos, A. Shashi, Y.-Z. Chou, J.-S. Caux, and A. Imambekov, Phys. Rev. B 88

Spin-1/2 XXZ chain

$$H = \frac{J}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)\right)$$



Bethe State

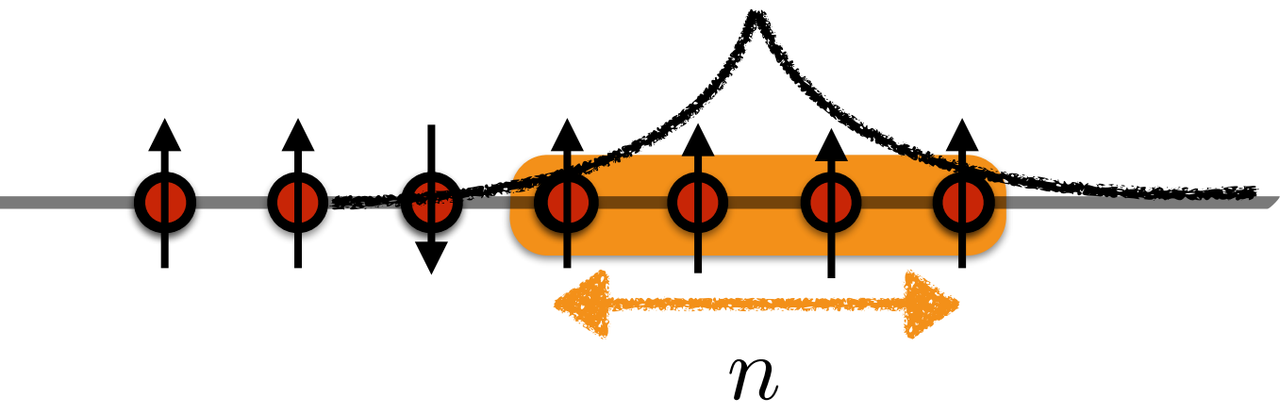
$$|\boldsymbol{\lambda}\rangle = \sum_{\boldsymbol{x}} \Psi_M(\boldsymbol{x}|\boldsymbol{\lambda})\, \sigma_{x_1}^- \ldots \sigma_{x_M}^- \, |\uparrow\uparrow \ldots \uparrow\rangle$$

$$\Psi_M(\boldsymbol{x}|\boldsymbol{\lambda}) = \sum_{Q\in\mathcal{S}_M} (-1)^{[Q]} \exp\left\{ -i\sum_{j=1}^{M} x_j\, p(\lambda_{Q_j}) - \frac{i}{2}\sum_{\substack{j,k=1\\ k>j}}^{M} \theta_2(\lambda_{Q_k} - \lambda_{Q_j}) \right\}$$

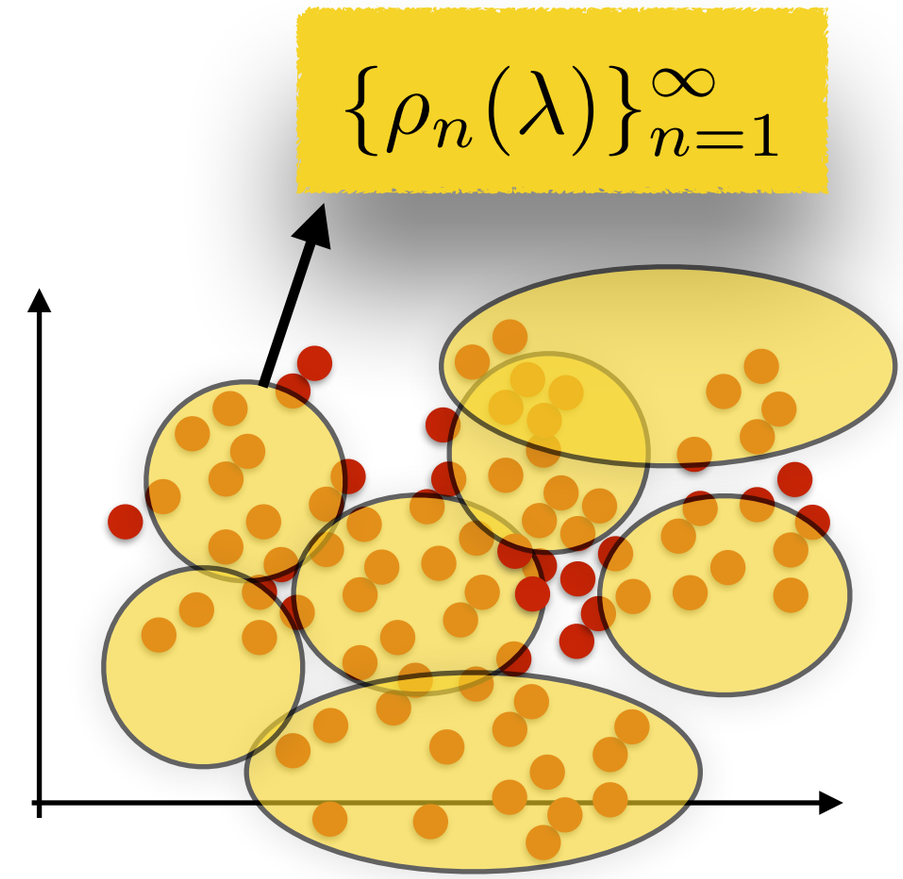
$$e^{-iNp(\lambda_j)} = \prod_{k\neq j} e^{-i\theta_2(\lambda_j - \lambda_k)} \qquad \forall j$$

Bethe equations

Spectrum of bound states



$$\Psi \sim e^{-\sum_{i<j} \frac{|x_i - x_j|}{2} \cosh^{-1} \Delta + i\lambda_{j,n} \frac{\sum_j x_j}{2}}$$



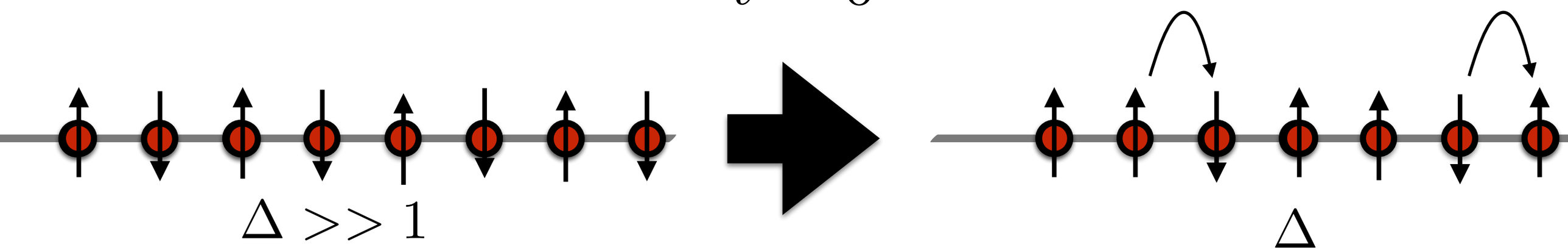
$$E[\rho] = L \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} d\lambda \rho_n(\lambda) \epsilon_n(\lambda) + O(1)$$

$$S_{YY}[\rho] = L \sum_{n=1}^{\infty} \int_{-\pi/2}^{\pi/2} d\lambda [\rho_n \ln(1 + \rho_{n,h}/\rho_n) + \rho_{n,h} \ln(1 + \rho_n/\rho_{n,h})]$$

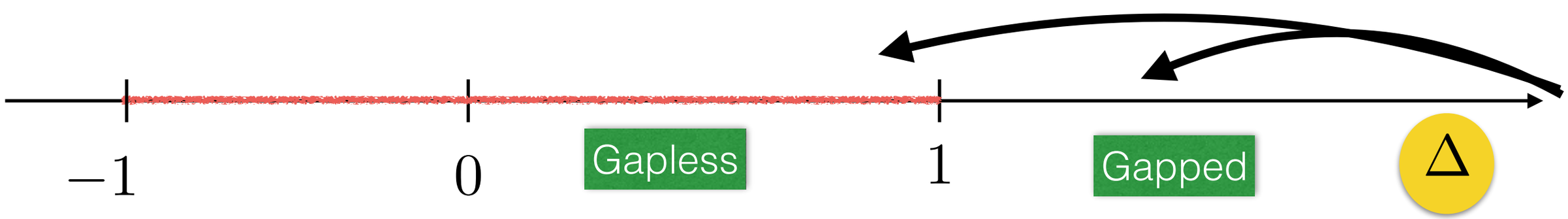
Quench from a Néel state

$$H = \frac{J}{4}\sum_{j=1}^{N}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta(\sigma_j^z\sigma_{j+1}^z - 1)\right)$$

$$t = 0$$



$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle^{\otimes N/2} + |\downarrow\uparrow\rangle^{\otimes N/2}\right)$$



Overlap of Bethe states with the Néel state

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle^{\otimes N/2} + |\downarrow\uparrow\rangle^{\otimes N/2}\right)$$

$$\langle\Psi_0|\{-\lambda\},\{\lambda\}\rangle = \sqrt{2}\left[\prod_{j=1}^{N/2}\frac{\sqrt{\tan(\lambda_j + i\eta/2)\tan(\lambda_j - i\eta/2)}}{2\sin(2\lambda_j)}\right]\sqrt{\frac{\det_{N/2}(G^+_{jk})}{\det_{N/2}(G^-_{jk})}}$$

$$G^{\pm}_{jk} = \delta_{jk}\left(NK_{\eta/2}(\lambda_j) - \sum_{l=1}^{N/2} K^{\pm}_{\eta}(\lambda_j,\lambda_l)\right) + K^{\pm}_{\eta}(\lambda_j,\lambda_k)$$

$$K^{\pm}_{\eta}(\lambda,\mu) = K_{\eta}(\lambda-\mu) \pm K_{\eta}(\lambda+\mu)$$

$$K_{\eta}(\lambda) = \frac{\sinh(2\eta)}{\sin(\lambda+i\eta)\sin(\lambda-i\eta)}$$

M. Brockmann, J. De Nardis, B. Wouters, and J.-S. Caux, J. Phys. A 47

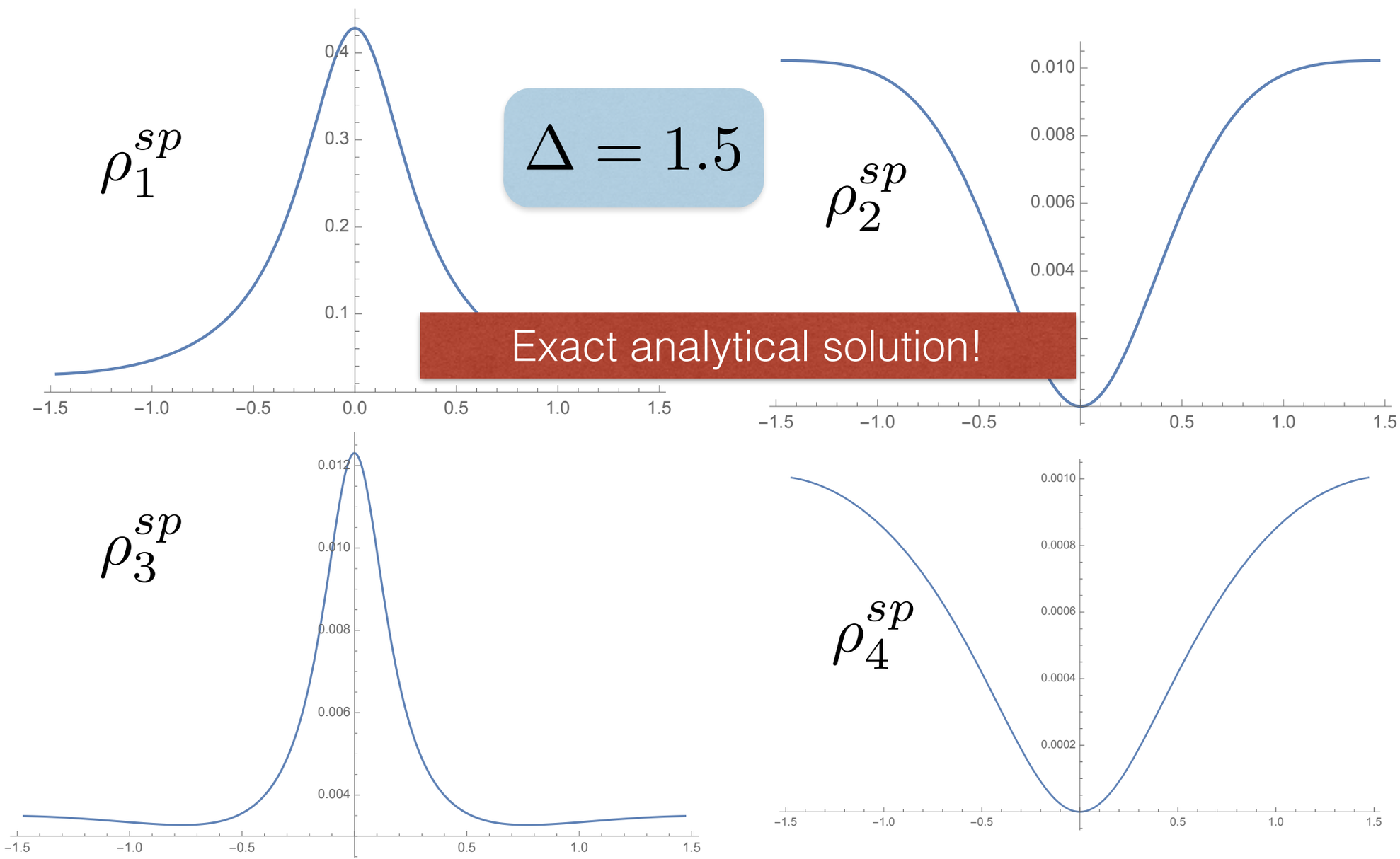
Post-quench saddle point distribution of quasi-momenta

$$\frac{\delta\left[\lim_{L\to\infty}|\log\langle\Psi_0|\{\lambda\}\rangle|^2 + S_{YY}[\rho]\right]}{\delta\rho_n}\Bigg|_{\rho_n=\rho_n^{sp}}$$

Wouters B, De Nardis J, Brockmann M, Fioretto D, Rigol M and Caux J-S 2014 Phys. Rev. Lett. 113

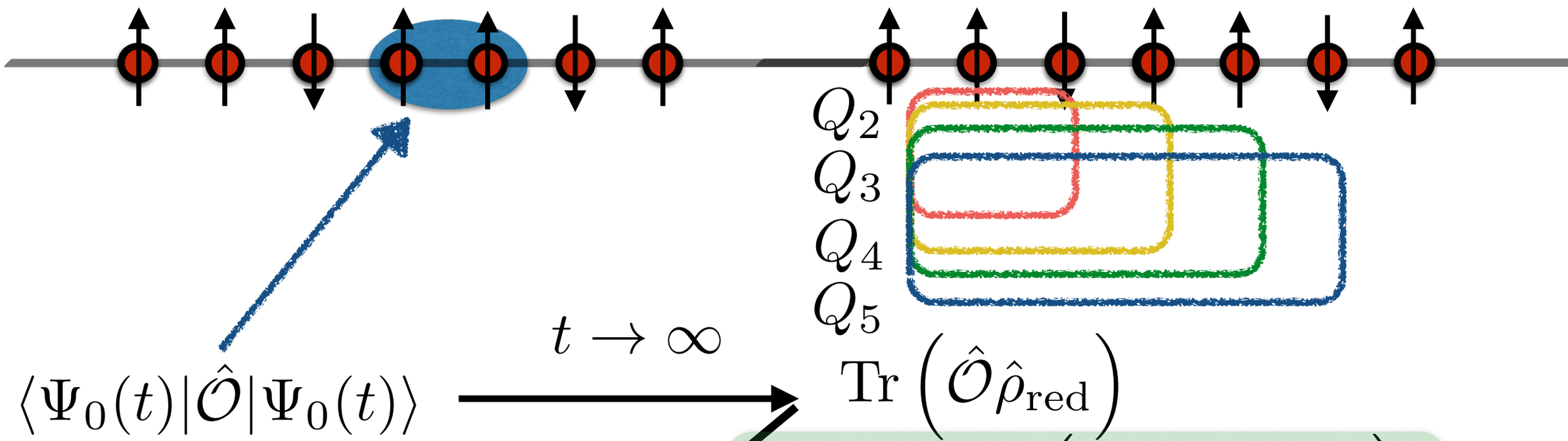
$\Delta = 1.5$

Exact analytical solution!



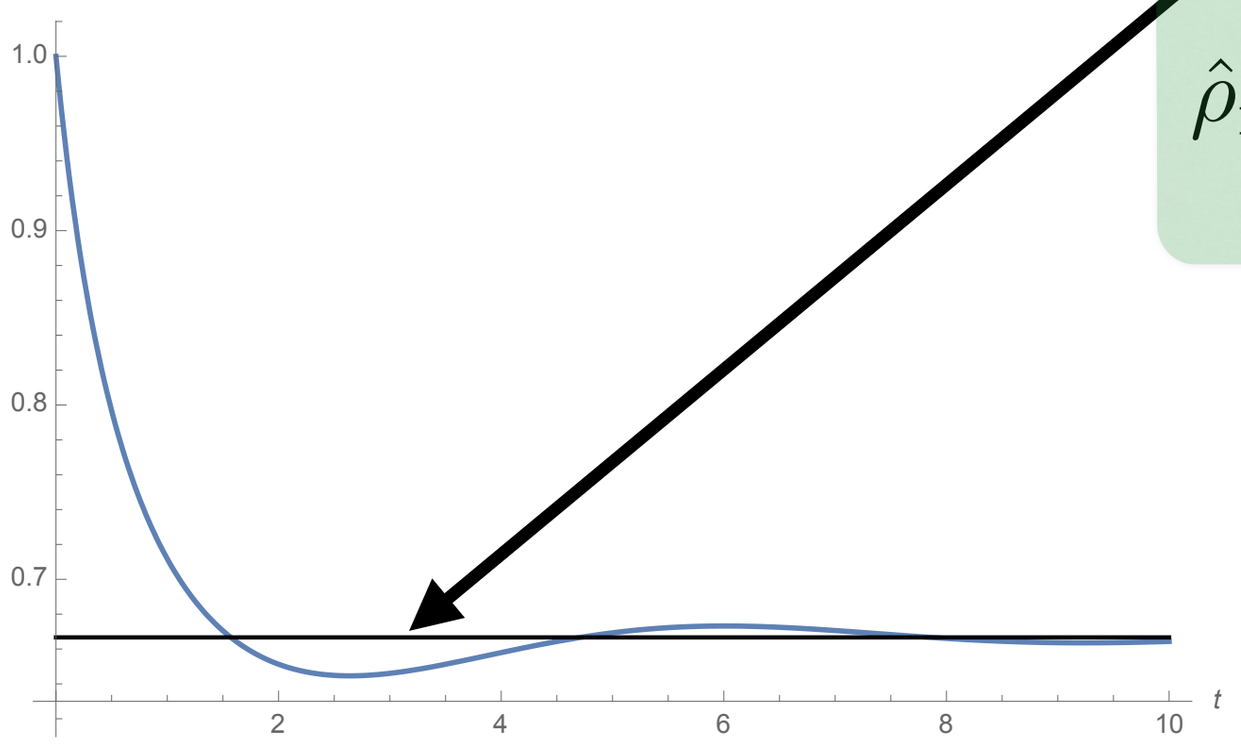
Generalized Gibbs Ensemble

$$\hat{\mathcal{O}} = \sigma_j^z \sigma_{j+1}^z$$



$$\langle \Psi_0(t) | \hat{\mathcal{O}} | \Psi_0(t) \rangle \xrightarrow{t \to \infty} \mathrm{Tr}\left(\hat{\mathcal{O}} \hat{\rho}_{\mathrm{red}}\right)$$

$$\hat{\rho}_{\mathrm{red}} = \exp\left(-\sum_n \beta_n Q_n\right)$$



Generalized Gibbs Ensemble

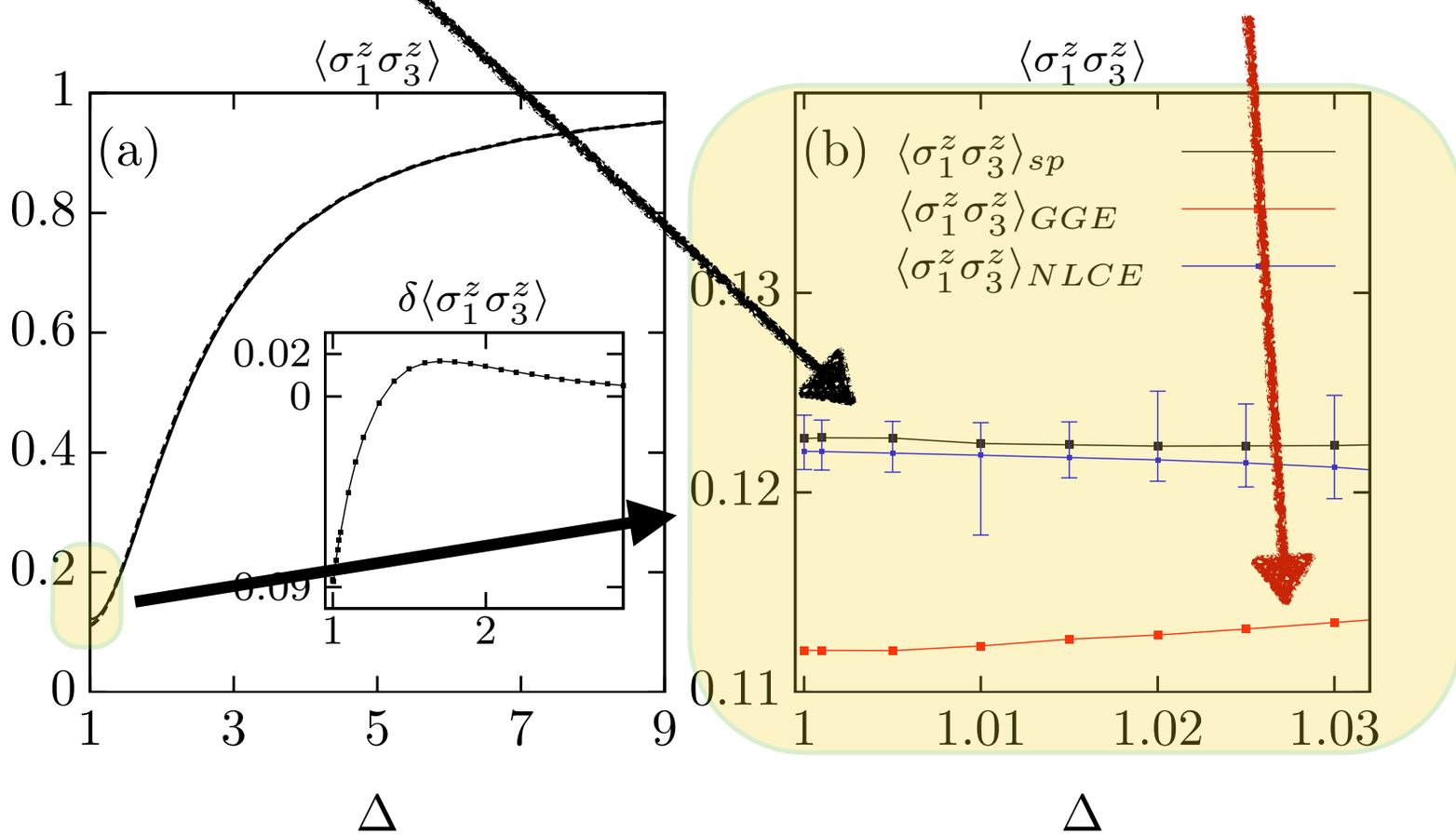
Failure of Generalized Gibbs Ensemble with only local charges

Local charges are not complete!

Exact result given by the saddle point of the **overlaps**

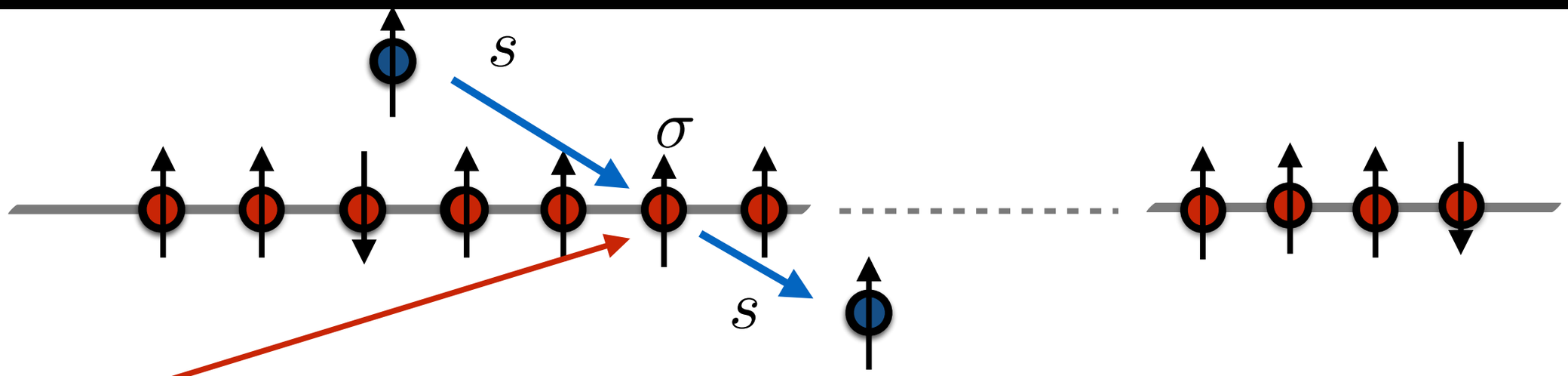
Generalized Gibbs with local conserved charges

$$\lim_{t\to\infty} \langle \Psi_0(t) | \sigma_1 \sigma_{1+\ell} | \Psi_0(t) \rangle \overset{?}{\neq} \mathrm{Tr}\left(\sigma_1 \sigma_{1+\ell} e^{-\sum_{n=1}^{\infty} \beta_n Q_n} \right)$$



Wouters B, De Nardis J, Brockmann M, Fioretto D, Rigol M and Caux J-S 2014 Phys. Rev. Lett. 113

Quasi-Local symmetries and higher spin

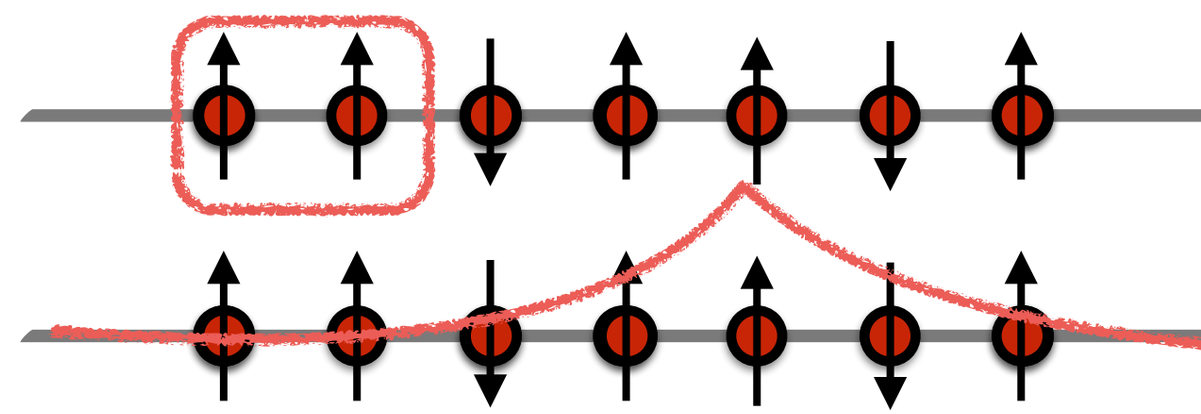


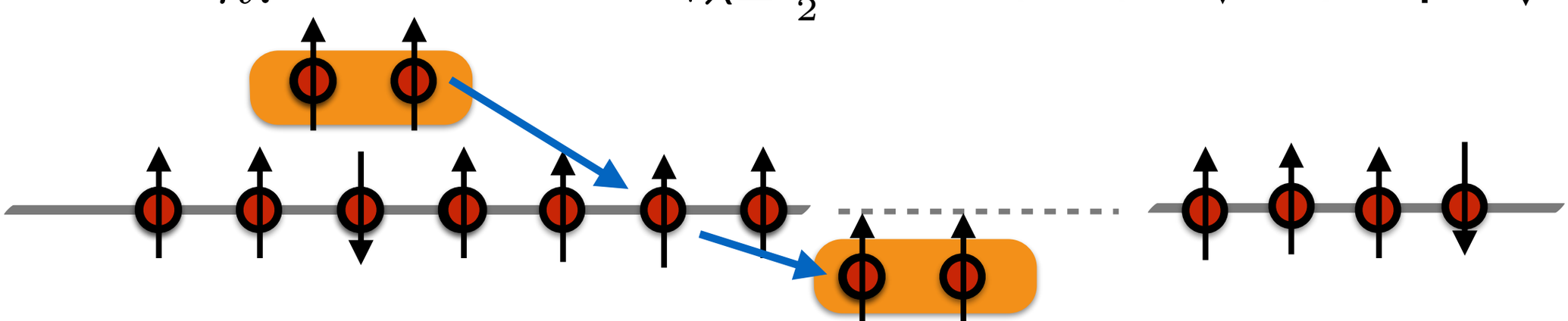
$$L(z,s) = \frac{1}{\sinh\eta}\Big(\sinh(z)\cosh(\eta s^z)\otimes\sigma^0 + \cosh(z)\sinh(\eta s^z)\otimes\sigma^z + \sinh(\eta)(s^-\otimes\sigma^+ + s^+\otimes\sigma^-)\Big)$$

$$T_s(z) = \mathrm{Tr}_a\left[L_{a,1}(z,s)\ldots L_{a,N}(z,s)\right]$$

$$Q_{n+1} = \frac{i}{n!}\partial_\lambda^n \ln T_{\frac{1}{2}}(-i\lambda)\Big|_{\lambda=\frac{i\eta}{2}}$$

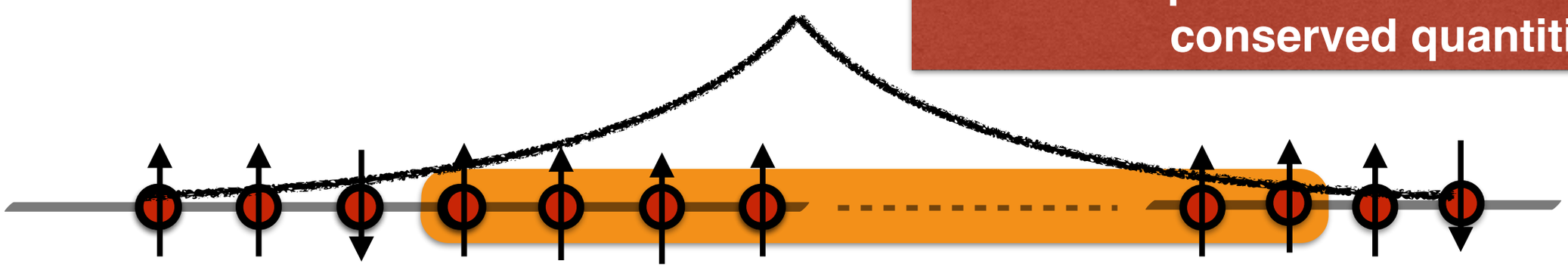
$$Q_{n+1}^{(s)} = \frac{i}{n!}\partial_\lambda^n \ln T_s(-i\lambda)\Big|_{\lambda=\frac{i\eta}{2}}$$





Quasi-Local symmetries

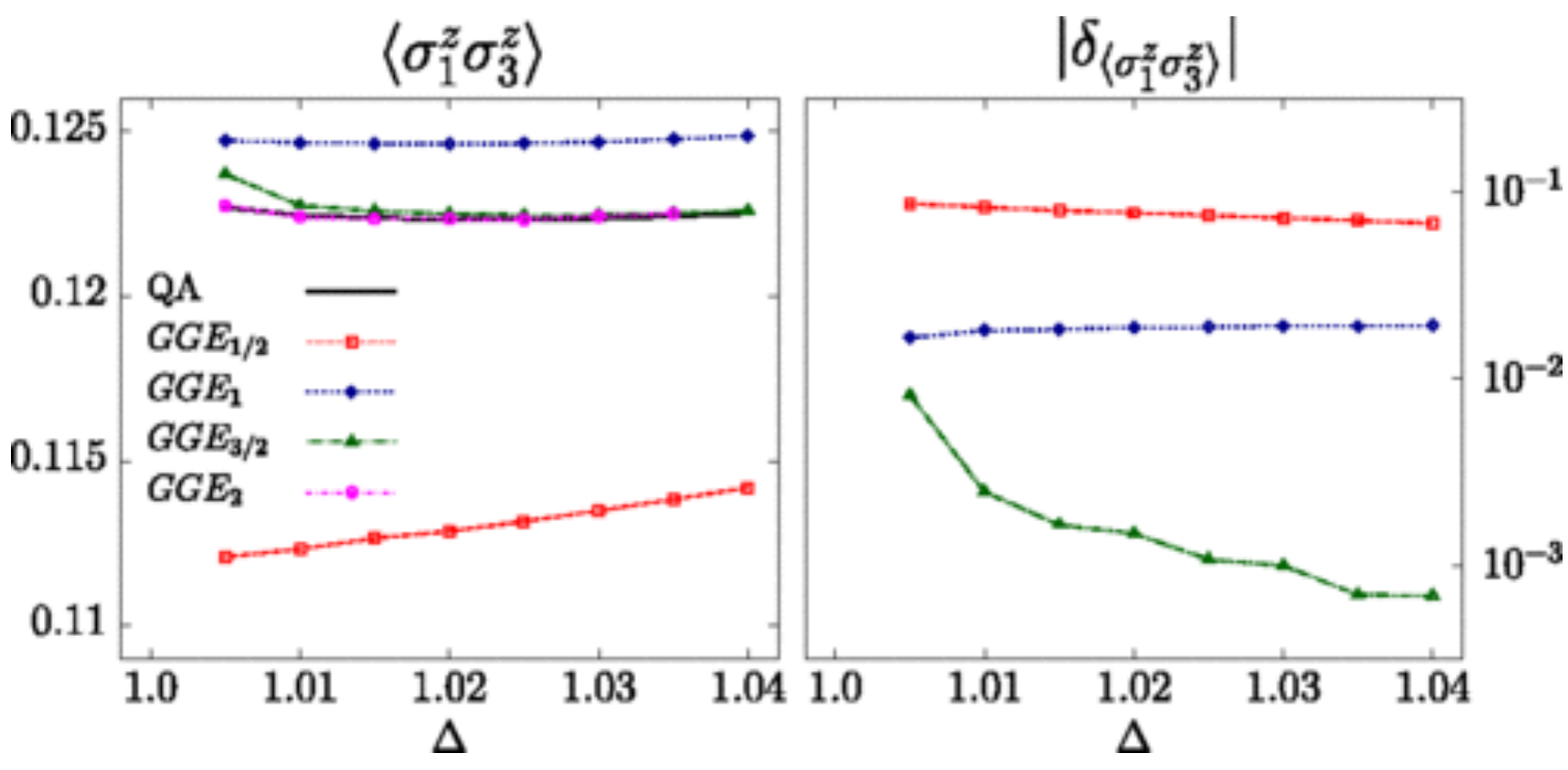
Now a complete set of local and quasi-local conserved quantities



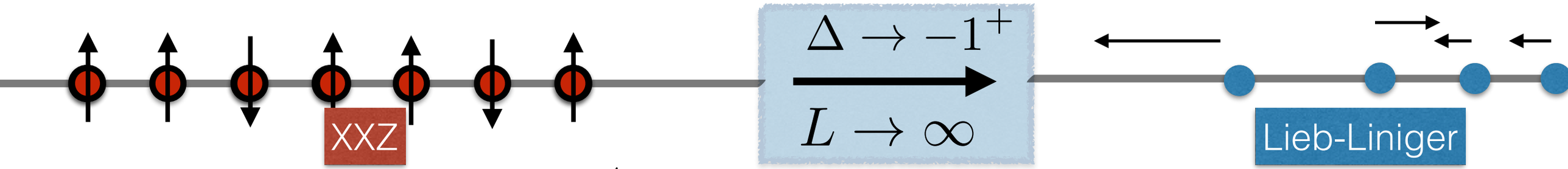
$$\hat{\rho}_{\mathrm{GGE}} = e^{-\sum_n \beta_n Q_n - \sum_{n,s>1} \beta_n^{(s)} Q_n^{(s/2)}}$$

$$\lim_{t\to\infty} \langle \Psi_0(t) | \sigma_1 \sigma_{1+\ell} | \Psi_0(t) \rangle = \mathrm{Tr}\left(\sigma_1 \sigma_{1+\ell} \hat{\rho}_{GGE}\right)$$

Ilievski et al 2015, PRL 115



BEC state and Néel state



$$|BEC\rangle = \binom{L}{N_{LL}}^{-1/2} \sum_{\{n_j\}_{j=1}^{N_{LL}}} \sigma^-_{n_1} \dots \sigma^-_{n_{N_{LL}}} |\uparrow \dots \uparrow\rangle$$

spatially uniformly distributed spins

$$S^+ \sim \sum_{n=1}^{L} (-1)^n \sigma^+_n$$

$$(S^+)^{L/2-N_{LL}} |\uparrow\downarrow \dots \uparrow\downarrow\rangle \sim \sum_{\substack{\{n_j\}_{j=1}^{N_{LL}} \\ n_j \text{ even}}} \sigma^-_{n_1} \dots \sigma^-_{n_{N_{LL}}} |\uparrow \dots \uparrow\rangle$$

$$\langle\{\pm\lambda\}|(S^+)^{L/2-N_{LL}}|\uparrow\downarrow\ldots\uparrow\downarrow\rangle \to \langle\{\lambda\}|\mathrm{BEC}\rangle$$

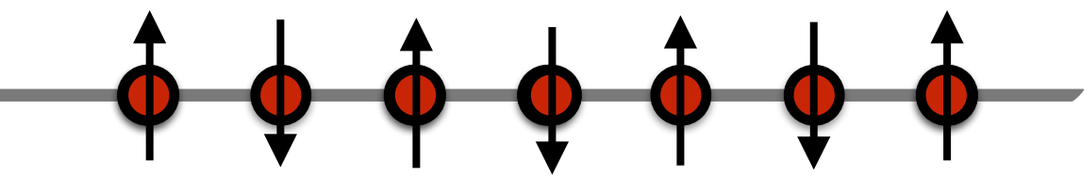
$$S^+|\{\lambda\}\rangle \sim \lim_{\lambda\to\infty} |\{\lambda\}\rangle$$

$$\prod_{j=1}^{L/4-N_{LL}/2} \frac{\sinh^2(\mu_j)}{\sinh^2(\eta)} \langle\text{Néel}|\{\pm\lambda_j\}_{j=1}^{N_{LL}/2} \cup \{\pm\mu_j\}_{j=1}^{L/4-N_{LL}/2}\rangle \xrightarrow[\{\mu\}\to\infty]{} \langle\mathrm{BEC}|\{\pm\lambda\}_{j=1}^{N_{LL}/2}\rangle$$

M. Brockmann J Stat Mech P05006

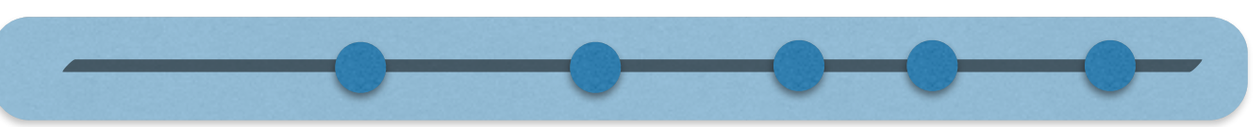
Exclusion processes and integrable models

XXZ spin chain



Néel initial state

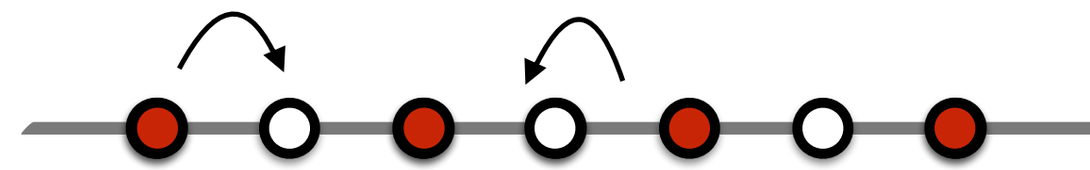
Lieb-Liniger



$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j>k} \delta(x_j - x_k)$$

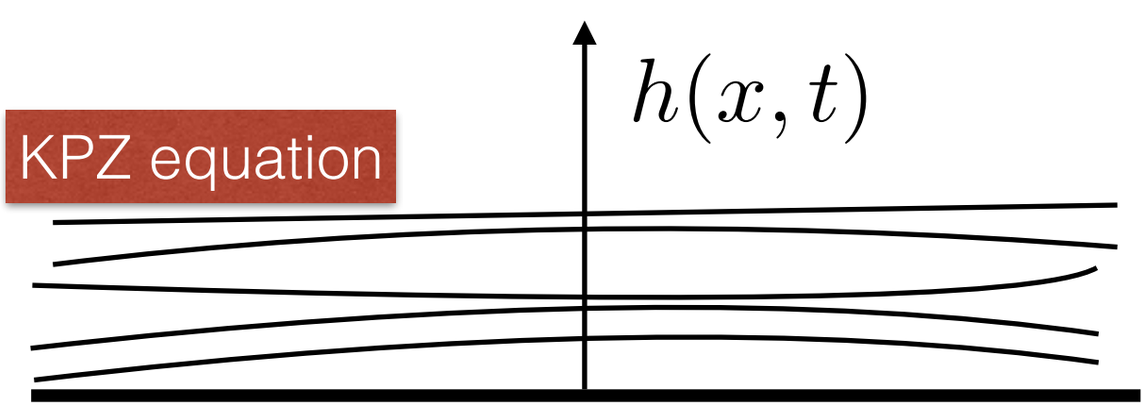
BEC initial state

ASEP



Flat initial condition

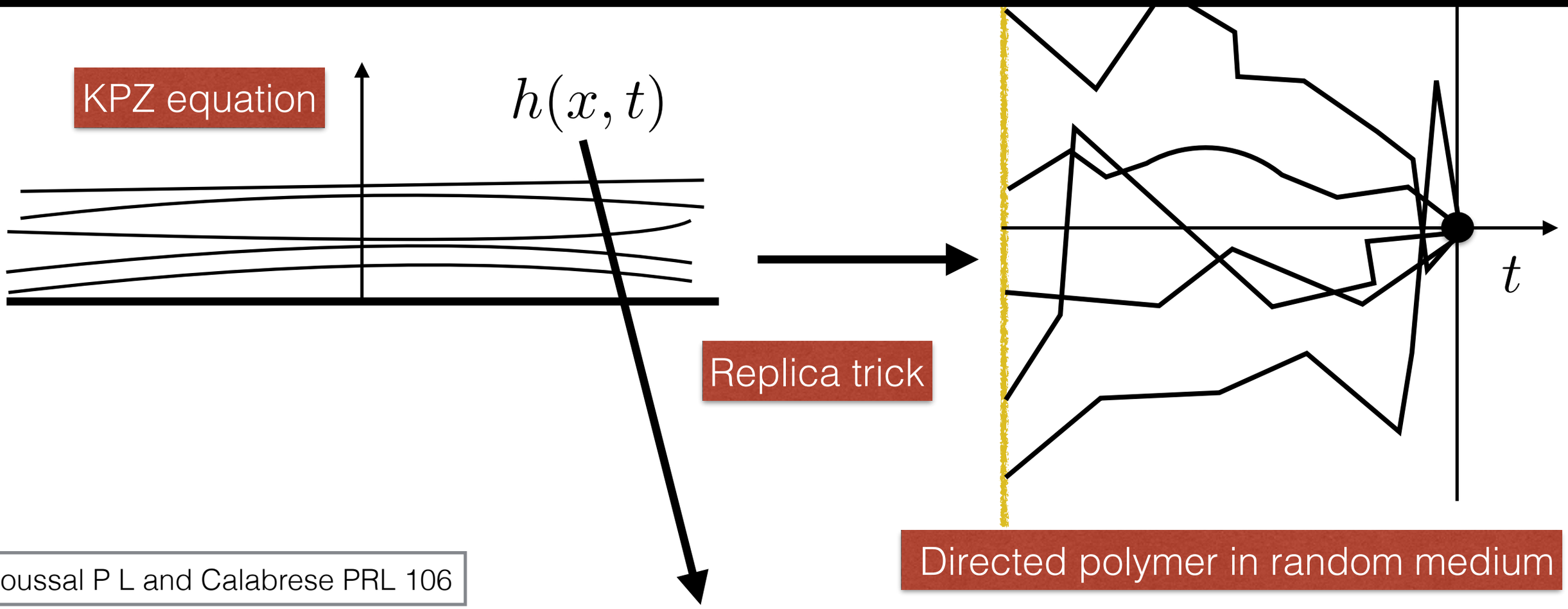
KPZ equation



$$\partial_t h = \nu \nabla^2 h + \frac{1}{2}\lambda_0 (\nabla h)^2 + \eta(x,t)$$

Flat initial condition

Directed polymer and Lieb-Liniger gas



Doussal P L and Calabrese PRL 106

Doussal P L and Calabrese P 2012
J. Stat. Mech.: Th. Exp. 2012 P06001

$$f = -\log Z \quad \rightarrow P_t(f)$$

$$\langle Z^n_{\text{flat}} \rangle = \sum_{\{\mu\}} \langle \text{BEC} | \{\mu\} \rangle \frac{\Psi^*(\{\mu\}|\{x_i = 0\})}{||\{\mu\}||} e^{-tE_\mu}$$

$$\lim_{t\to\infty} P_t(f) = F_1(f)$$

GOE Tracy-Widom

Conclusions

Out of equilibrium quantum physics in presence of **strong correlations** is a mostly unexplored territory

Need **exact methods** (not energy dependent) to study integrable system in and out of equilibrium

The basis of integrable systems is ideal

Symmetries in the system: find the relevant ones to fix the whole post-quench dynamics

Quasi-local charges and out-of-equilibrium critical models

Relation between equilibrium states and non-equilibrium steady states

Towards an equilibrium-like description of relaxation dynamics:
minimal information
to recover the whole time evolution

Big questions

Out of equilibrium **quantum** physics ⟷ Out of equilibrium **classical** stochastic models

Integrability in classical stochastic models?

a **GGE** for the **KPZ** equation?

Universality in **Out of equilibrium quantum** physics ⟷ Universality in Out of equilibrium **classical** physics

Thank you.