The asymmetric KMP model, and its duality.

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Outline

- The symmetric KMP model
- Lie algebraic approach to duality theory
- $\mathfrak{su}_q(1, 1)$ algebra: ASIP, ABEP, AKMP
- Applications
Non-equilibrium in 1d: particle transport

- density reservoirs

- current reservoirs

- asymmetry
Non-equilibrium in 1d: energy transport

Fourier law \( J = \kappa \nabla T \)

KMP model (1982)

Energies at every site: \( z = (z_1, \ldots, z_N) \in \mathbb{R}_+^N \)

\[
L^{KMP} f(z) = \sum_{i=1}^{N} \int_{0}^{1} dp \left[ f(z_1, \ldots, p(z_i + z_{i+1}), (1 - p)(z_i + z_{i+1}), \ldots, z_N) - f(z) \right]
\]

\( \rightarrow \) conductivity \( 0 < \kappa < \infty \); model solved by duality.
(Stochastic) Duality

Definition

$(\eta_t)_{t \geq 0}$ Markov process on $\Omega$ with generator $L$

$(\xi_t)_{t \geq 0}$ Markov process on $\Omega_{dual}$ with generator $L_{dual}$

$\xi_t$ is dual to $\eta_t$ with duality function $D : \Omega \times \Omega_{dual} \rightarrow \mathbb{R}$ if $\forall t \geq 0$

$$\mathbb{E}_{\eta}(D(\eta_t, \xi)) = \mathbb{E}_{\xi}(D(\eta, \xi_t)) \quad \forall (\eta, \xi) \in \Omega \times \Omega_{dual}$$

$\eta_t$ is self-dual if $L_{dual} = L$.

In terms of generators:

$$L D(\cdot, \xi)(\eta) = L_{dual} D(\eta, \cdot)(\xi)$$
Duality

Why is it useful:
- The dual process is simpler: “from many to few”
- Duality is a signature of integrability

Questions:
- How to find a dual process & duality function?
- How to construct processes with duality?
Lie algebraic approach to duality theory
Algebraic approach

1. The Markov generator, in abstract form, is an element of a (quantum) Lie algebra.

2. Duality is related to a change of representation. Duality functions are the intertwiners.

3. Self-duality is associated to symmetries.

Conversely, the approach can be turned into a constructive method.
Duality

Abstract generator

\[ L \]

Original generator \[ L \]

Dual generator \[ L^{\text{Dual}} \]

\[ D \]
Trivial self-duality

Consider Markov chains with countable state space $\Omega$ and with a reversible measure $\mu$.

A trivial (i.e. diagonal) self-duality function is

$$d(\eta, \xi) = \frac{1}{\mu(\eta)} \delta_{\eta, \xi}$$

Indeed

$$\sum_{\eta' \in \Omega} L(\eta, \eta') d(\eta', \xi) = \sum_{\xi' \in \Omega} L(\xi, \xi') d(\eta, \xi')$$
Trivial self-duality

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$$\frac{L(\eta, \xi)}{\mu(\xi)} = \sum_{\eta' \in \Omega} L(\eta, \eta') d(\eta', \xi) = \sum_{\xi' \in \Omega} L(\xi, \xi') d(\eta, \xi') = \frac{L(\xi, \eta)}{\mu(\eta)}$$
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Thus

$$Ld = dL^T$$
Symmetries and self-duality

$S$: symmetry of the Markov generator, i.e. $[L, S] = 0$

d: trivial self-duality function

$\rightarrow \quad D = Sd$ is a self-duality function

Indeed

$L D = L S d = S L d = S d L^T = D L^T$
Construction of Markov generators with algebraic structure

Ingredients:

- (Algebra): Start from a Lie algebra $g$.
- (Casimir): Pick an element $C$ in the center of $g$, e.g. the Casimir.
- (Co-product): Consider a co-product $\Delta : g \to g \otimes g$ conserving the commutation relations.

Steps:

(i) (Quantum Hamiltonian): Compute $H = \Delta(C)$.

(ii) (Symmetries): $S = \Delta(X)$ with $X \in g$

$$[H, S] = [\Delta(C), \Delta(X)] = \Delta([C, X]) = \Delta(0) = 0.$$ 

(iii) (Markov generator): Apply a “positive ground state transformation” to turn $H$ into a Markov generator $L$. 
Quantum $\mathfrak{su}_q(1, 1)$ algebra
\textit{q-numbers}

For $q \in (0, 1)$ and $n \in \mathbb{N}_0$ introduce the \textit{q}-number

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Remark: $\lim_{q \to 1} [n]_q = n$.

The first \textit{q}-number’s are:

$[0]_q = 0, \quad [1]_q = 1, \quad [2]_q = q + q^{-1}, \quad [3]_q = q^2 + 1 + q^{-2}, \ldots$
Quantum Lie algebra $\mathfrak{su}_q(1, 1)$

For $q \in (0, 1)$ consider the generators $K^+, K^-, K^0$ with

$$[K^0, K^\pm] = \pm K^\pm, \quad [K^+, K^-] = -[2K^0]_q$$

where

$$[2K^0]_q := \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

Irreducible representations are infinite dimensional. E.g., for $n \in \mathbb{N}$

$$\begin{cases}
K^+ e^{(n)} &= \sqrt{[n + 2k]_q[n + 1]_q} \ e^{(n+1)} \\
K^- e^{(n)} &= \sqrt{[n]_q[n + 2k - 1]_q} \ e^{(n-1)} \\
K^0 e^{(n)} &= (n + k) \ e^{(n)}
\end{cases}$$

Casimir element

$$C = [K^0]_q [K^0 - 1]_q - K^+ K^-$$

In this representation

$$C \ e^{(n)} = [k]_q [k - 1]_q \ e^{(n)} \quad k \in \mathbb{R}_+$$
Co-product

A co-product $\Delta : U_q(\mathfrak{su}(1, 1)) \to U_q(\mathfrak{su}(1, 1)) \otimes^2$

\[
\Delta(K^\pm) = K^\pm \otimes q^{-K^0} + q^{K^0} \otimes K^\pm
\]
\[
\Delta(K^0) = K^0 \otimes 1 + 1 \otimes K^0
\]

and it extends via

\[
\Delta[A \cdot B] = \Delta[A] \cdot \Delta[B] \quad \Delta[A + B] = \Delta[A] + \Delta[B]
\]

The co-product conserves the commutation relations:

\[
[\Delta(K^0), \Delta(K^\pm)] = \pm \Delta(K^\pm) \quad [\Delta(K^+), \Delta(K^-)] = [2\Delta(K^0)]_q
\]

Iteratively $\Delta^n : U_q(\mathfrak{su}(1, 1)) \to U_q(\mathfrak{su}(1, 1)) \otimes^{(n+1)}$, i.e. for $n \geq 2$

\[
\Delta^n(K^\pm) = \Delta^{n-1}(K^\pm) \otimes q^{-K^0_{n+1}} + q^{\Delta^{n-1}(K^0)} \otimes K^\pm_{n+1}
\]
\[
\Delta^n(K^0) = \Delta^{n-1}(K^0) \otimes 1 + 1 \otimes^n \otimes K^0_{n+1}
\]
Quantum Hamiltonian

\[ \Delta(C_i) = q^{K_0^i} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_0^{i+1}} \]
Quantum Hamiltonian

\[
\Delta(C_i) = q^{K_i^0} \left\{ K_i^+ \otimes K_{i+1}^- + K_i^- \otimes K_{i+1}^+ - B_i \otimes B_{i+1} \right\} q^{-K_{i+1}^0}
\]

out-of-diagonal: \( \geq 0 \)

\[
B_i \otimes B_{i+1} = \frac{(q^k + q^{-k})(q^{k-1} + q^{-(k-1)})}{2(q - q^{-1})^2} \left( q^{K_i^0} - q^{-K_i^0} \right) \otimes \left( q^{K_{i+1}^0} - q^{-K_{i+1}^0} \right) + \frac{(q^k - q^{-k})(q^{k-1} - q^{-(k-1)})}{2(q - q^{-1})^2} \left( q^{K_i^0} + q^{-K_i^0} \right) \otimes \left( q^{K_{i+1}^0} + q^{-K_{i+1}^0} \right)
\]
Quantum Hamiltonian

\[ \Delta(C_i) = q^{K^0_i} \left\{ K^+_i \otimes K^-_{i+1} + K^-_i \otimes K^+_{i+1} - B_i \otimes B_{i+1} \right\} q^{-K^0_{i+1}} \]

\[ H^{(L)} := \sum_{i=1}^{L-1} \left( 1^{\otimes(i-1)} \otimes \Delta(C_i) \otimes 1^{\otimes(L-i-1)} + c_{q,k} 1^{\otimes L} \right) \]

\[ c_{q,k} = \frac{(q^{2k} - q^{-2k})(q^{2k-1} - q^{-(2k-1)})}{(q - q^{-1})^2} \]

s.t. \[ H \cdot (\otimes_{i=1}^{L} e^{(0)}_i) = 0 \]
Symmetries

Lemma
Let \( a \in \{+, -, 0\} \), then \( K^a = \Delta^{L-1}(K^a_1) \) are symmetries:

\[
[H^{(L)}, K^a] = 0
\]

Explicitly

\[
K^\pm := \sum_{i=1}^{L} q^{K^0_1} \otimes \cdots \otimes q^{K^0_{i-1}} \otimes K^\pm_i \otimes q^{-K^0_{i+1}} \otimes \cdots \otimes q^{-K^0_L}
\]
\[
K^0 := \sum_{i=1}^{L} 1 \otimes \cdots \otimes 1 \otimes K^0_i \otimes 1 \otimes \cdots \otimes 1
\]

Proof:
For \( n = 2 \) : \( [H^{(2)}, K^a] = [\Delta(C_1), \Delta(K^a_1)] = \Delta([C_1, K^a_1]) = \Delta(0) = 0 \).
For \( n > 2 \) : induction.
Markov processes with $su_q(1, 1)$ symmetry
Lemma
Let $H$ be a matrix with $H(\eta, \eta') \geq 0$ if $\eta \neq \eta'$.
Suppose $g$ is a positive ground state, i.e. $Hg = 0$ and $g(\eta) > 0$.
Let $G$ be the matrix $G(\eta, \eta') = g(\eta)\delta(\eta, \eta')$. Then

$$L = G^{-1}H \cdot G$$

is a Markov generator.

Indeed

$$L(\eta, \eta') = \frac{H(\eta, \eta')g(\eta')}{g(\eta)}$$

Therefore

$$L(\eta, \eta') \geq 0 \quad \text{if} \quad \eta \neq \eta' \quad \quad \sum_{\eta'} L(\eta, \eta') = 0$$
Exponential symmetries

- \( g = \bigotimes_{i=1}^{L} e_{i}^{(0)} \) is a ground state, i.e. \( Hg = 0 \).
- For every symmetry \([H, S] = 0\) another ground state is \( g_{S} := Sg \).
- The exponential symmetry

\[
S^{+} = \exp_{q^{2}}(E) = \sum_{n \geq 0} \frac{(E)^{n}}{[n]q^{!}} q^{-n(n-1)/2}
\]

with

\[
E = \Delta^{(L-1)}(q^{K_{0}^{0}}) \cdot \Delta^{(L-1)}(K_{1}^{+})
\]

gives a positive ground state

\[
g_{S^{+}} := S^{+} g = \sum_{\ell_{1}, \ldots, \ell_{L} \geq 0} \bigotimes_{i=1}^{L} \left( \sqrt{\left( \frac{\ell_{i} + 2k - 1}{\ell_{i}} \right)} q^{\ell_{i}(1-k+2ki)} \right) e^{(\ell_{i})}
\]
Asymmetric Inclusion Process: ASIP(q,k)

For \( k \in \mathbb{R}_+ \) the interacting particle system \( \text{ASIP}(q, k) \) on \([1, L] \cap \mathbb{Z}\) with state space \( \{0, 1, \ldots\}^L \) is defined by

\[
(L^{\text{ASIP}(q,k)}f)(\eta) = \sum_{i=1}^{L-1} (L_{i,i+1} f)(\eta)
\]

with

\[
(L_{i,i+1} f)(\eta) = q^{\eta_i-\eta_{i+1}+(2k-1)[\eta_i]}q[2k+\eta_{i+1}]q(f(\eta^{i,i+1}) - f(\eta)) + q^{\eta_i-\eta_{i+1}-(2k-1)[2k+\eta_i]}q[\eta_{i+1}]q(f(\eta^{i+1,i}) - f(\eta))
\]

- \( q = 1 \rightarrow \text{SIP}(k) \): symmetric inclusion
  - jump right at rate \( \eta_i(2k + \eta_{i+1}) \), jump left at rate \( (2k + \eta_i)\eta_{i+1} \)
Properties of ASIP\((q,k)\)

- The \(ASIP(q, k)\) on \([1, L] \cap \mathbb{Z}\) has a family (labeled by \(\alpha > 0\)) of inhomogeneous reversible product measures with marginals

\[
P_\alpha(\eta_i = x) = \frac{\alpha^x}{Z_{i,\alpha}} \binom{x + 2k - 1}{x} \cdot q^{4kix}
\]

- \(q = 1\): the reversible measure is homogeneous and product of Negative Binomials \((2k, \alpha)\)
(2) Asymmetric Brownian Energy Process: ABEP($\sigma$, $k$)

For $\sigma > 0$, let $(\eta^{(\epsilon)}(t))_{t \geq 0}$ be the ASIP($1 - \epsilon \sigma$, $k$) process initialized with $\epsilon^{-1}$ particles. The scaling limit (weak asymmetry)

$$z_i(t) := \lim_{\epsilon \to 0} \epsilon \eta_i^{(\epsilon)}(t)$$

is the diffusion ABEP($\sigma$, $k$) with generator $L^{ABEP(\sigma,k)} = \sum_{i=1}^{L-1} \mathcal{L}_{i,i+1}$

$$\mathcal{L}_{i,i+1} = -\frac{1}{2\sigma}\left\{ (1 - e^{-2\sigma z_i})(e^{2\sigma z_{i+1}} - 1) + 2k (2 - e^{-2\sigma z_i} - e^{2\sigma z_{i+1}}) \right\} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)$$

$$+ \frac{1}{4\sigma^2} (1 - e^{-2\sigma z_i})(e^{2\sigma z_{i+1}} - 1) \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2$$
Properties of ABEP($\sigma, k$)

$\sigma \to 0^+$

$$\mathcal{L}_{i,i+1} = -2k (z_i - z_{i+1}) \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right) + z_i z_{i+1} \left( \frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_{i+1}} \right)^2$$

The reversible measures are given by product of i.i.d. $\text{Gamma}(2k; \gamma)$.

$\sigma \neq 0$

the process is truly asymmetric, i.e. on the 1-d torus it has a non-zero current.

On $\mathbb{Z}_+$ it has inhomogeneous reversible product measures (labeled by $\gamma > -4\sigma k$) with marginal density

$$\mu(z_i) = \frac{1}{Z_{i,\gamma}} (1 - e^{-2\sigma z_i})^{(2k-1)} e^{-(4\sigma ki + \gamma) z_i}$$
(3) KMP\((k)\) process

Instantaneous thermalization limit:

\[
L_{i,j}^{\text{KMP}(k)} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{tL_{i,j}^{\text{BEP}(k)}} - 1 \right) f(z_i, z_j)
\]

\[
= \int_0^1 dp \nu^{(k)}(p) [f(p(z_i + z_j), (1 - p)(z_i + z_j)) - f(z_i, z_j)]
\]

\[Z_i, Z_j \sim \text{Gamma} \left( 2k, \theta \right) \quad \text{i.i.d.} \quad \Rightarrow \quad P = \frac{Z_i}{Z_i + Z_j} \sim \text{Beta} \left( 2k, 2k \right)\]

\[
\nu^{(k)}(p) = \frac{p^{2k-1}(1 - p)^{2k-1}}{B(2k, 2k)}
\]

For \(k = \frac{1}{2}\): uniform redistribution, original KMP
AKMP($\sigma, k$) process

1. **AKMP($\sigma, k$)**

\[
L_{i,j}^{AKMP(\sigma,k)} f(z_i, z_j) := \lim_{t \to \infty} \left( e^{tL_{i,j}^{ABEP(\sigma,k)}} - 1 \right) f(z_i, z_j)
\]

\[
= \int_0^1 dp \, \nu^{(k)}_\sigma (p|z_i + z_j) \left[ f(p(z_i + z_j), (1 - p)(z_i + z_j)) - f(z_i, z_j) \right]
\]

with

\[
\nu^{(k)}_\sigma (p|E) = \frac{1}{Z_{\sigma,k}(E)} e^{2\sigma pE} \left\{ (e^{2\sigma pE} - 1) \left( 1 - e^{-2\sigma(1-p)E} \right) \right\}^{2k-1}
\]

2. **Th-ASIP($q, k$)**

\[
(n, m) \to (R_q, n + m - R_q)
\]

with $R_q$ a q-deformed Beta-Binomial $(n + m, 2k, 2k)$
Duality relations
Self-duality of ASIP\((q, k)\)

Theorem [Carinci,G., Redig, Sasamoto (2015)]

The ASIP\((q, k)\) is self-dual on

\[
D(\eta, \xi^{(\ell_1, \ldots, \ell_n)}) = \frac{q^{-4k} \sum_{m=1}^{n} \ell_m - n^2}{(q^{2k} - q^{-2k})^n} \cdot \prod_{m=1}^{n} (q^{2N_{\ell_m}(\eta)} - q^{2N_{\ell_{m+1}}(\eta)})
\]

where \(\xi^{(\ell_1, \ldots, \ell_n)}\) is the configuration with \(n\) particles at sites \(\ell_1, \ldots, \ell_n\) and

\[
N_i(\eta) := \sum_{k=i}^{L} \eta_k
\]

It follows from the explicit knowledge of the reversible measure and from the exponential symmetry \(S_+\)
Duality between \textit{ABEP}(\(\sigma, k\)) and \textit{SIP}(\(k\))

Theorem [Carinci,G., Redig, Sasamoto (2015)]

- For every \(\sigma\) (including \(0^+\), the process \(\{z(t)\}_{t \geq 0}\) with generator \(L^{\text{ABEP}(\sigma, k)}\) and the process \(\{\eta(t)\}_{t \geq 0}\) with generator \(L^{\text{SIP}(k)}\) are dual on

\[
D(z, \xi) = \prod_{i=1}^{L} \frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} \left( \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma} \right)^{\xi_i}
\]

with

\[
E_i(z) = \sum_{l=i}^{L} z_l \quad \quad E_{L+1}(z) = 0
\]

- Same duality holds between \textit{AKMP}(\(\sigma, k\)) and Th-\textit{SIP}(\(k\))
symmetric case $\sigma = 0^+$

\[
L = \sum_{i=1}^{L-1} \left( K_{i+1}^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + 2k^2 \right)
\]

Two representations:

\[
\begin{align*}
K_i^+ \varepsilon^{(\eta_i)} &= (\eta_i + 2k) \varepsilon^{(\eta_i + 1)} \\
K_i^- \varepsilon^{(\eta_i)} &= \eta_i \varepsilon^{(\eta_i - 1)} \\
K_i^o \varepsilon^{(\eta_i)} &= (\eta_i + 4k) \varepsilon^{(\eta_i)}
\end{align*}
\]

\[
\begin{align*}
\mathcal{K}_i^+ &= z_i \\
\mathcal{K}_i^- &= z_i \partial_{z_i}^2 + 2k \partial_{z_i} \\
\mathcal{K}_i^o &= z_i \partial_{z_i} + k
\end{align*}
\]

\[
L = L^{SIP(k)} \quad \quad \quad \quad \quad \quad L = L^{BEP(k)}
\]

\[
\frac{\Gamma(2k)}{\Gamma(2k + \xi_i)} z_i^\xi_i
\]

Duality fct $\equiv$ intertwiner
asymmetric case $\sigma \neq 0$

- The $ABEP(\sigma, k)$ can be mapped to $BEP(k)$ via the non-local transformation

$$z \mapsto g(z) \quad g_i(z) := \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{2\sigma}$$

Equivalently

$$L^{ABEP(\sigma,k)} = C_g \circ L^{BEP(k)} \circ C_{g^{-1}}$$

with

$$(C_g f)(z) = (f \circ g)(z)$$

- Therefore, despite the asymmetry, the symmetry group of $ABEP(\sigma, k)$ is the same as for $BEP(k)$, namely $su(1, 1)$. The representation is a non-local conjugation of the differential operator representation.
Applications
Example 1: Current of ABEP($\sigma, k$)

Definition

The current $J_i(t)$ during the time interval $[0, t]$ across the bond $(i - 1, i)$ is defined as:

$$J_i(t) = E_i(z(t)) - E_i(z(0))$$

where

$$E_i(z) := \sum_{k \geq i} z_k$$

Remark: let $\xi(i)$ be the configuration with 1 dual particle:

$$\xi^{(i)}_m = \begin{cases} 
1 & \text{if } m = i \\
0 & \text{otherwise}
\end{cases}$$

then

$$D(z, \xi^{(i)}) = \frac{e^{-2\sigma E_{i+1}(z)} - e^{-2\sigma E_i(z)}}{4k\sigma}$$
Example 1: Current of $\text{ABEP}(\sigma, k)$

Using duality between $\text{ABEP}(\sigma, k)$ and $\text{SIP}(k)$

$$\mathbb{E}_z(e^{-2\sigma J_i(t)}) = e^{-4kt} \sum_{n \in \mathbb{Z}} e^{-2\sigma (E_n(z) - E_i(z))} I_{|n-i|}(4kt)$$

$I_{n}(t)$ modified Bessel function.

The computation requires a single dual SIP particle, which is a simple symmetric random walk jumping at rate $2k$:

$$\mathbb{P}_i(X_t = n) = e^{-4kt} I_{|n-i|}(4kt).$$
Brownian Momentum Process with reservoirs

\[ L_{LM}^{res} = T_L \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \]

\[ L_{BMP} = \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1} \]

\[ L_{LR}^{res} = T_R \frac{\partial^2}{\partial x_N^2} - x_N \frac{\partial}{\partial x_N} \]
Example 2: Energy covariance in the boundary driven BEP

If $\vec{\xi} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \Rightarrow D(x, \vec{\xi}) = x_i^2 x_j^2$

site $i \uparrow$ site $j \uparrow$

In the dual process we initialize two SIP walkers $(X_t, Y_t)_{t \geq 0}$ with $(X_0, Y_0) = (i, j)$
Inclusion Process with absorbing reservoirs

\[ L_{abs}^1 f(\xi) = 2\xi_1 \left( f(\xi^{1,0}) - f(\xi) \right) \]
\[ E(x_i^2 x_j^2) = T_L^2 P(\bullet) + T_R^2 P(\bullet) + T_L T_R (P(\bullet) + P(\bullet)) \]
Example 2: Energy covariance in the boundary driven BEP

\[ \mathbb{E}\left(x_i^2 x_j^2\right) - \mathbb{E}\left(x_i^2\right) \mathbb{E}\left(x_j^2\right) = \frac{2i(N + 1 - j)}{(N + 3)(N + 1)^2}(T_R - T_L)^2 \]

Remark: Long range correlations

\[ \lim_{N \to \infty} N \text{Cov}(x_{s_1 N}^2, x_{s_2 N}^2) = 2s_1(1 - s_2)(T_R - T_L)^2 \]
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