KPZ 1:2:3 scaling $h_\epsilon(t, x) = \epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x)$

$$\partial_t h_\epsilon = (\partial_x h_\epsilon)^2 + \epsilon^{1/2} \partial_x^2 h_\epsilon + \epsilon^{1/4} \xi$$

All KPZ class models have analogue of $h$ and 1:2:3 scaling

Goal to understand the limit $\hat{h}$ of $h_\epsilon$ as $\epsilon \to 0$ KPZ fixed point

Or $u = \partial_x h$, $\hat{u} = \partial_x \hat{h}$ stochastic Burgers

Philosophy: Understand KPZ universality by showing it is stable fixed point of 1:2:3 scaling.

But we don’t even know what it is! What do we know?

- Depends on only one free parameter (= 1 by rescaling time)
- Depends on the initial data: Markov process
- Local dynamics
- KPZ 1:2:3 scaling invariance
- $u$ weak solution of Burgers equation $\partial_t u = \partial_x u^2$. Not unique.
  
  Dissipative limit $\partial_t u = \partial_x u^2 + \epsilon \partial_x^2 u \neq$ dispersive limit $\partial_t u = \partial_x u^2 + \epsilon \partial_x^3 u$ (Lax, Levermore, Venakidis, Deift, Zhou, …)
Conjectural KPZ Universality Class

Markov processes $\rightarrow$ fluctuations depend on initial data
Special initial data determined by $\epsilon^{1/2} h_0(\epsilon^{-1} x)$ invariance
Fractional Burgers equation

Goncalves-Jara conjecture the KPZ fixed point

\[ \partial_t u = \partial_x (u^2) - (-\Delta)^{3/4} u dt + \sqrt{(-\Delta)^{3/4}} \xi \]

status of the equation is that Gubinelli-Jara can prove “energy solutions” exist, preserve white noise, but not that they are unique

✓ Invariant under 1:2:3 scaling

X Nonlocal Ornstein-Uhlenbeck \[-(\Delta)^{3/4} u dt + \sqrt{(-\Delta)^{3/4}} \xi\]
looks rather unnatural

X more than one free parameter

\[ \partial_t u = \partial_x (u^2) - \delta(-\Delta)^{3/4} u dt + \sqrt{\delta(-\Delta)^{3/4}} \xi \]

Conclusion: one parameter family of non-local perturbations off KPZ fixed point

Conjecture: converges to KPZ fixed point as \( \delta \searrow 0 \)
What we do know are Airy process limits from special scale invariant initial data

**Point-to-point** directed random polymers, random growth on curved substrate, stochastic heat eq (KPZ) starting from delta function $Z(0, x) = \delta_0(x)$, etc

$$h(t, x) \sim c_1 t + c_2 \frac{x^2}{t} - c_3 t^{1/3} A(c_4 t^{-2/3} x)$$

Airy process stationary, marginals $F_{\text{GUE}}$

$t \to \infty$ \quad $c_i$ non-universal, in some cases computable

**Point-to-line** directed random polymers, random growth on flat substrate, stochastic heat eq (KPZ) starting from a constant $Z(0, x) \equiv 1$, etc

$$h(t, x) \sim c_5 t - c_6 t^{1/3} A_1(c_7 t^{-2/3} x)$$

Airy$_1$ process stationary, marginals $F_{\text{GOE}}$

"**Stationary**" case e.g. stochastic Burgers equation with invariant white noise, KPZ with Brownian motion $A_{\text{stat}}(x)$

**Mixed cases** start with one of the above to the right of the origin, and another to the left. $A_{1 \to 2}(x)$, $A_{1 \to \text{stat}}(x)$, $A_{2 \to \text{stat}}(x)$

6(+) universality subclasses $\iff$ 6(+) Airy processes, determinental f.d.d.'s

Prahofer, Spohn, Sasamoto, Borodin, Baik, Johansson, ... 00's
Airy process

$\mathcal{A}(\cdot)$ is the most basic. Here is a new formula for it (Q-Remenik, in prep) $g(t) \geq c - \kappa t^2$, $\kappa < 3/4$, $\int_{-L}^{L} |g'|^2 < \infty$ for all $L$

$$\mathbb{P}(\mathcal{A}(t) \leq g(t) + t^2, \ t \in \mathbb{R}) = \det \left( I - K_{Ai} + B_0 P_0 \mathcal{S} P_0 B_0 \right)$$

$$S(x_1, x_2) = \int_{-\infty}^{g(0)} dy G_g(x_1, y, g) G_g(-t)(y, x_2)$$

$$G_g(x, y) = B_0(x_1, y) - \int_{0}^{\infty} \mathbb{P}_y(\tau_g \in dt)(e^{-t\Delta} B_0)(x, g(t))$$

$$B_0(x, y) = \text{Ai}(x + y), \ \tau_g = \inf\{t \geq 0: \sqrt{2}B_1(t) \geq g(t)\}$$

Our old formula with Ivan was only for $\mathbb{P}(\mathcal{A}(t) \leq g(t) + t^2, \ t \in [-L, L])$ and then one had to take a (highly singular) limit as $L \nearrow \infty$
KPZ with more general initial data $h_0(x)$

### E.g. Initial data $h_0$ for KPZ

By linearity of stochastic heat eqn

$$h(t, x) \sim \log \int e^{-(x-y)^2/2t + t^{1/3}A(t^{-2/3}(x-y)) - h^0(y)} dy$$

$$h_\epsilon(t, x) \sim \epsilon^{1/2} \log \int e^{\epsilon^{-1/2}(-(x-y)^2/2t + t^{1/3}A(t^{-2/3}(x-y)) - \epsilon^{1/2}h^0(\epsilon^{-1}y))} dy$$

so if $\epsilon^{-1/2} h_0(\epsilon^{-1}y) \to h_0(y)$

$$h(t, x) = \sup_y \left\{ -(x-y)^2/2t + t^{1/3}A(t^{-2/3}(x-y)) - h^0(y) \right\}$$

- **only in sense of 1 − d distributions**
- limit class only depends on $h_0(x) = \lim_{\epsilon \to 0} \epsilon^{-1/2} h_0(\epsilon^{-1}x)$
  (can prove this using Corwin-Hammond),
- eg. $h_0(x) = -|x|^{1/2+\delta}$ gives $F_{\text{GUE}}$, whereas $h_0(x) = c|x|^{1/2-\delta}$ gives flat. Boundary between curved and flat is $h_0(x) = c|x|^{1/2}$, and it has an exact formula.
Exact formula for the square root class [Q-Remenik, in prep]

\[
\mathbb{P}\left( \sup_{x \in \mathbb{R}} \left\{ A(x) - x^2 - \alpha_1 |x - a|^{1/2} \mathbf{1}_{x < a} - \alpha_2 |x - a|^{1/2} \mathbf{1}_{x \geq a} \right\} \leq r \right) = \det \left( I - K_{A_1} + B_0 P_0 S_{\infty}^{sq,a,\alpha_1,\alpha_2} P_0 B_0 \right)
\]

\[
S_{\infty}^{sq,a,\alpha_1,\alpha_2}(x_1, x_2) = \int_{-\infty}^{r} dy \left[ (e^{a \Delta B_0})(x_1, y) - \int_{0}^{\infty} dt_1 \rho_{r-y,\alpha_1}(t_1)(e^{(a-t_1)\Delta B_0})(x_1, \alpha_1 \sqrt{t_1}) \right] \times \left[ (e^{-a \Delta B_0})(y, x_2) - \int_{0}^{\infty} dt_2 \rho_{r-y,\alpha_2}(t_2)(e^{-(a+t_2)\Delta B_0})(\alpha_2 \sqrt{t_2} + r, x_2) \right].
\]

\[
\rho_{r-y,\alpha}(t) = \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}(y - r)\right)^{2\nu_n(\alpha)}}{\partial_{\nu} \Upsilon(\nu, \frac{\alpha^2}{4})}_{\nu=\nu_n(\alpha)} t^{\nu_n(\alpha)-1},
\]

where \((\nu_n(\alpha))_{n \geq 1}\) are the negative zeros of the function \(\Upsilon(\cdot, \frac{\alpha^2}{4}).\)

\[
\Upsilon(\nu, \frac{z^2}{2}) = 2^\nu e^{z^2/4} D_{-2\nu}(z)
\]

\(D_\nu(z)\) is the parabolic cylinder function, \(B_0(x, y) = \text{Ai}(x + y).\)
Airy sheet

For fixed point at time $t$ as a process in $x$ need extra parameter

$$Z(0, y, t, x) \sim c_t e^{-\frac{(x-y)^2}{2t}} + t^{1/3} A(t^{-2/3}x, t^{-2/3}y)$$

$$h(t, x) = \sup_y \left\{ -\frac{(x-y)^2}{2t} + t^{1/3} A(t^{-2/3}x, t^{-2/3}y) - h^0(y) \right\}$$

Lax-Oleinik formula for Burgers eq driven by Airy noise

Space-time Airy sheet

$$A(s, y; t, x) = \lim_{\epsilon \to 0} \epsilon^{1/2} \log Z(\epsilon^{-3/2} s, \epsilon^{-1} y, \epsilon^{-3/2} t, \epsilon^{-1} x)$$

1. **Independent increments.** $A(u, y; t, x)$ is independent of $A(u', y; t', x)$ if $(u, t) \cap (u', t') = \emptyset$;

2. **Space and time stationarity.**

   $$A(u, y; t, x) \overset{\text{dist}}{=} A(u + h, y; t + h, x) \overset{\text{dist}}{=} A(u, y + z; t, x + z);$$

3. **Scaling.** $A(0, y; t, x) \overset{\text{dist}}{=} t^{1/3} A(0, t^{-2/3}y; 1, t^{-2/3}x);$  

4. **Semi-group property.** $u < s < t, \hat{A}(u, y; t, x) := A(u, y; t, x) - \frac{(x-y)^2}{2(t-u)},$

$$\hat{A}(u, y; t, x) = \sup_{z \in \mathbb{R}} \{ \hat{A}(u, y; s, z) + \hat{A}(s, z; t, x) \}$$
Prelimiting Airy sheets are tight \textit{Corwin-Pimentel-Q, in prep} in

1. Hammersley LPP;
2. Exponential and geometric (lattice) LPP;
3. Brownian semi-discrete LPP;
4. Log-gamma polymer;
5. Brownian semi-discrete polymer;

Unfortunately, we have no uniqueness, so cannot even prove, eg. scaling property of the limit

\textbf{Polymer fixed point}

From $x$ at time $s$ to $y$ at time $t$ travels through $z_1, \ldots, z_n$ at times $s < s_1 < \cdots < s_n < t$ which optimize

$$\hat{A}(s, x; s_1, z_1) + \hat{A}(s_1, z_1; s_2, z_2) + \cdots + \hat{A}(s_n, z_n; t, y)$$

Conjectured universal limit of directed polymers in $1+1$ dimensions
Enough to study \( t \to \infty \) and then fine mesh limit \( \mu, \nu \to \infty \) of the generating function

\[
G(s, x; c, y) = \left< \exp \left( - e^{t/24} \left[ \sum_{k,l=1}^{\mu,\nu} e^{-s_k - c_l} Z(0, y_l, t, x_k) \right] \right) \right>.
\]

Expand exp, write it using replicas as

\[
\sum_{N=0}^{\infty} \frac{(-1)^N e^{tN/24}}{N!} \sum_r e^{-tE_r} |\psi_r(0)|^2 \Phi_r(x, s) \Phi^*_r(y, c)
\]

\[
\Phi_r(x, s) := \sum_{k_1,\ldots,k_N=1}^{\mu} e^{-(s_{k_1} + \cdots + s_{k_N})} \frac{\psi_r(x_{k_1}, \ldots, x_{k_N})}{\psi_r(0)}
\]

(1)

\( E_r, \psi_r \) are the eigenvalues, eigenfunctions of the attractive (symmetric) \( \delta \)-Bose gas \(-\frac{1}{2} \sum_{i=1}^{N} \partial^2_{x_i} - \frac{1}{2} \sum_{i \neq j=1}^{N} \delta(x_i - x_j)\)

Problem: Summation (1) only explicit in very special cases.
Eigenfunctions

\[ \psi_{q,n}(x_1, \ldots, x_N) = \]
\[ C \sum_{p \in \mathcal{P}} \text{sgn}(p) \prod_{1 \leq a < b \leq N} \left( \xi_{p(a)} - \xi_{p(b)} + i \text{sgn}(x_a - x_b) \right) e^{i \sum_{c=1}^{N} \xi_{p(c)} x_c} \]

\[ \xi_a = q_\alpha - \frac{i}{2} \left( n_\alpha + 1 - 2r_\alpha \right) \quad \text{for} \quad a = \sum_{\beta=1}^{\alpha-1} n_\beta + r_\alpha , \quad \alpha = 1, \ldots, M \]

\[ n_\alpha = \text{number of particles in the } \alpha\text{-th cluster} \]
\[ N = \sum_{\alpha=1}^{M} n_\alpha = \text{total number of particles} \]

Prolhac-Spohn factorization assumption

\[ \Phi_r(x, s) \approx \prod_{\alpha=1}^{M} e^{-\frac{1}{4} \sum_{\mu} \sum_{a, b=1}^{\mu} |x_a - x_b| \partial_s a \partial_s b} \left( \sum_{k=1}^{\mu} e^{-s_k + i q_\alpha x_k} \right)^{n_\alpha} \]
Factorization assumption

- Allows one to put generating function in form so that one can take \( t \to \infty \)
- For narrow wedge initial data get full Airy process \((\text{Prolhac-Spohn})\)
- For Brownian initial data get correct two point function of \( \mathcal{A}_{\text{stat}} \) \((\text{Imamura-Sasamoto})\)
- For narrow wedge two points Dotsenko’s summation formula,

\[
\frac{\langle x_1, \ldots, x_1, x_2, \ldots, x_2 | \psi_z \rangle}{\langle 0 | \psi_z \rangle} \delta^{\sum_{\alpha=1} M} n_{\alpha}, L + R = \\
(L + R)^{-1} \sum_{m_{\alpha} + l_{\alpha} > 0} \prod_{\alpha = 1}^M \left( \frac{n_{\alpha}}{m_{\alpha}} \right) e^{i x_1 m_{\alpha} q_{\alpha} + i x_2 l_{\alpha} q_{\alpha} - m_{\alpha} l_{\alpha} x / 2} \delta^{m_{\alpha} + l_{\alpha}, n_{\alpha}} \delta^{\sum_{\alpha=1}^M m_{\alpha}, L} \delta^{\sum_{\alpha=1}^M l_{\alpha}, R} \\
\times \prod_{\alpha \neq \beta} \frac{\Gamma[1 + \frac{1}{2}(m_{\alpha} + m_{\beta} + l_{\alpha} + l_{\beta}) + i(q_{\alpha} - q_{\beta})] \Gamma[1 + \frac{1}{2}(-m_{\alpha} + m_{\beta} + l_{\alpha} - l_{\beta}) + i(q_{\alpha} - q_{\beta})]}{\Gamma[1 + \frac{1}{2}(-m_{\alpha} + m_{\beta} + l_{\alpha} + l_{\beta}) + i(q_{\alpha} - q_{\beta})] \Gamma[1 + \frac{1}{2}(m_{\alpha} + m_{\beta} + l_{\alpha} - l_{\beta}) + i(q_{\alpha} - q_{\beta})]}
\]

Setting double product = 1 \((\text{value at steepest descent point})\)
equivalent to factorization assumption \((\text{Imamura-Sasamoto-Spohn})\)

On this evidence we proceed and obtain transition probabilities.
KPZ fixed pt transition probs from replica + factorization assumption  
(Corwin-Q-Remenik)

\[ P(h(1, x) \leq g(x) \text{ on } [a, b] \mid h(0, x) = f(x) \text{ on } [c, d]) = \det(I - K + L) \]

where \( L(z, z') \) is given by

\[
\int \int dmdu \ e^{(d-c)H} K(\partial_m Y_{[c,d]}^\tilde{f}(\cdot) + m)(z', u) Y_{[a,b]}^{\tilde{g}(\cdot) + u - m} e^{(b-a)H} K(u, z)
\]

\( Y_{[a,b]}^{\tilde{f}} \) is solution operator of
\[
\begin{cases}
\partial_t u = -Hu & a < t < b \\
u(t, x) = 0 & x \geq f(t)
\end{cases}
\]

\( H = -\partial_x^2 + x \) is the Airy operator

Hard to verify.

Test: Flat \( \mapsto \) multipoint should give \( \mathcal{A}_1 \). One pt ok but 2 point function unclear. Passes some non-trivial tests, but even numerics is inconclusive