MACDONALD OPERATORS: FROM CLUSTER ALGEBRA TO ELLIPTIC HALL ALGEBRA

[PDF + Rinat Kedem][ArXiv:1505.01657]

1. Macdonald theory from Double Affine Hecke Algebra
   Torus knot invariants, Macdonald Polynomials
2. Generalized Macdonald operators and q-Whittaker limit
   Cluster Algebra, discrete integrable systems
3. Graded characters and Difference equations
   generalized q-difference Toda eigenvectors
4. (q, t) deformation and Elliptic Hall Algebra
   isomorphisms & the big picture

1.A. DOUBLE AFFINE HECKE ALGEBRA

Def: $A_r(q,t)$ - DAHA $\quad q\,,\,t \quad \theta = t^{1/2}$

Ⓐ Hecke Algebra

$$\boxed{\begin{array}{c} T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \\ (T_i-\theta)(T_i+\theta^{-1}) = 1 \end{array}} \qquad (i=1,\dots,r)$$

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$(i=1,\dots r)$

Ⓒ Y_i $i=1 \dots r+1$ commuting variables $\quad \boxed{T_i^{-1}Y_iT_i^{-1} = Y_{i+1}}$

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Ⓒ Y_i $\; i = 1 \dots r+1$ commuting variables $\quad \boxed{T_i^{-1} Y_i T_i^{-1} = Y_{i+1}}$

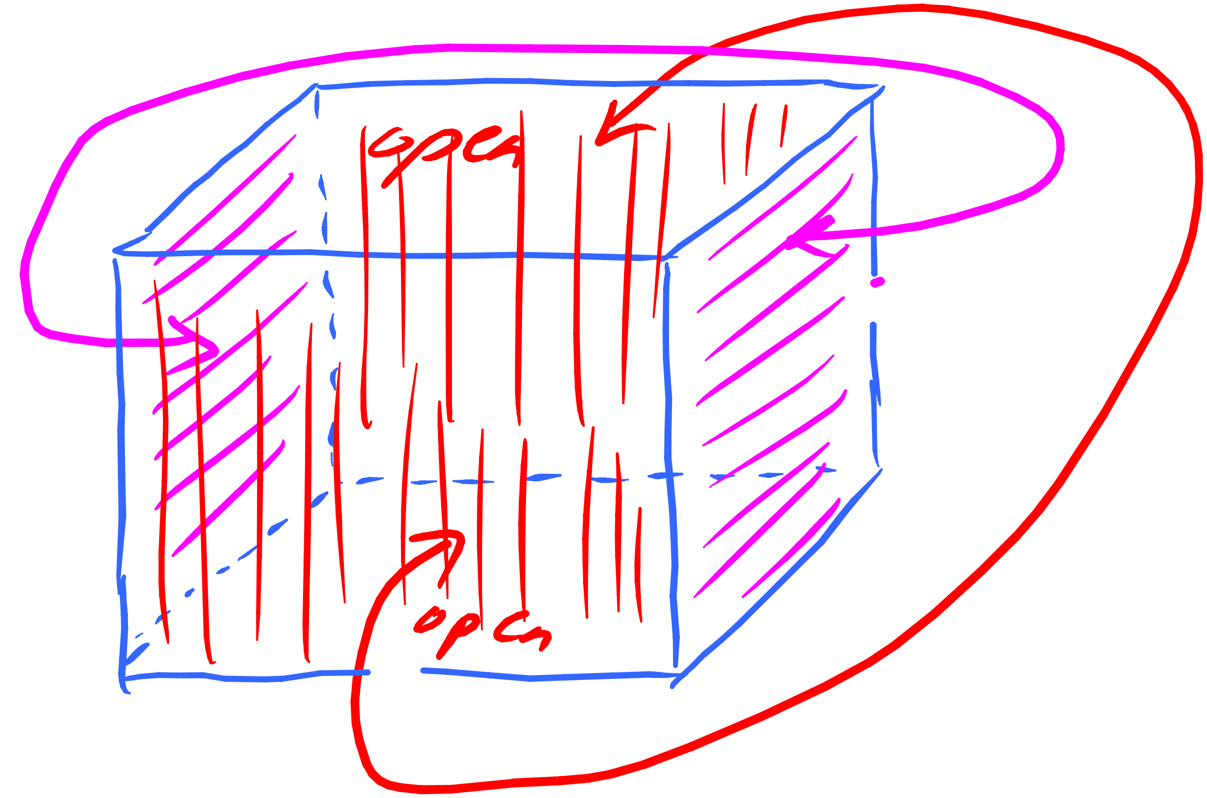
and commutations:

$$\boxed{\left\{ \begin{array}{l} T_i X_j = X_j T_i \\ T_i Y_j = Y_j T_i \end{array} \right. \quad (j \neq i, i+1)} \qquad (i = 1 \dots r)$$

$$\boxed{X_1 Y_2 = Y_2 T_1^2 X_1} \qquad \boxed{\left(\prod_1^{r+1} X_i\right) Y_j = q^{-1} Y_j \left(\prod_1^{r+1} X_i\right)} \qquad \boxed{\left(\prod_{i=1}^{r+1} Y_i\right) X_j = q X_j \left(\prod_1^{r+1} Y_i\right)}$$

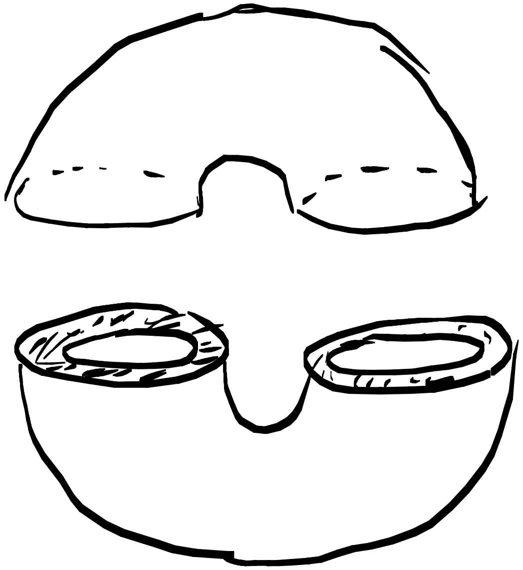
$(j = 1 \dots r+1) \qquad (j = 1 \dots r+1)$

1. B. Geometric Picture: Torus knots

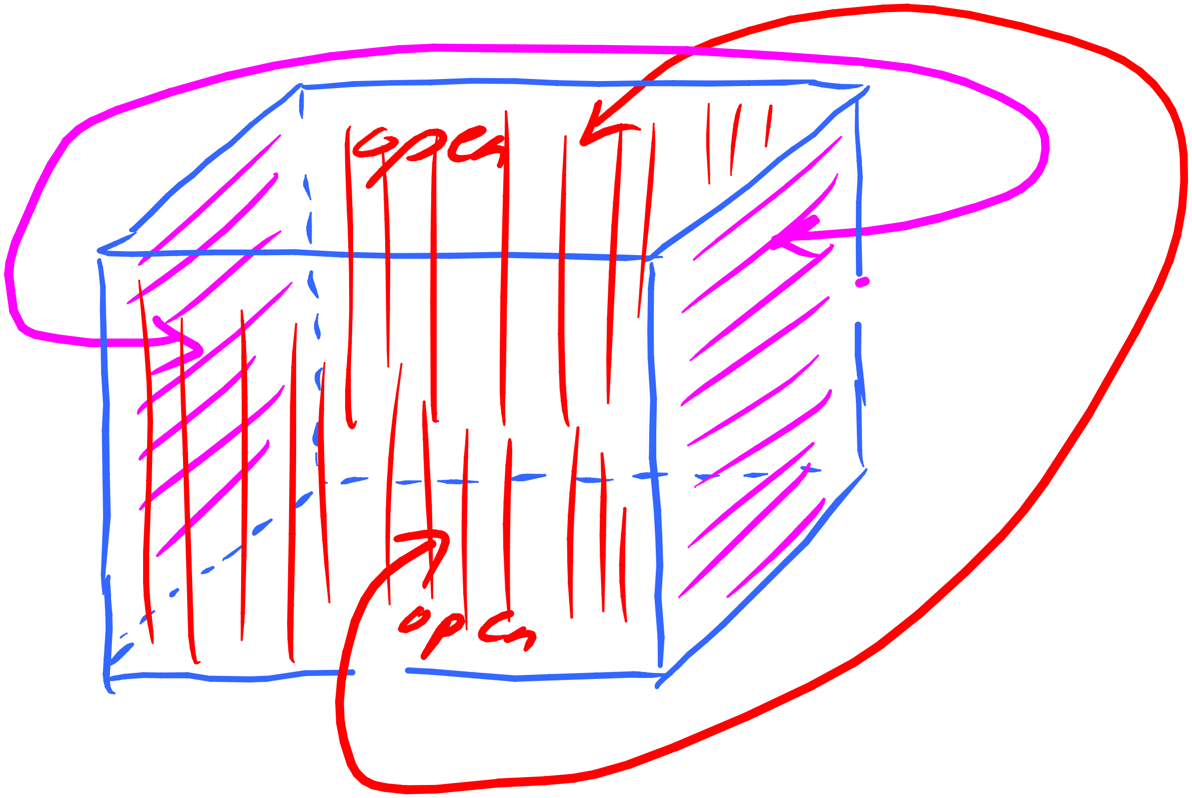


solid Torus

[Burella, Watts, Pasquier, Vala '13]



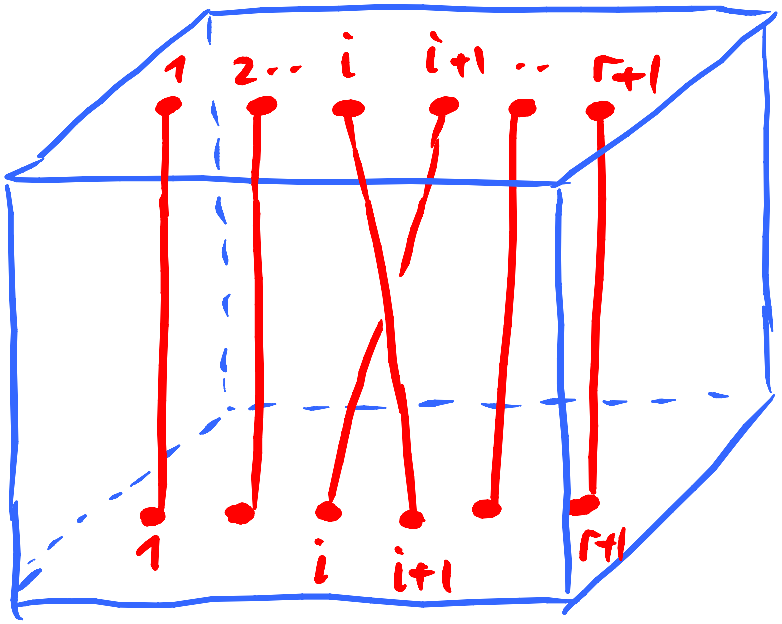
generators connect both open sides



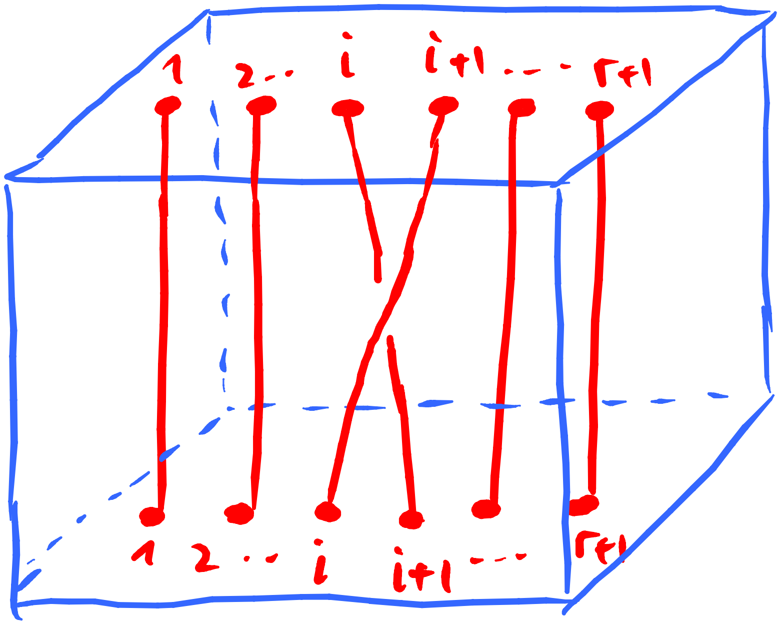
Solid Torus

[Burella, Watts, Pasquier, Vala: 13]

Product = "Matriochka" of genus 1



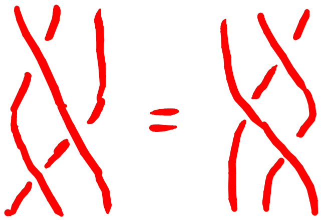
T_i

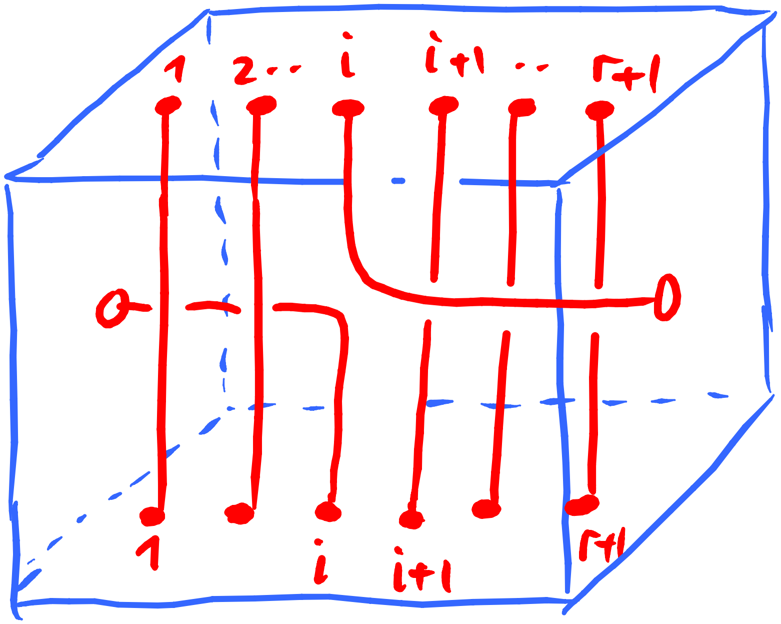


T_i^{-1}

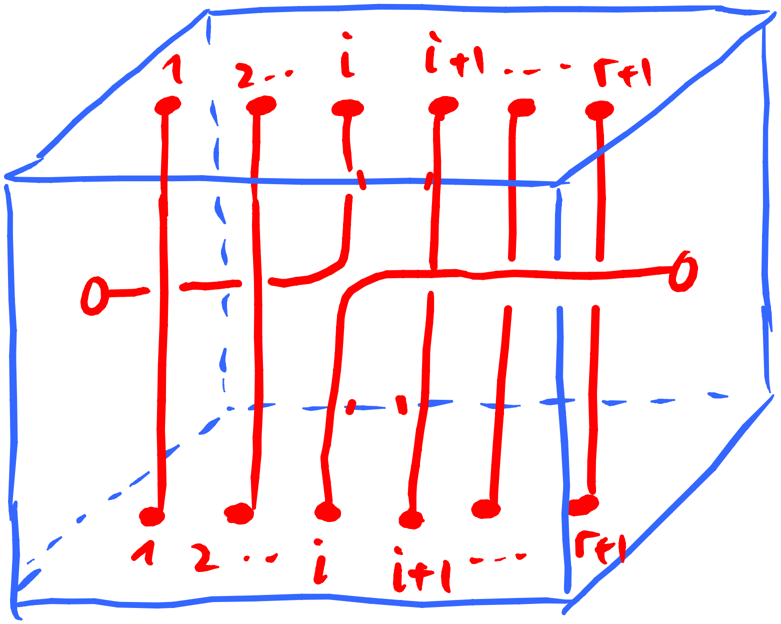
$i \in [1, r]$

- Action : top → bottom
- Braid relation = pull strings





Y_i



Y_i^{-1}

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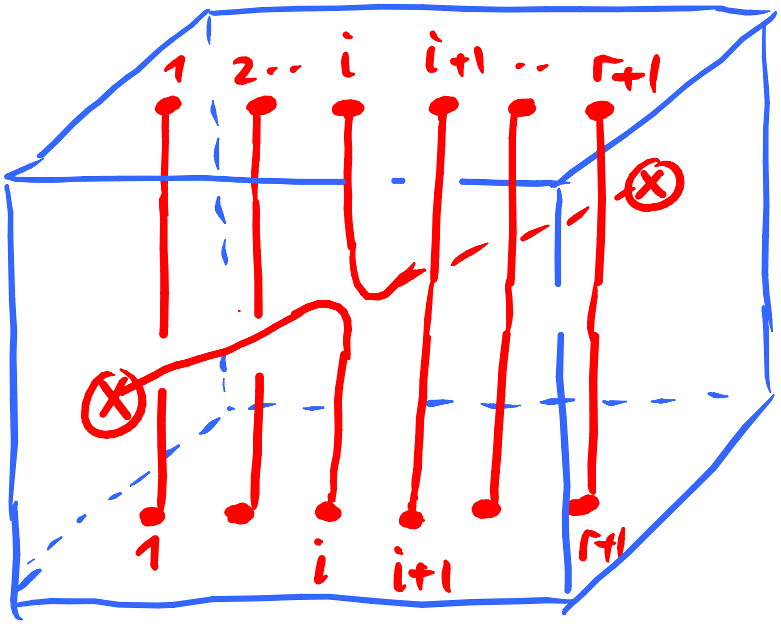
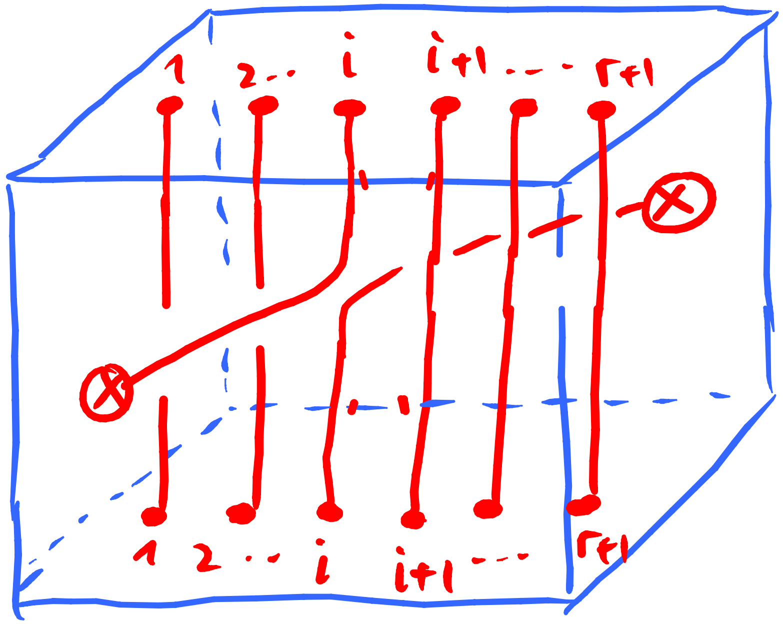


X_i



X_i^{-1}

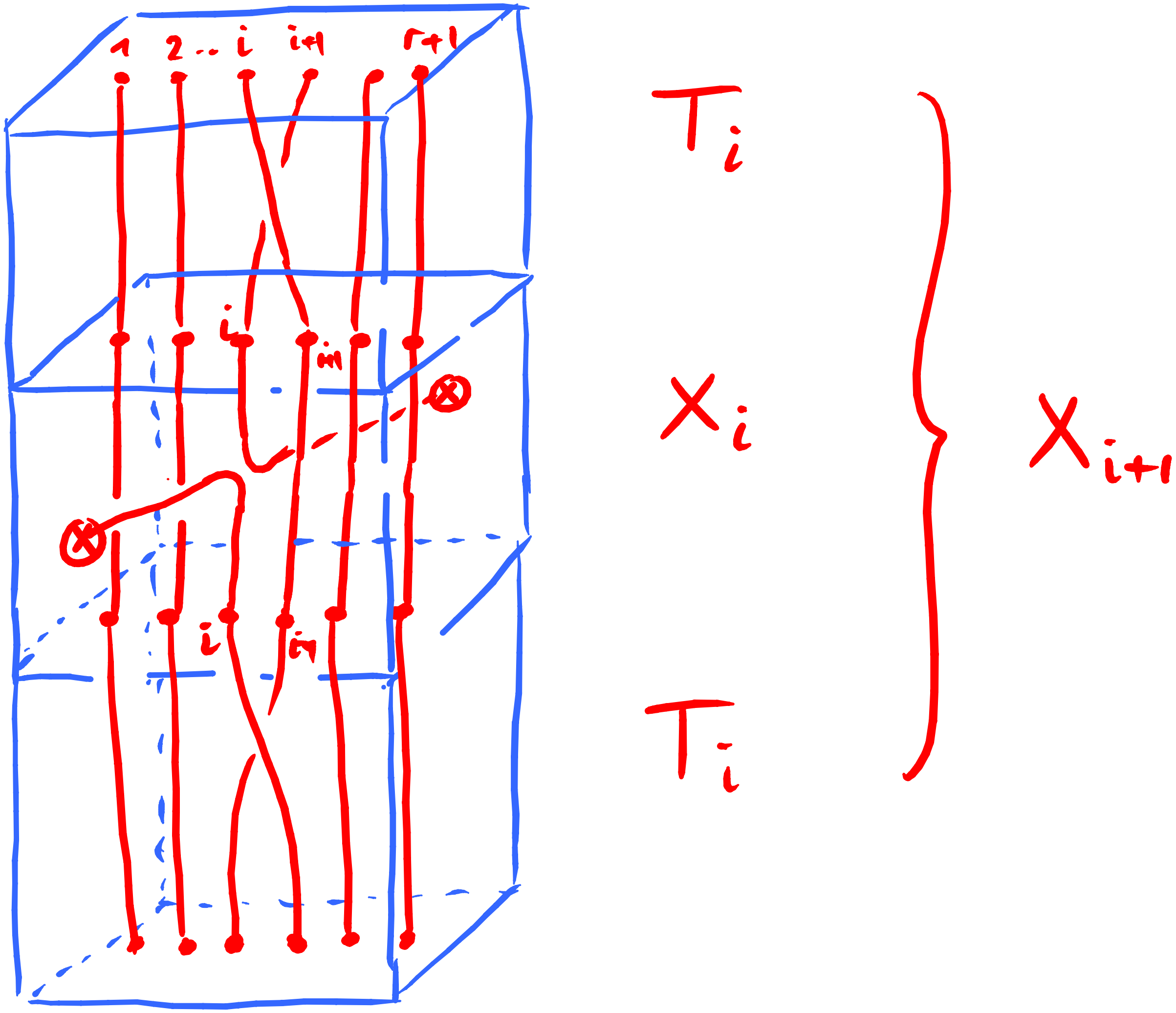
$i \in [1, r]$

and π = cyclic permutation of the points.

then: All the relations are pictorial!

Example 1

$T_i X_i T_i = X_{i+1}$

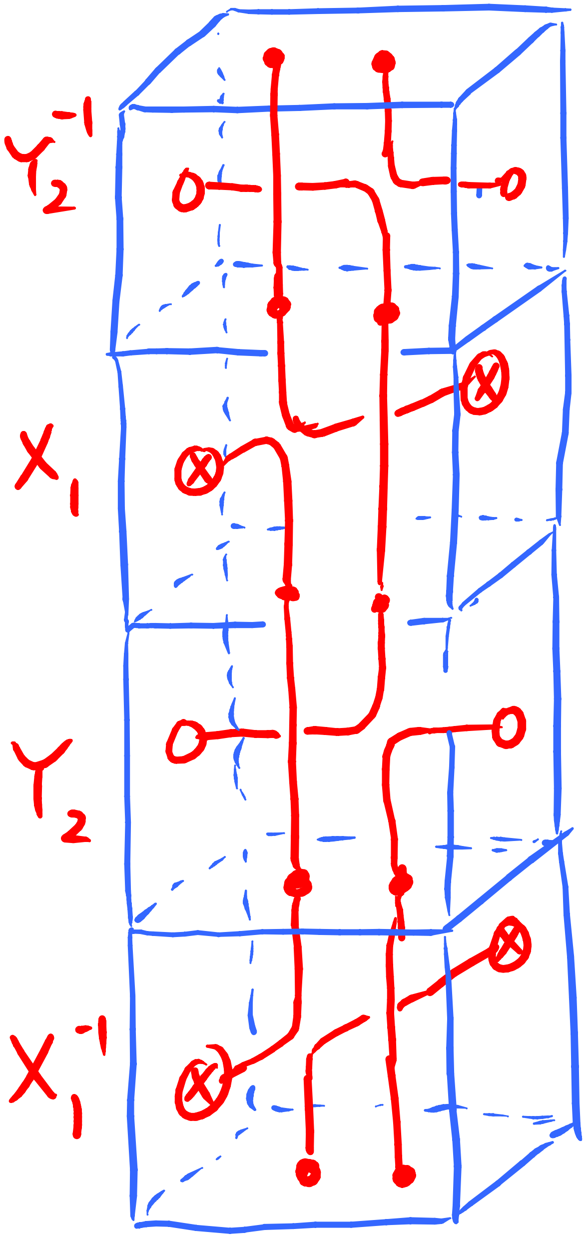


Example 2

$$Y_2 T_1^2 X_1 = X_1 Y_2$$

$$\downarrow$$

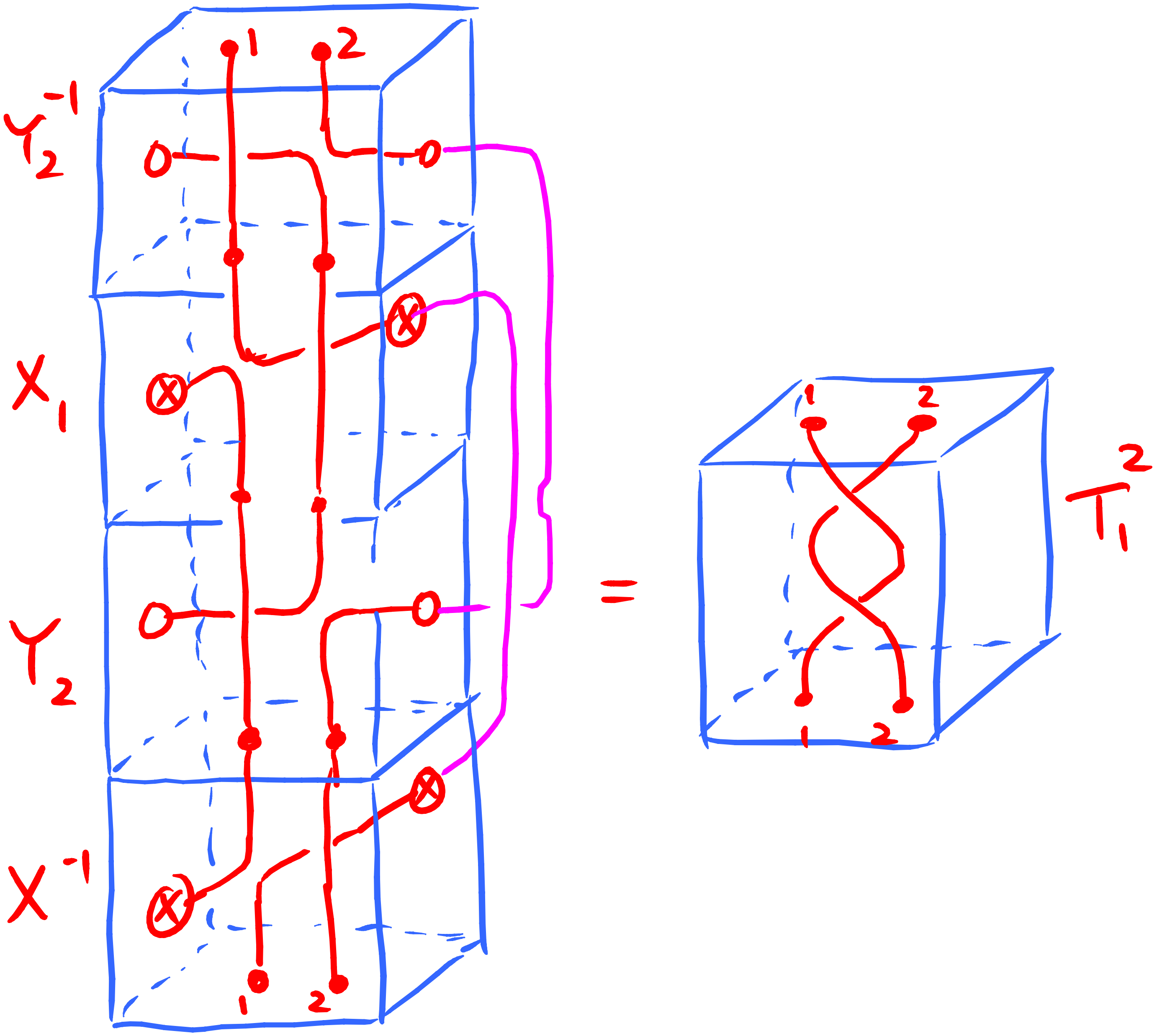
$$Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2$$



$$Y_2 T_1^2 X_1 = X_1 Y_2$$

or

$$Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2$$



1.C. MACDONALD THEORY (Cherednik)

Polynomial representation of sDAHA on $\mathbb{C}[X_1 .. X_{r+1}]^{\mathfrak{S}_{r+1}}$

- $$\boxed{\begin{cases} T_i = \theta s_i + X_{i+1}\dfrac{\theta-\theta^{-1}}{X_i - X_{i+1}}(s_i - 1) \\ T_i^{-1} = T_i - \theta + \theta^{-1} \end{cases}}$$

$s_i : X_i \leftrightarrow X_{i+1}$

$(\theta^2 = t)$

- $\pi X_i = X_{i+1} \pi$ $(i \in [1,r])$ & $\pi X_{r+1} = q^{-1} X_1 \pi$

- $Y_i = (T_i T_{i+1} \cdots T_r) \pi^{-1} (T_1^{-1} T_2^{-1} \cdots T_{i-1}^{-1})$ $i \in [1, r+1]$

- MACDONALD OPERATORS

THM

$$\boxed{\mathcal{D}_\alpha^{q,t} = e_\alpha(Y_1,\dots Y_{r+1}) = \sum_{|I|=\alpha} \prod_{\substack{i\in I\\ j\notin I}} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} D_I}$$

$$D_i f(x_1,\dots,x_i,\dots,x_{r+1}) = f(x_1,\dots,q x_i,\dots,x_{r+1})$$

$$D_I = \prod_{i\in I} D_i$$

- MACDONALD POLYNOMIALS

THM $\exists!$ $P_\lambda(x_1\dots x_{r+1})$ such that $\mathcal{D}_\alpha P_\lambda = \left(\sum_1^{r+1} q^{\lambda_i} t^{\frac{r}{2}+1-i}\right) P_\lambda$ and $P_\lambda = m_\lambda$ + lower terms (monic).

$$P_\lambda \equiv P_\lambda^{q,t} \qquad \left(= P_\lambda^{q^{-1},t^{-1}}\right)$$

2. GENERALIZED MACDONALD OPERATORS

2.A. Theorem-definition

THM [DFK15] Let $Y_{i,n} = (X_1 \cdots X_{i-1})^{-n} Y_i (X_1 \cdots X_i)^n$

Then the $\{Y_{i,n}\}_{1 \le i \le r+1}$ form a commuting family

and
$$\boxed{\mathcal{D}^{q,t}_{\alpha,n} := e_\alpha(Y_{1,n} \cdots Y_{r+1,n}) = \sum_{|I|=\alpha} (X_I)^n \prod_{\substack{i \in I \\ j \notin I}} \frac{\theta X_i - \theta^{-1} X_j}{X_i - X_j} D_I}$$

when acting on symmetric polynomials $\mathbb{C}[X_1 \cdots X_{r+1}]^{\mathfrak{S}_{r+1}}$

where does this come from?

2.B. $SL_2(\mathbb{Z})$ ACTION ON sDAHA

Define $\begin{cases} X_i = q^{x_i} \\ Y_i = q^{y_i} \end{cases}$ $\quad \begin{matrix} \curlyvee = q^{\frac{1}{2}\sum_1^{r+1} x_i^2} \\ \nabla = q^{\frac{1}{2}\sum_1^{r+1} y_i^2} \end{matrix}$ (in suitable completion)

2.B. $SL_2(\mathbb{Z})$ ACTION ON sDAHA

Define $\begin{cases} X_i = q^{x_i} \\ Y_i = q^{y_i} \end{cases}$ $\quad \gamma = q^{\frac{1}{2}\sum_1^{r+1} x_i^2}$ $\quad \nabla = q^{\frac{1}{2}\sum_1^{r+1} y_i^2}$ (in suitable completion)

THM $\{\mathrm{ad}_\gamma, \mathrm{ad}_\nabla\}$ generate an $SL_2(\mathbb{Z})$ action on sDAHA, with $\begin{cases} \mathrm{ad}_\gamma \longrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \mathrm{ad}_\nabla \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{cases}$

2.B. $SL_2(\mathbb{Z})$ ACTION ON sDAHA

Define $\begin{cases} X_i = q^{x_i} \\ Y_i = q^{y_i} \end{cases}$ $\quad \gamma = q^{\frac{1}{2}\sum_1^{r+1} x_i^2}$ $\quad \nabla = q^{\frac{1}{2}\sum_1^{r+1} y_i^2}$ (in suitable completion)

THM $\{ad_\gamma, ad_\nabla\}$ generate an $SL_2(\mathbb{Z})$ action on sDAHA, with $\begin{cases} ad_\gamma \longrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ ad_\nabla \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{cases}$

THM

$$\mathcal{D}^{q,t}_{\alpha,n} = q^{-\frac{n\alpha}{2}} \cdot \gamma^{-n} \mathcal{D}^{q,t}_{\alpha} \gamma^{n} \qquad (n \in \mathbb{Z})$$

↑ generalized MACDO

↑ usual MACDO

"NABLA OPERATOR"

THM

$$\nabla P_\lambda^{q,t} = \left(C_r \cdot \prod_{i=1}^{r+1} q^{\frac{\lambda_i^2}{2}} t^{(\frac{r}{2}+1-i)\lambda_i}\right) P_\lambda^{q,t}$$

$$\mathrm{Log}(C_r) = \frac{r(r+1)(r+2)}{24} (\log t)^2 / \log q$$

- usually defined on the modified Macdonald polynomials $\tilde{H}_\lambda^{q,t}$ [Garsia, Haiman, Bergeron, ...]
- Cherednik's "superpolynomial" → refined torus knot invariants

2.C. USEFUL PROPERTIES

let $p_k = \sum^{r+1} x_i^k$ power sums

then $[p_k , \mathcal{D}^{q,t}_{1,n}] = (1-q^k)\, \mathcal{D}^{q,t}_{1,n+k}$

Proof: $D_i \, x_j^k = (1+(q^k-1)\,\delta_{i,j})\, x_j^k \, D_i$

(discrete time-translation operators)
(cf relation to plethysm $X \to X + \frac{1-q}{z}$)

2.D. q-Whittaker limit

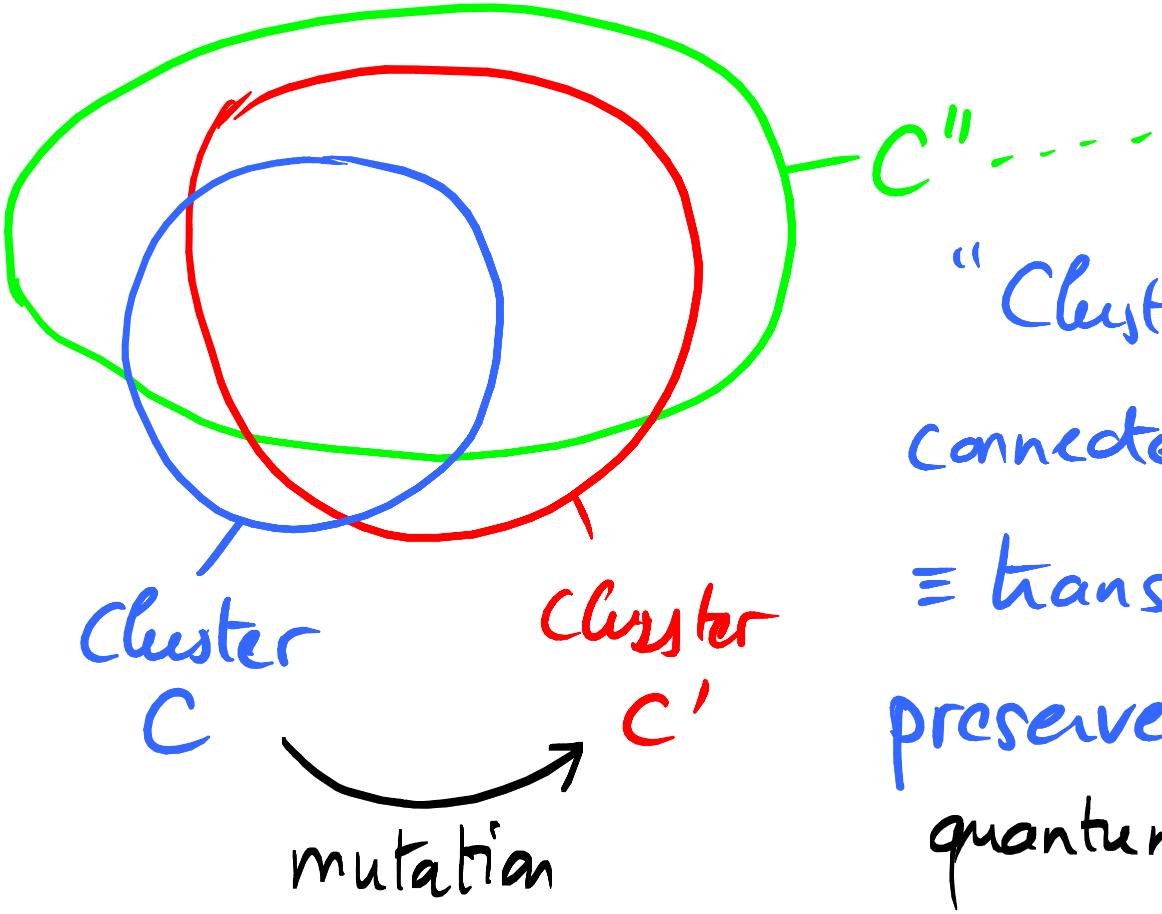
- corresponds to $t \to +\infty$ (ie $\theta \to \infty$)
- define $M_{\alpha,n} := \lim\limits_{\theta\to\infty} \theta^{-\alpha(r+1-\alpha)} \mathcal{D}^{q,t}_{\alpha,n}$

ie.

$$\boxed{M_{\alpha,n} = \sum_{\substack{|I|=\alpha \\ I\subset[1,r+1]}} (x_I)^n \prod_{\substack{i\in I \\ j\notin I}} \frac{x_i}{x_i-x_j} D_I}$$

- very special → obey the quantum-A_r-Q-system-cluster algebra relations.

2.E. CLUSTER ALGEBRA CONNECTION



"Clusters" of variables connected via mutations ≡ transformations that preserve the Laurent property

quantum → q-commutations

$$\text{Cluster} = M_A := \{M_{\alpha,n}\}_{\substack{\alpha\in[1,r]\\ n\in A}}$$

2.E. CLUSTER ALGEBRA CONNECTION

THM The generalized Macdonald operators in the q-Whittaker limit satisfy the following:

- $M_{\alpha,n} \; M_{\beta,n+p} = q^{\mathrm{Min}(\alpha,\beta)p} M_{\beta,n+p} M_{\alpha,n}$ $(|p| \leq |\beta-\alpha|+1)$
- $q^{\alpha} M_{\alpha,n+1} M_{\alpha,n-1} = M_{\alpha,n}^2 - M_{\alpha+1,n} M_{\alpha-1,n}$ (mutation)
- $M_{r+2,n} = 0 \quad M_{r+1,n} = A^n \Delta^{-1}$

  $A = X_1 \cdots X_{r+1} \; ; \; \Delta^{-1} = D_1 D_2 \cdots D_{r+1}$

EXAMPLE: SL_2 $(r=1)$

only $\alpha=1$ $\qquad M_{1,n} \equiv M_n = \frac{1}{x_1-x_2}\left(x_1^{n+1} D_1 - x_2^{n+1} D_2\right)$

ie $\qquad M_n f(x_1,x_2) = \frac{x_1^{n+1} f(qx_1,x_2) - x_2^{n+1} f(x_1,qx_2)}{x_1-x_2}$

2 relations:

$$\boxed{\begin{array}{c} M_n M_{n+1} = q M_{n+1} M_n \\ q M_{n+1} M_{n-1} = M_n^2 - A^n \Delta^{-1} \end{array}}$$

$$\Delta^{-1} = D_1 D_2 \qquad A = x_1 x_2$$

$$\left(\begin{array}{l} \Delta^{-1} M_n = q^n M_n \Delta^{-1} \\ A \; M_n = q^{-2} M_n A \end{array}\right)$$

The relations form a discrete integrable system in the discrete time variable n

$$M_{\alpha,n} = \sum_{\substack{I\subset[1,r+1]\\ |I|=\alpha}} (X_I)^n \underbrace{\left(\prod_{\substack{i\in I\\ j\notin I}} \frac{X_i}{X_i-X_j} \qquad D_I\right)}_{\text{Coefficient}}$$

↑ linear combination of "plane waves"

X_I = angle variables

X_i = conserved

Notation: use $M_n \equiv M_{1,n}$ $\qquad M_{0,n} = 1$

2.F. INTEGRABILITY

- $\zeta_{\alpha,i} = M_{\alpha,i} M^{-1}_{\alpha,i+1}$; $X_{\alpha,i} = \zeta^{-1}_{\alpha,i} \zeta_{\alpha-1,i}$
- Discrete shift D : $D f(n) = f(n+1) D$
- Miura operator

$$\mu_n = (D - X_{r+1,n})(D - X_{r,n}) \cdots (D - X_{1,n})$$

THM

1. $\mu_n = \mu$ independent of n
2. $\mu_n(M_n) = 0$
3. $\mu = \mu_\infty = \prod_{i=1}^{r+1} (D - X_i) = \sum_{i=0}^{r+1} (-1)^i e_i D^{r+1-i}$

explicit Laurent polynomials of M's = conserved quantities

EXAMPLE SL_2 $(r=1)$

$$\begin{cases} \zeta_{1,n} = M_n M_{n+1}^{-1} = \zeta_n \\ \zeta_{2,n} = A^{-1} = (X_1X_2)^{-1} \\ \zeta_{0,n} = 1 \end{cases} \qquad \begin{cases} X_{2,n} = \zeta_{2,n}^{-1}\zeta_{1,n} = A\zeta_n \\ X_{1,n} = \zeta_{1,n}^{-1}\zeta_{0,n.} = \zeta_n^{-1} \end{cases}$$

- $\mu_n = (D - A\zeta_n)(D - \zeta_n^{-1}) = D^2 - e_1 D + e_2$

$$\begin{cases} e_1 = A\zeta_n + \zeta_{n+1}^{-1} \\ e_2 = A = X_1X_2 \end{cases}$$

independent of n!

$$\boxed{e_1 = (AM_n + M_{n+2})M_{n+1}^{-1} = AM_nM_{n+1}^{-1} + M_{n+1}M_n^{-1} - A^{n+1}\Delta^{-1}M_{n+1}^{-1}M_n^{-1}}$$

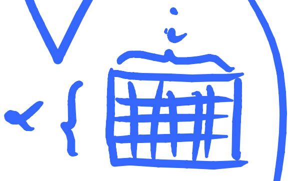
$$\boxed{e_1(n \to \infty) = X_1 + X_2}$$

← ↑ identical!

3. GRADED CHARACTERS & DIFFERENCE EQUATIONS

Def $\underline{n} = \{n_{\alpha,i}\}_{\substack{1\leq\alpha\leq r\\ 1\leq i}}$ grading on tensor products $\mathfrak{gl}_{r+1}$

$$\bigotimes_{i=1}^{k}\bigotimes_{\alpha=1}^{r}\left(V\right)^{\otimes n_{\alpha,i}} \longleftarrow V_{\alpha,i}$$

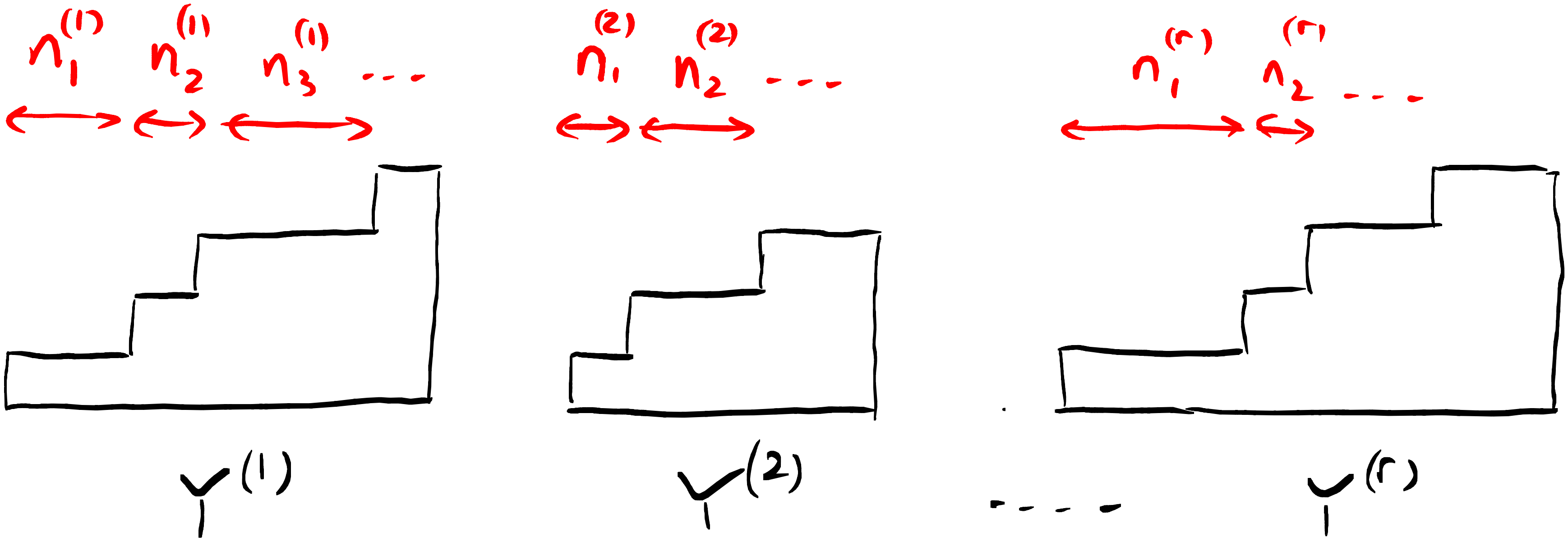


$$\chi_{\underline{n}}(q,z) = \sum_{\lambda} \mathrm{Mult}_q\left(\otimes V_{\alpha i}^{\otimes n_{\alpha i}}, V_\lambda\right)\cdot s_\lambda(\underline{z})$$

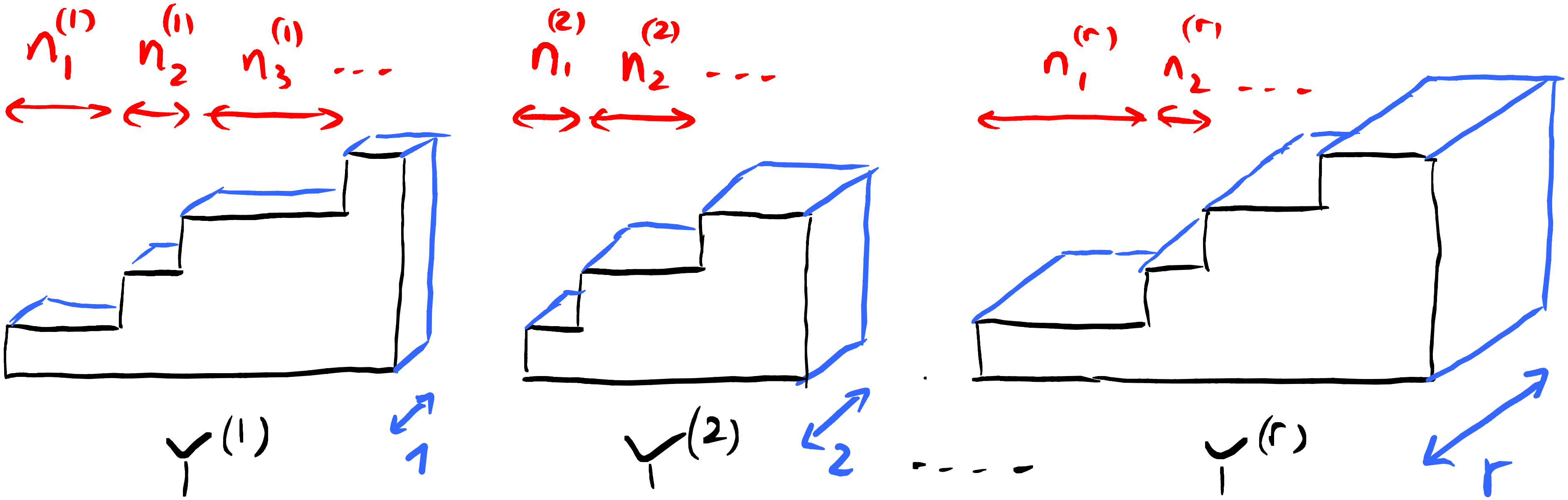
THM [DF-Kedem 15]

$$\chi_{\underline{n}}(q,z) = q^{-a(\underline{n})} \prod_{j=k}^{1}\prod_{\alpha=1}^{r}\left(M_{\alpha,j}\right)^{n_{\alpha,j}} \cdot 1$$

STATES = Collection of Young diagrams $Y^{(1)}, .., Y^{(r)}$

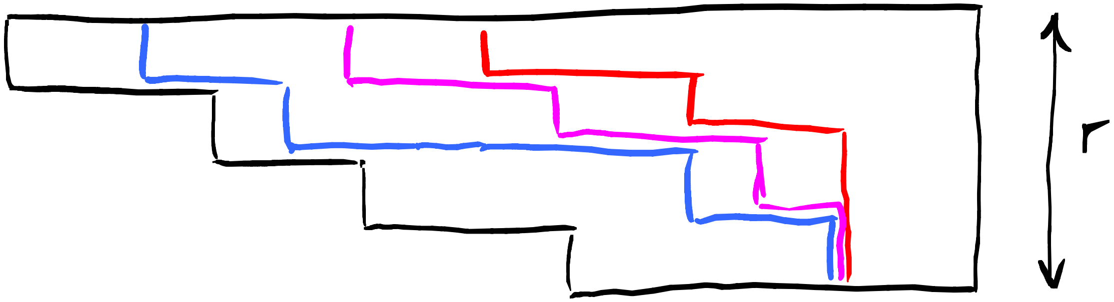


STATES = Collection of Young diagrams $Y^{(1)} \dots Y^{(r)}$ with "thickness" $1, 2, \dots, r$



⇒ PLANE PARTITION

(view from above)



THM The graded characters satisfy difference equations of the form: $e_m \cdot \chi_{\underline{n}} = \mathcal{L}_m \, \chi_{\underline{n}}$

↑ eigenvalue (≡ conserved qty)

↑ difference operator by use of Laurent polynomial expression in M's

↓

Hamiltonians acting on the states

Proof:

EXAMPLE : SL_2 $(r=1)$

$$e_1 = (AM_n + M_{n+2})M_{n+1}^{-1} = AM_nM_{n+1}^{-1} + M_{n+1}M_n^{-1} - A^{n+1}\Delta^{-1}M_{n+1}^{-1}M_n^{-1}$$

$$e_2 = A = x_1x_2$$

$$\chi_{\underline{n}} = q^{-a(\underline{n})} M_k^{n_k} M_{k-1}^{n_{k-1}} \cdots M_1^{n_1} \cdot 1$$

$$\boxed{\begin{aligned}(x_1+x_2)\chi_{n_1 \ldots n_k} = {} & \chi_{n_1,\ldots,n_{k-1}-1,n_k+1} + x_1x_2\,\chi_{n_1,\ldots,n_{k-1}+1,n_k-1} \\ & - (x_1x_2)^k q^{k-1-\Sigma i n_i}\,\chi_{n_1,\ldots,n_{k-1}-1,n_k-1}\end{aligned}}$$

Another set of difference equations:

SYMMETRIC POWER REPRESENTATIONS

$(\alpha=1)$ $\quad \otimes\left(V_{\underbrace{\boxed{\;|\;|\;|\;|\;|\;}}_{j}}\right)^{\otimes n_j} \qquad \chi_{\underline{n}} = q^{-a(\underline{n})} \prod_{j=k}^{1} (M_j)^{n_j} \cdot 1$

use
$$\boxed{\begin{cases} 1. \quad M_j M_{j+1} = q\, M_{j+1} M_j \\ 2. \quad [e_1, M_j] = (1-q)\, M_{j+1} \end{cases}}$$

$$\Rightarrow \quad [e_1, (M_j)^n] = (1-q^n)\, M_{j+1} M_j^{n-1}$$

$e_1 \chi_{\underline{n}} = ?$ use $\varphi_{\underline{n}} = M_k^{n_k} M_{k-1}^{n_{k-1}} \cdots M_1^{n_1} \cdot 1$

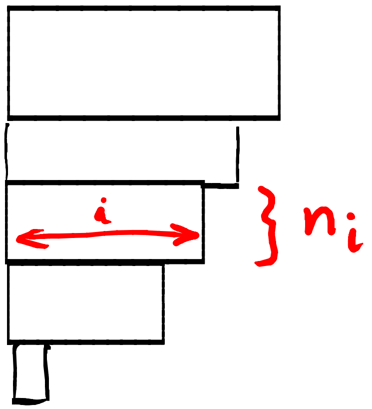
$$e_1 \cdot \varphi_{\underline{n}} = \varphi_{n_1+1, n_2 \ldots n_k \ldots} + \sum_{i=1}^{\infty} (1-q^{n_i}) \varphi_{\ldots n_i-1, n_{i+1}+1 \ldots}$$

$$= \underbrace{\left(S_1 + \sum_{i=1}^{\infty} (1-q^{n_i}) S_i^{-1} S_{i+1}\right)}_{\text{q-difference q-boson / Toda operator}} \varphi_{\underline{n}}$$

$\chi_{\underline{n}}$ is HL polynomial up to plethysm

$\underline{n} \longleftrightarrow$ Young diagram

$(\equiv Y^{(1)}; n_i^{(1)} \equiv n_i)$

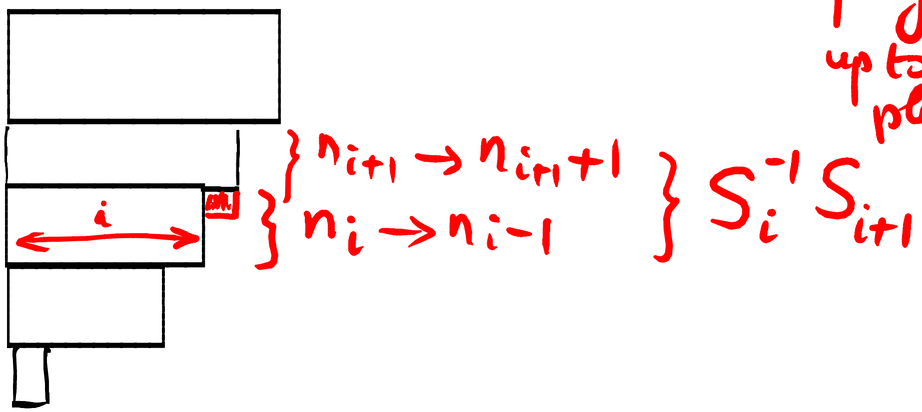


$$e_1 \cdot \varphi_{\underline{n}} = \varphi_{n_1+1, n_2 \dots n_k} + \sum_{i=1}^{\infty} (1-q^{n_i})\, \varphi_{\dots n_i-1, n_{i+1}+1 \dots}$$

$$= \underbrace{\left(S_1 + \sum_{i=1}^{\infty} (1-q^{n_i})\, S_i^{-1} S_{i+1} \right)}_{\text{q-difference q-boson/Toda operator}} \varphi_{\underline{n}}$$

χ_n is HL polynomial up to plethysm

$\underline{n} \longleftrightarrow$ Young diagram



- Pieri Rule: weight $(1-q^{n_i})$ for box added on top of $i \times n_i$ rectangle

4. (q, t) - deformation and ELLIPTIC HALL ALGEBRA

4. A. EHA : definition

- Multiplication rules for classes in equivariant K-theory of certain varieties (NQVs).

- generators : $u_{a,b}$; $(a,b) \in \mathbb{Z}^2$
- parameters : q, t $\alpha_k = \frac{1}{k}(1-q^k)(1-t^{-k})(1-q^{-k}t^k)$

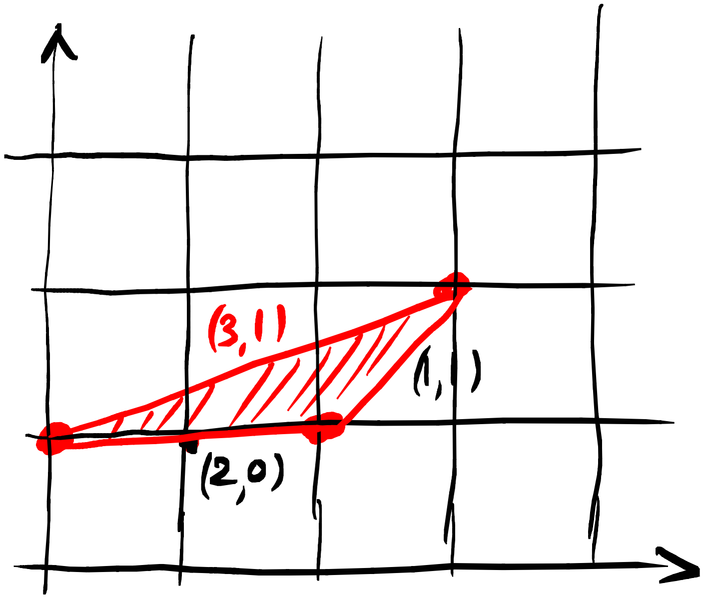
1. $[u_{c,d}, u_{a,b}] = 0$ if $(0,0)(a,b)(c,d)$ aligned

2. $[u_{c,d}, u_{a,b}] = \frac{\theta_{a+c,b+d}}{\alpha_1}$ if $\widehat{(0,0)(a,b)(c,d)}$ quasi-empty

3. $a \wedge b = 1 \Rightarrow \sum \theta_{ka,kb} z^k = e^{\sum \alpha_k z^k u_{ka,kb}}$

EXAMPLES

(A)



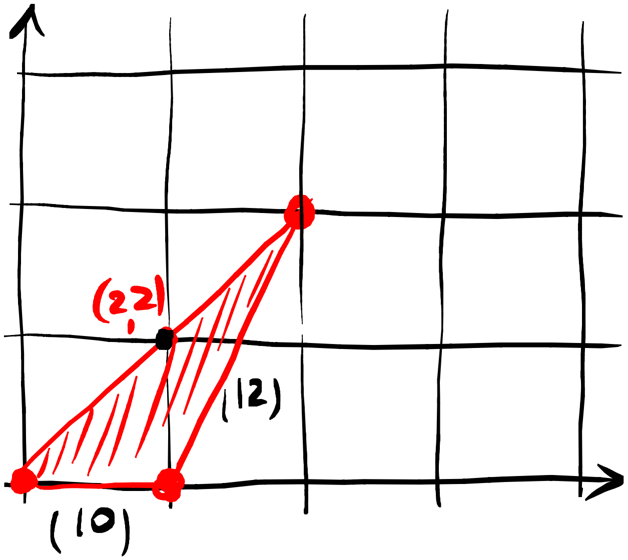
$$\left(\begin{array}{l}\alpha_1 = (1-q)(1-t^{-1})(1-q^{-1}t)\\ \alpha_2 = \frac{1}{2}(1-q^2)(1-t^{-2})(1-q^{-2}t^2)\end{array}\right)$$

coprime

$$[u_{11}, u_{2,0}] = \frac{\theta_{31}}{\alpha_1} = u_{31}$$

triangle is quasi-empty

(B)



$$[u_{1,2}, u_{1,0}] = \frac{\theta_{22}}{\alpha_1} \leftarrow \text{non coprime}$$

$\Downarrow$

$$1 + z\theta_{11} + z^2\theta_{22} + \dots = e^{z\alpha_1 u_{11} + z^2\alpha_2 u_{22} + \cdots}$$

$$\theta_{22} = \frac{\alpha_1^2}{2} u_{11} + \alpha_2 u_{22}$$

4.B. Isomorphisms

1. $\mathrm{sDAHA}_{r\to\infty} \cong \mathrm{EHA}$

2. $\mathrm{sDAHA}_r \underset{\text{surj Hom}}{\hookleftarrow} \mathrm{EHA}$

The isomorphism uses power sums:

$P_k = \sum Y_i^k \longleftrightarrow u_{k,0}$ (N.B.: $\mathcal{D}_{\alpha,0} = e_\alpha(Y)$
$\mathcal{P}_\alpha = p_\alpha(Y)$)

$p_k = \sum X_i^k \longleftrightarrow u_{0,k}$ (Conserved quantities)

4.C. EHA and generalized Macdonald operators

THM

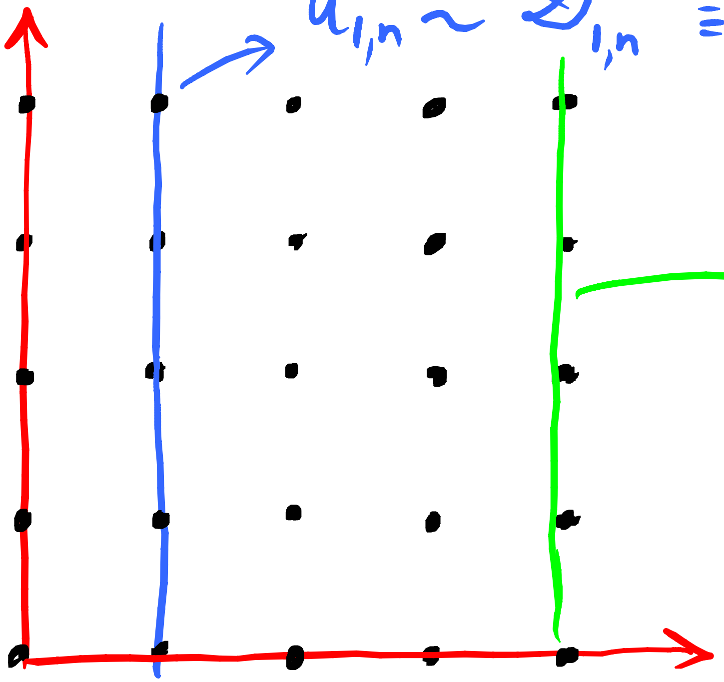
1. $u_{1,k} \leftrightarrow \mathcal{D}^{q,t}_{1,k}$ (gen. Macdonald operators)

2. $SL_2(\mathbb{Z})$ action is same, $\begin{cases} \mathrm{ad}_{\gamma} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ \mathrm{ad}_{\nabla} \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{cases}$

$$\begin{pmatrix} u & v \\ w & x \end{pmatrix} \cdot u_{a,b} = u_{ua+vb,\, wa+xb}$$

Proof: $u_{0,1} = e_1 \qquad [e_1, \mathcal{D}_{1,k}] = (1-q)\mathcal{D}_{1,k+1}$

4.D. APPLICATIONS



$u_{1,n} \sim \mathcal{D}^{q,t}_{1,n}$ = gen. Macdonald

$u_{0,n} \sim p_n(X)$ ←
power sums

→ $\mathcal{D}_{k,n}$ ← $SL_2(\mathbb{Z})$

↳ $u_{k,0} = p_k(Y)$

$$e^{-\sum \frac{z^k}{k} p_k(Y)} = \prod_{k=1}^{r+1}(1 - zY_k) = \sum_{0}^{r+1} (-z)^k \mathcal{D}_{k,0}$$

← Macdonald operators

t – deformed quantum cluster algebra relations

$$\boxed{(1+t^{-1})(1+q^{-1}t)\,\mathcal{D}_{2,n} = \frac{[\mathcal{D}_{1,n+1},\mathcal{D}_{1,n-1}]}{1-q} - (1+q^{-1})\,\mathcal{D}_{1,n}^{2}}$$

Proof: $\theta_{2,2n} \sim [u_{1,n+1}, u_{1,n-1}]$

$$1+z\,\theta_{1,n}+z^2\,\theta_{2,2n}+\dots = e^{z\alpha_1 u_{1,n} + z^2\alpha_2 u_{2,2n}+\dots}$$

$$= 1+z\alpha_1 u_{1,n} + z^2\left(\alpha_2 u_{2,2n} + \frac{\alpha_1^2}{2}u_{1,n}^2\right)+\dots$$

(recall $M_{2,n} = q\,M_{n+1}M_{n-1} - M_n^2$ in $t\to\infty$ limit)

CONSTANT TERM IDENTITIES For $\mathcal{D}_{\alpha,n}$

$$\mathcal{D}(u) = \sum_{n\in\mathbb{Z}} u^n \mathcal{D}_{1,n}$$ "current"

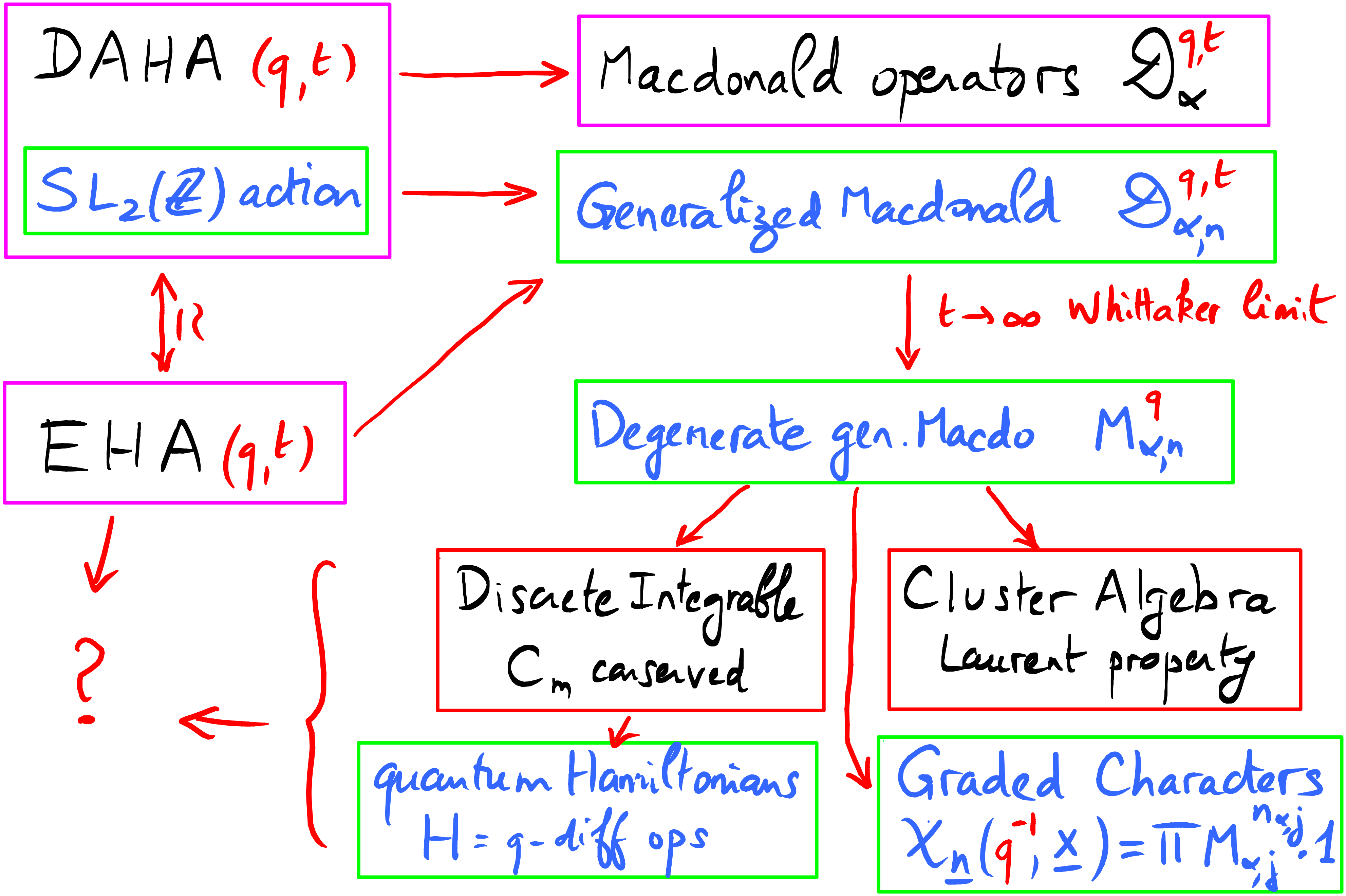
$$\mathcal{D}_{\alpha,n} = \frac{1}{\alpha!} CT_{u_1\cdots u_\alpha}\left(\prod_{i<j}\frac{(u_i-u_j)(u_i-qu_j)}{(u_i-tu_j)(u_i-qt^{-1}u_j)}\,\frac{\mathcal{D}(u_1)\cdots\mathcal{D}(u_\alpha)}{(u_1u_2\cdots u_\alpha)^n}\right)$$

$t\to\infty$ q-Whittaker limit: "quantum determinant"

$$M_{\alpha,n} = CT_{u_1\cdots u_\alpha}\left(\prod_{i<j}\left(1-q\frac{u_j}{u_i}\right) M(u_1)\cdots M(u_\alpha)\,(u_1u_2\cdots u_\alpha)^{-n}\right)$$

$$M_{2,n} = M_n^2 - q M_{n+1}M_{n-1} =: \begin{vmatrix} M_n & M_{n+1} \\ M_{n-1} & M_n \end{vmatrix}_q$$

$$\begin{aligned} M_{3,n} = {} & M_n^3 - q(1+q)M_{n+1}M_nM_{n-1} - q M_{n+1}M_{n-1}M_n - q M_nM_{n+1}M_{n-1} \\ & + q^2 M_{n+2}M_{n-1}^2 + q^2 M_{n+1}^2 M_{n-2} - q^3 M_{n+2}M_nM_{n-2} =: \begin{vmatrix} M_n & M_{n+1} & M_{n+2} \\ M_{n-1} & M_n & M_{n+1} \\ M_{n-2} & M_{n-1} & M_n \end{vmatrix}_q \end{aligned}$$



DAHA (q,t)

$SL_2(\mathbb{Z})$ action

Macdonald operators $\mathcal{D}_{\alpha}^{q,t}$

Generalized Macdonald $\mathcal{D}_{\alpha,n}^{q,t}$

EHA (q,t)

$\wr$

$t \to \infty$ Whittaker limit

Degenerate gen. Macdo $M_{\alpha,n}^{q}$

Discrete Integrable
C_m conserved

Cluster Algebra
Laurent property

quantum Hamiltonians
$H = q$-diff ops

Graded Characters
$\chi_{\underline{n}}(q^{-1};\underline{x}) = \prod M_{\alpha,j}^{n_{\alpha,j}} \cdot 1$

?

THE BIG PICTURE...