Convergent Weak-Coupling Expansions for non-Abelian Field Theories from DiagMC Simulations

[1705.03368, 1510.06568]

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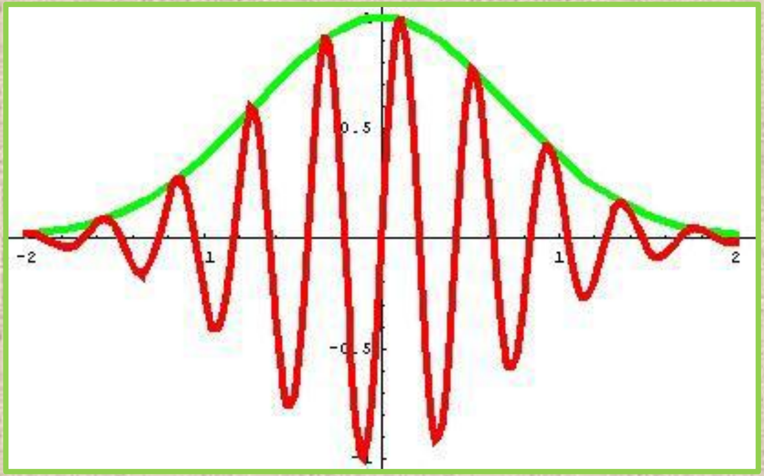
Sign problem in QCD

$$\int dA_\mu \det\left(\mathcal{D}\left[A_\mu\right]\right)^{N_f} e^{-S[A_\mu]},$$

$$\mathcal{D}^\dagger\left[A_\mu\right] \neq -\mathcal{D}\left[A_\mu\right] \Rightarrow \arg\det\left(\mathcal{D}\left[A_\mu\right]\right) \neq 0$$

Lattice QCD
@ finite baryon density:

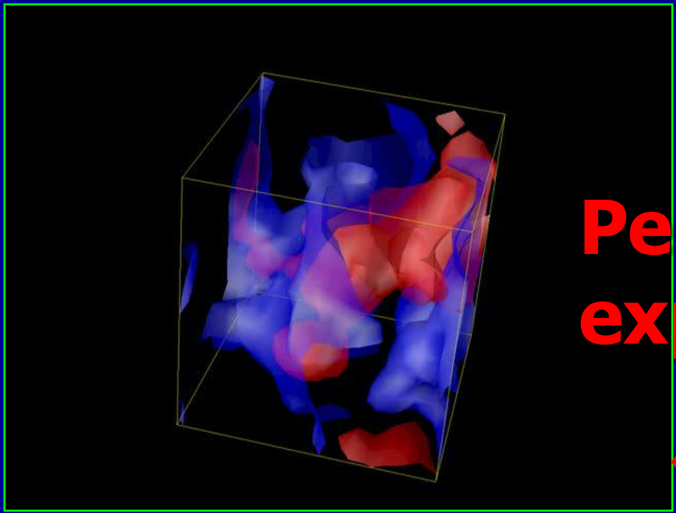
- **Complex path integral**
- **No positive weight for Monte-Carlo**



This motivates alternative approaches → DiagMC

Diagrammatic Monte-Carlo in QFT

Sum over fields

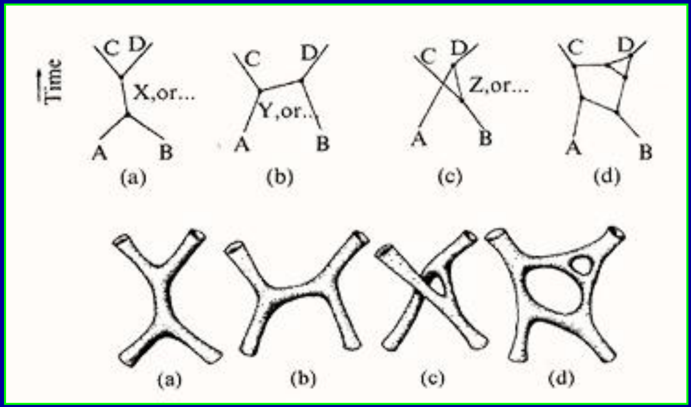


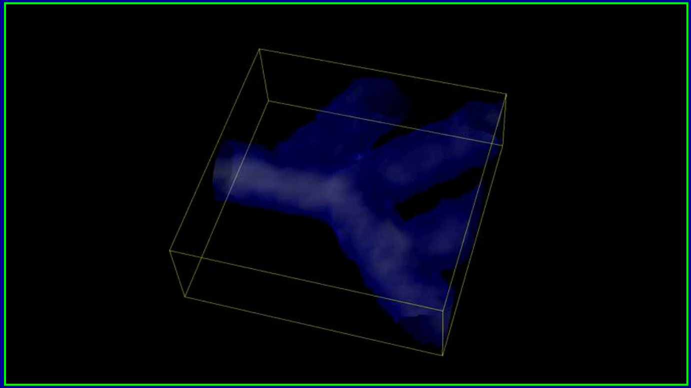
$$\mathcal{Z} = \mathrm{Tr}\, e^{-\hat{\mathcal{H}}/kT} =$$

$$= \int \mathcal{D}\phi\left(x^{\mu}\right)\, \exp\left(-S_E\left[\phi\left(x^{\mu}\right)\right]\right)$$

Perturbative expansions

Sum over interacting paths





Works inspiringly well in cond-mat models (e.g. Hubbard)

[Prokof'ev,Svistunov & Co]

Series expansions as a solution to fermionic sign problem

[Rossi,Prokof'ev,Svistunov, Van Houcke,Werner 1703.10141]

Assume sufficiently fast convergence

$$|f(\lambda) - f_N(\lambda)| \sim \epsilon^N,$$

$$f_N(\lambda) = \sum_{i=0}^{N} f_i \lambda^i$$

Assume f_i can be computed in time

$$\tau_i \sim A^i$$

(nontrivial, as diagram number grows as factorial of i)

$$\tau(err) = (err)^{A/\ln\epsilon}$$

Ideal case: convergent geometric series

What can we do for real (lattice) gauge theory?

Strong-coupling expansion:

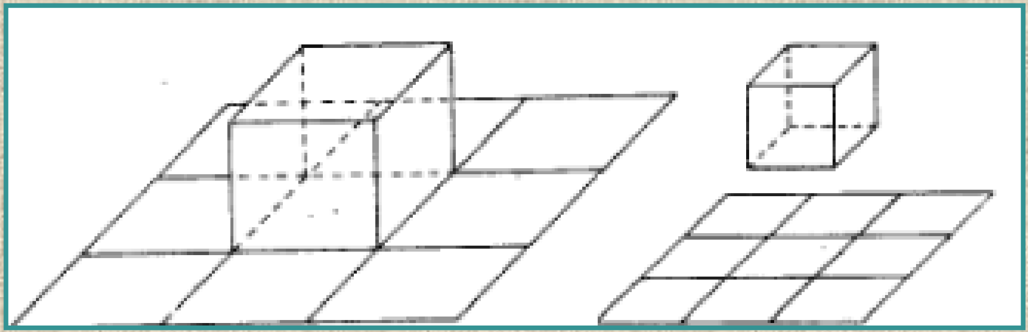
(Lattice) [de Forcrand,Philipsen,Unger,…]

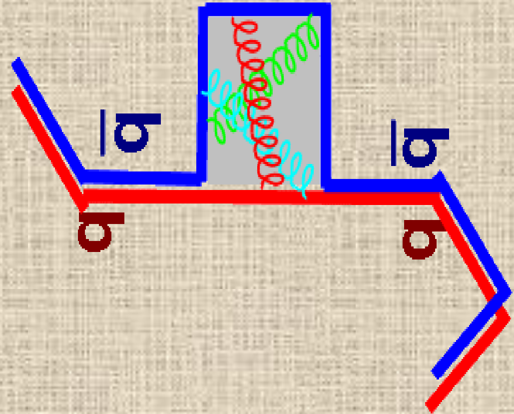
V Convergent (finite volume, any coupling)

V Intuitive and physical: confinement, hadrons, chiral symmetry breaking…

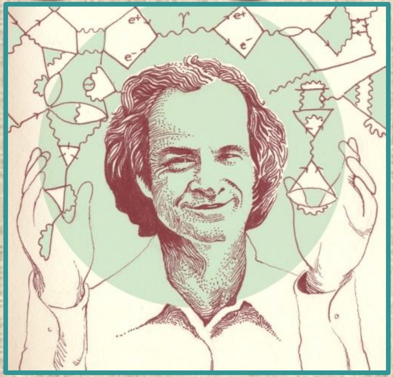
X Difficult to generate automatically

X In infinite volume, bad at weak coupling





What can we do for real gauge theory?



Weak-coupling expansion:

V Works well at weak coupling

? Automatic generation easier (Feynman)

X Intermediate kinematic IR divergences

X Non-Borel-summable series divergences

BUT

V Resurgent structure of series

X Difficult to generate automatically: seem to need all saddles of the action, or complicated resummations…

In this talk

Construction of "weak-coupling" expansion for non-Abelian lattice field theories:

V Explicitly Infrared finite

V No factorial divergences

? Something similar to trans-series

X In fact, not yet trans-series: convergence sub-optimal

V Sampling by Diagrammatic Monte-Carlo possible

X … But not yet "polynomial complexity" solution

Approaching QCD:
SU(N) principal chiral model

$$\mathcal{Z} = \int_{U(N)} dg_x \exp\left(-\frac{N}{\lambda} \sum_{<x,y>} \mathrm{Tr}\,\left(g_x^\dagger g_y\right) \right)$$

- **Non-Abelian theory**
- **Asymptotic freedom**
- **Classical action is scale invariant**
- **Dynamical non-perturbative mass gap generation**

$$M \sim \exp\left(-\frac{8\pi}{\lambda}\right)$$

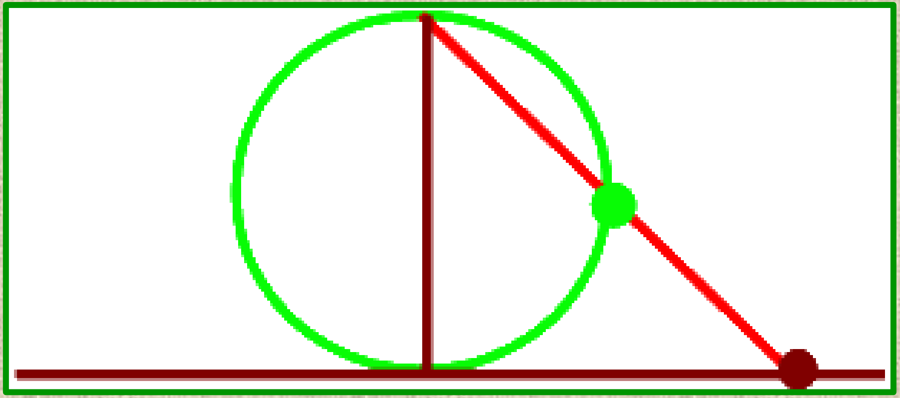
- **Admits large-N limit**
- **Perturbative series**

Perturbative expansion

- **Take N->∞ to reduce diagram space**
- **Small fluctuations of SU(N) fields**
- **Map SU(N) to Hermitian matrices**

Cayley map

$$g = \frac{1 + i\alpha\phi}{1 - i\alpha\phi}$$



$$\int\limits_{SU(N)} dg \Rightarrow \int\limits_{\mathbb{H}_{N\times N}} d\phi \det\left(1 + \alpha^2\phi^2\right)^{-N} =$$

$$= \int\limits_{\mathbb{H}_{N\times N}} d\phi \exp\left(-N\alpha^2 \operatorname{Tr}\phi^2 + O\left(\alpha^4\phi^4\right)\right)$$

"Perturbative" action

$$\mathcal{Z} = \int\limits_{U(N)} dg_x \exp\left(-\frac{N}{\lambda}\sum_{<x,y>} \mathrm{Tr}\,\left(g_x^\dagger g_y\right)\right)$$

$$\alpha^2 = \frac{\lambda}{8}$$

Expand action and Jacobian in φ

Infinitely many interaction vertices

$$S[\phi_x] = \frac{1}{2}\sum_{x,y}\left(D_{xy} + \frac{\lambda}{4}\delta_{xy}\right)\mathrm{Tr}\,\left(\phi_x\phi_y\right) +$$

$$+ \sum_{n=2}^{+\infty}\left(-\frac{\lambda}{8}\right)^{n-1}\left(\frac{\lambda}{8n}\sum_x \mathrm{Tr}\,\phi_x^{2n} + \right.$$

$$\left.\frac{1}{2}\sum_{l=1}^{2n-1}(-1)^{l-1}\sum_{x,y} D_{xy}\mathrm{Tr}\,\left(\phi_x^{2n-l}\phi_y^l\right)\right)$$

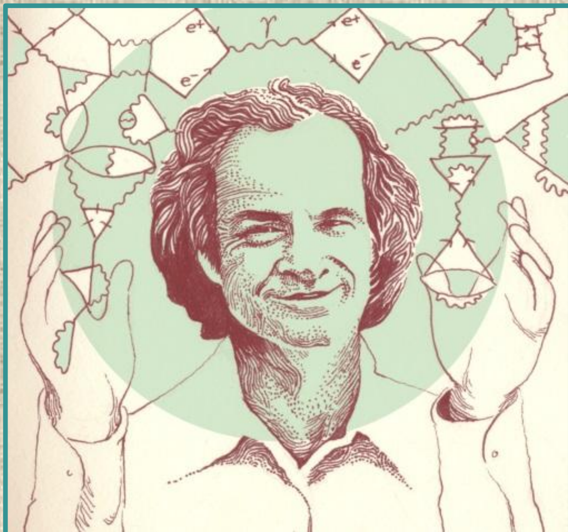
Setting up an expansion

Power series in t'Hooft λ?
Factorial growth even at large N
due to IR renormalons … [Bali, Pineda,di Renzo]
Can be sampled, but resummation difficult

…Bare mass term ~λ from Jacobian???
[a-la Fujikawa for axial anomaly]

- ✓ **Massive planar fields**
- ✓ **Suitable for DiagMC**
- ? **How to expand in λ?**

➡ **Count vertices !??**



Setting up an expansion

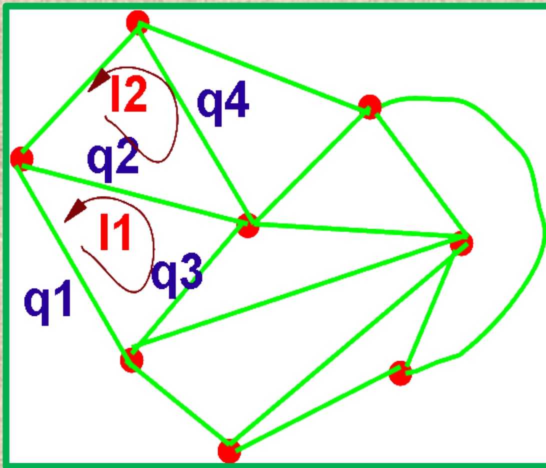
$$S[\phi_x] = \frac{1}{2}\sum_{x,y}\left(D_{xy}+\frac{\lambda}{4}\delta_{xy}\right)\mathrm{Tr}\,(\phi_x\phi_y) + \sum_{n=2}^{+\infty}\left(-\frac{\lambda}{8}\right)^{n-1}\left(\frac{\lambda}{8n}\sum_x \mathrm{Tr}\,\phi_x^{2n} + \frac{1}{2}\sum_{l=1}^{2n-1}(-1)^{l-1}\sum_{x,y}D_{xy}\mathrm{Tr}\,\left(\phi_x^{2n-l}\phi_y^l\right)\right)$$

$$m_0^2 \equiv \frac{\lambda}{4}$$

$$\xi = 1$$

$$S[\phi_x] = \frac{1}{2}\sum_{x,y}\left(D_{xy}+m_0^2\delta_{xy}\right)\phi_x\phi_y + \sum_{n=2}^{+\infty}\left(-\frac{\lambda}{8}\right)^{n-1}\xi^{n-1} \times \left(\frac{m_0^2}{2n}\mathrm{tr}\,\phi_x^{2n} + \frac{1}{2}\sum_{l=1}^{2n-1}(-1)^{l-1}\sum_{x,y}D_{xy}\mathrm{tr}\,\left(\phi_x^{2n-l}\phi_y^l\right)\right)$$

Counting powers of λ



Consider a planar diagram:

- ***f*** **faces = loop momenta**
- ***v*** **vertices** ***~ λ ~*** m_0^2
- ***l*** **bare propagators=lines**
- **In planar limit,** ***f – l + v = 2***

$$W_K \sim \int d^2q_1 \dots d^2q_{f-1} \frac{V_1 \dots V_v}{(Q_1^2 + m_0^2) \dots (Q_l^2 + m_0^2)}$$

- **Standard power counting**
- W_k **can only contain:**
- Λ^2_{UV}, m^2_0, m^4_0 / Λ^2_{UV}, ….
- **… Times probably logs !!!???**

IR finiteness and series properties

In the large-N limit:

- V **Number of diagrams grows exponentially with order**
- V **All diagram weights are by construction finite**

➡ **No factorially divergent terms can appear!!!**

(Remember that renormalon divergences persist even at N->∞ [David,Wiegmann,Braun,...])

Minimal working example:
2D O(N) sigma model @ large N

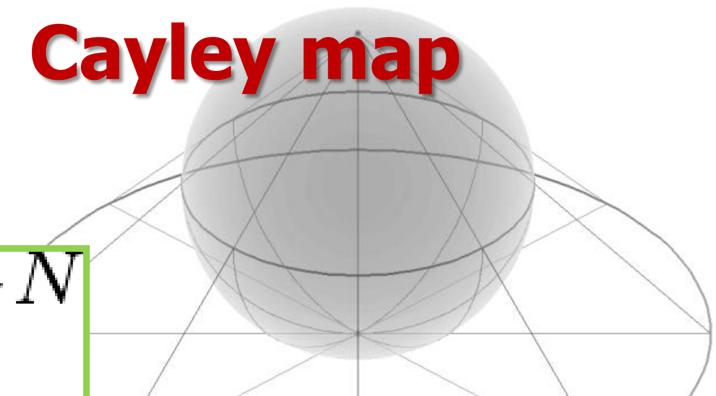
$$\int_{S_N} d\vec{n}_x \exp\left(-\frac{1}{\alpha^2}\sum_{<x,y>} \vec{n}_x\cdot\vec{n}_y\right)$$

$$\langle\, \vec{n}_x \cdot \vec{n}_y \,\rangle \sim \\ \sim \exp\left(-m^2|x-y|\right)$$

Non-perturbative mass gap

$$m^2 = 32\exp\left(-\frac{4\pi}{\alpha^2}\right)$$

Cayley map



Jacobian reads

$$\mathcal{D}n_x = \mathcal{D}\phi_x\left(1+\frac{\lambda}{4}\phi_x^2\right)^{-N}$$

$$S_N \rightarrow \mathbb{R}^{N-1}$$

Again, bare mass term from the Jacobian…

[PB, 1510.06568]

O(N) sigma model @ large N

Full action in new coordinates

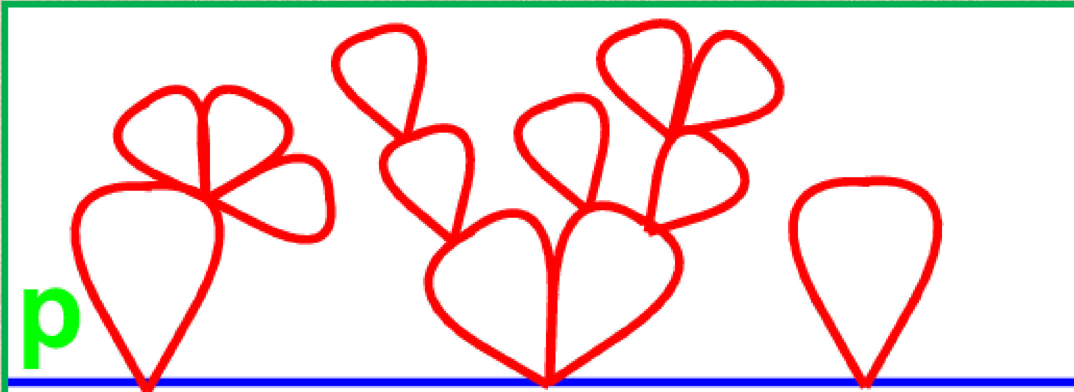
$$S[\phi_x] = \frac{1}{2}\sum_{x,y}\left(D_{xy} + \frac{\lambda}{2}\delta_{xy}\right)\phi_x \cdot \phi_y +$$

$$+ \sum_{k=2}^{+\infty} \frac{(-1)^{k-1}\lambda^k}{4^k\, k} \sum_x \left(\phi_x^2\right)^k +$$

$$+ \sum_{\substack{k,l=0 \\ k+l\neq 0}}^{+\infty} \frac{(-1)^{k+l}\lambda^{k+l}}{2\cdot 4^{k+l}} \sum_{x,y} D_{xy}\left(\phi_x^2\right)^k \left(\phi_y^2\right)^l (\phi_x \cdot \phi_y)$$

We blindly do perturbation theory [with A.Davody]

Only cactus diagrams @ large N



Analytic structure of expansion

$$m^2 = m_0^2 z^2 - \frac{\lambda\xi}{2} z^2 + \frac{\lambda\xi}{2} z m^2 I_0(m)\,,$$
$$z = 1 + \frac{\lambda\xi}{4} z^2 I_0(m)\,,$$

$$I_0(m) \approx -\frac{1}{4\pi}\log\left(\frac{m_0^2}{32}\right)$$

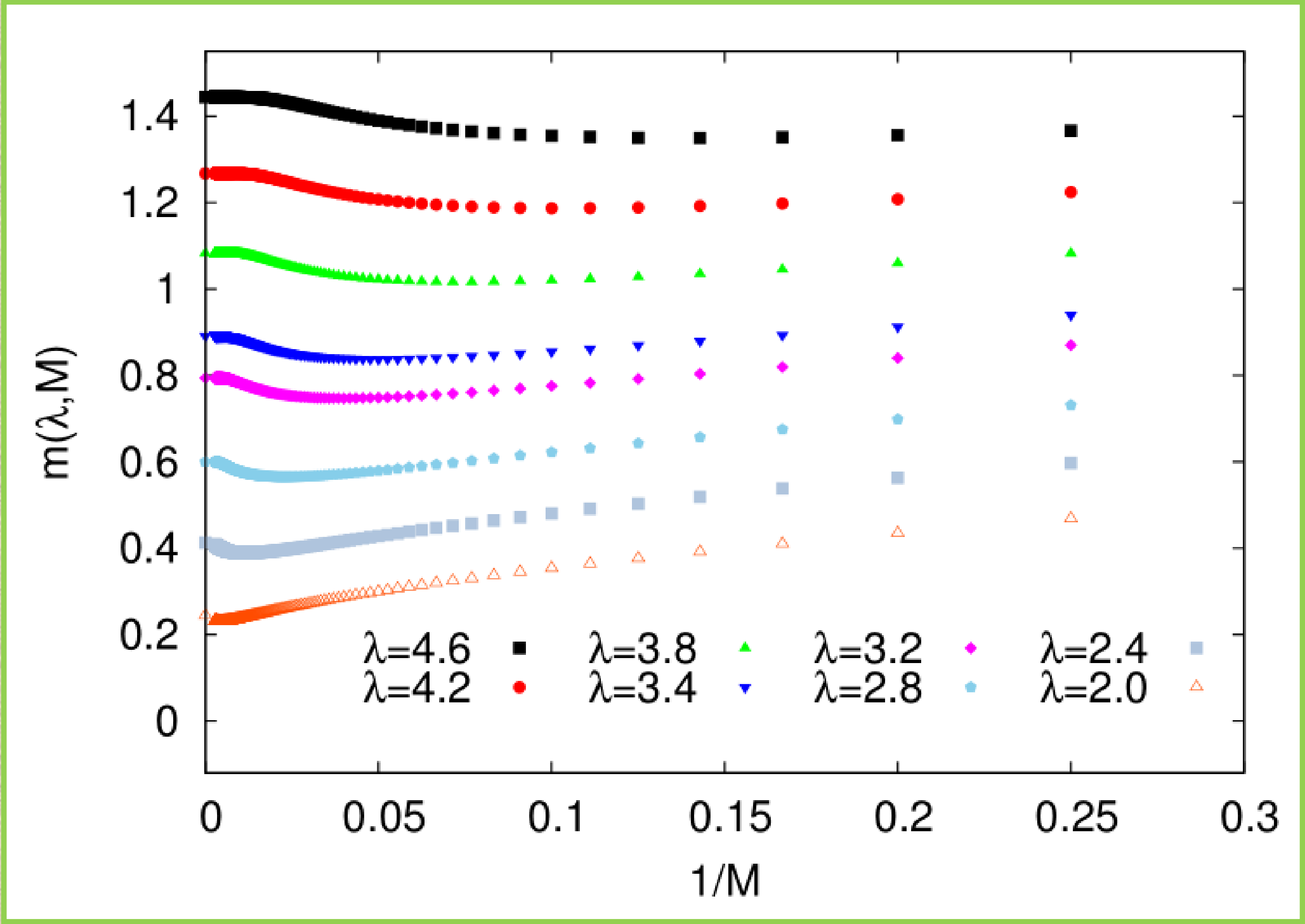
$$m^2 = m_0^2 + \lambda\,\xi\,\sigma_1(m_0) + \lambda^2\,\xi^2\,\sigma_2(m_0) + \ldots,$$
$$z = 1 + \lambda\,\xi\,z_1(m_0) + \lambda^2\,\xi^2\,z_2(m_0) + \ldots,$$

From our perturbative expansion we get

$$m^2 = \sum_{p,q=0}^{+\infty} c_{p,q}\,\lambda^p\,(\log\lambda)^q$$

$$\exp\left(-\frac{1}{\lambda}\right) = \exp\left(-e^{-\log(\lambda)}\right) = \sum_k c_k\,(\log\lambda)^k$$

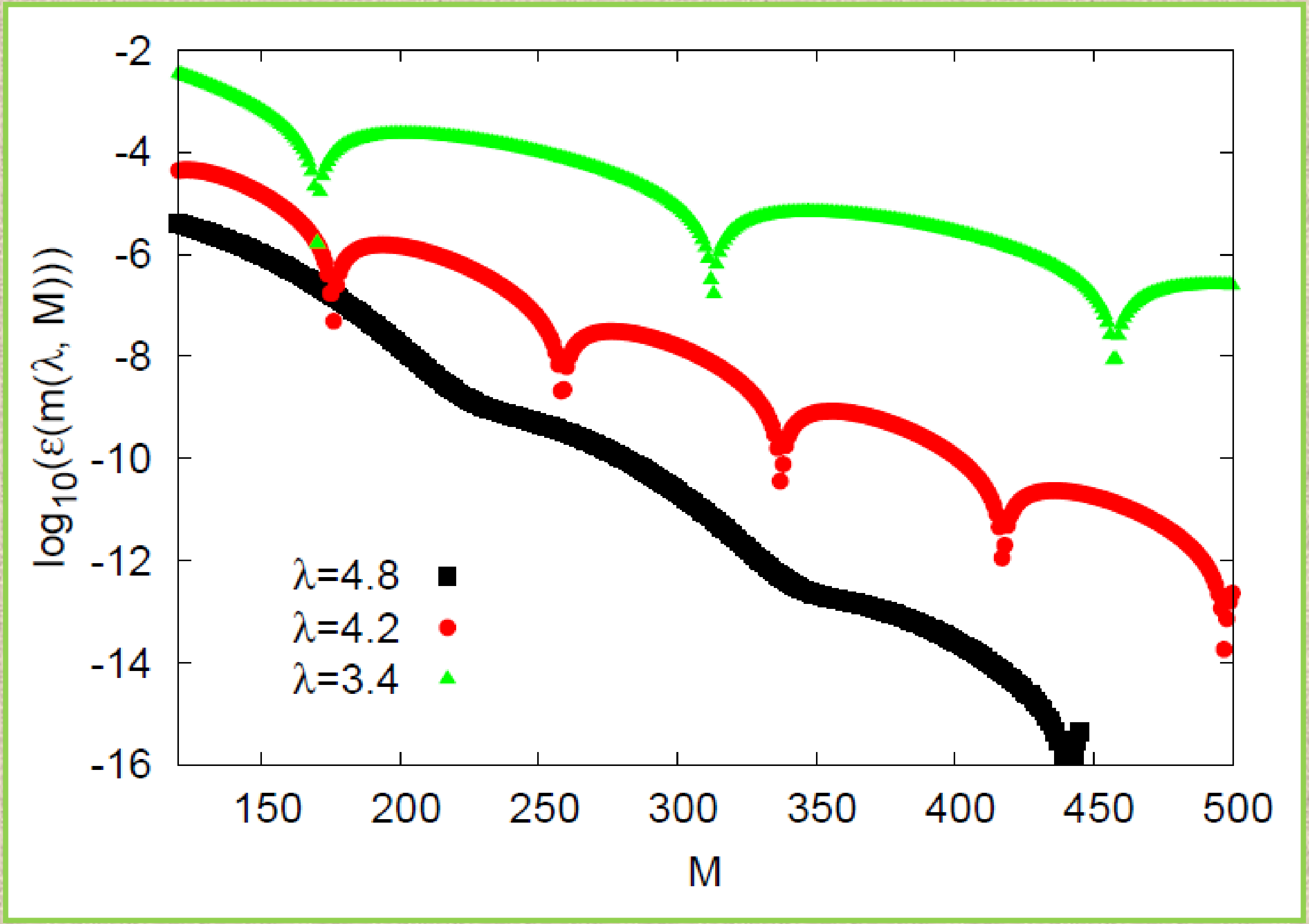
O(N) sigma model @ large N



Relative error of mass vs. order M

Numerical evidence of convergence!!!

O(N) sigma model @ large N

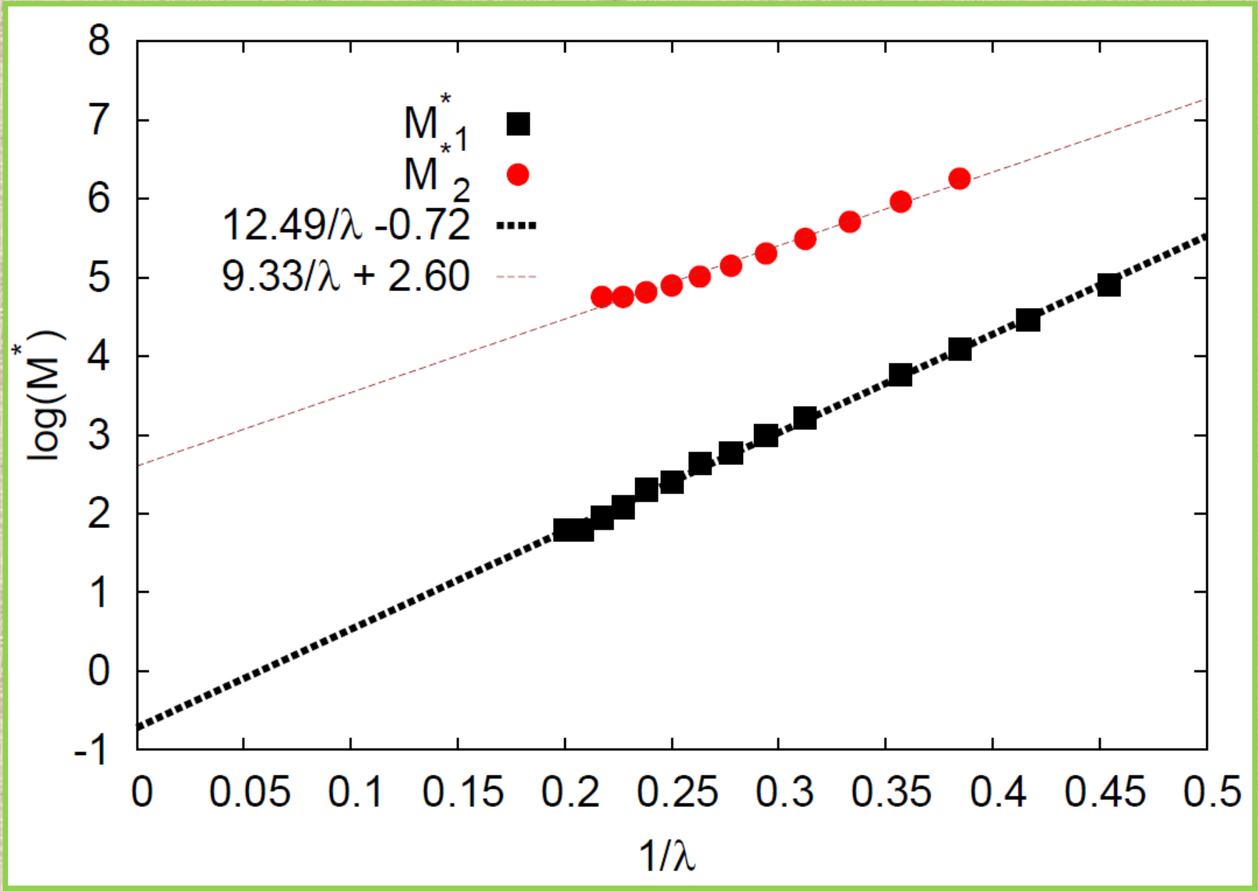


Relative error of mass vs. order M

Numerical evidence of convergence!!!

[Similar results by Chandrasekharan, 1709.06048]

O(N) sigma model @ large N



Convergence rate:
first extremum in ε(M)

Has physical scaling!!!

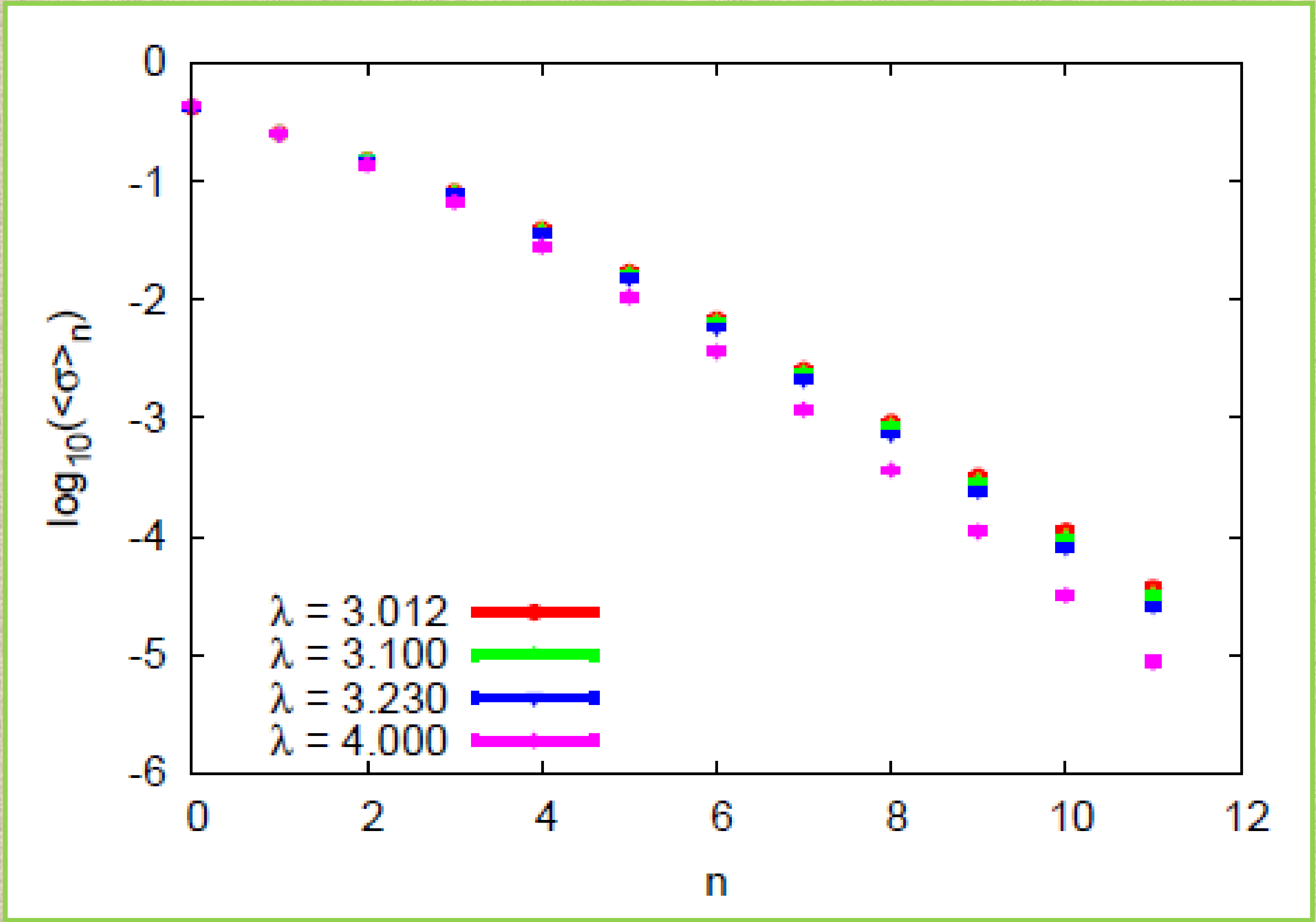
Similar to critical slowing-down in Monte-Carlo

$$M_1^\star \sim l_c^2 \sim m^{-2} \sim \exp\left(\frac{4\pi}{\lambda}\right)$$

Back to Principal Chiral Model

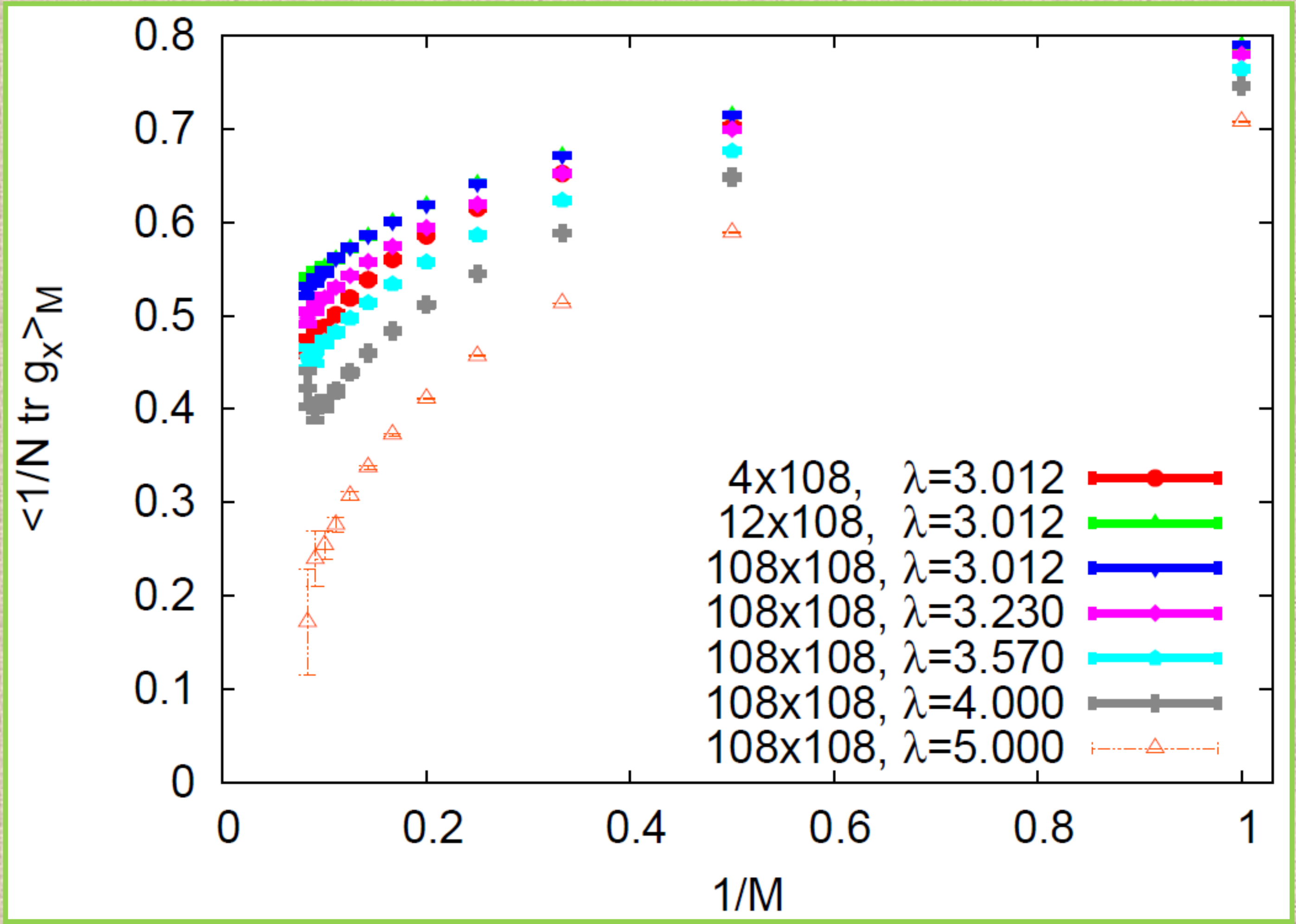
- Use DiagMC to generate (sample) expansion coefficients
- DiagMC based on Schwinger-Dyson equations [PB 1705.03368,1609.08833]
- Other methods, e.g. NSPT, possible
- BUT DiagMC works @ N->∞, V->∞
- Main problem: sign cancellations between different diagrams

Sign problem at high orders



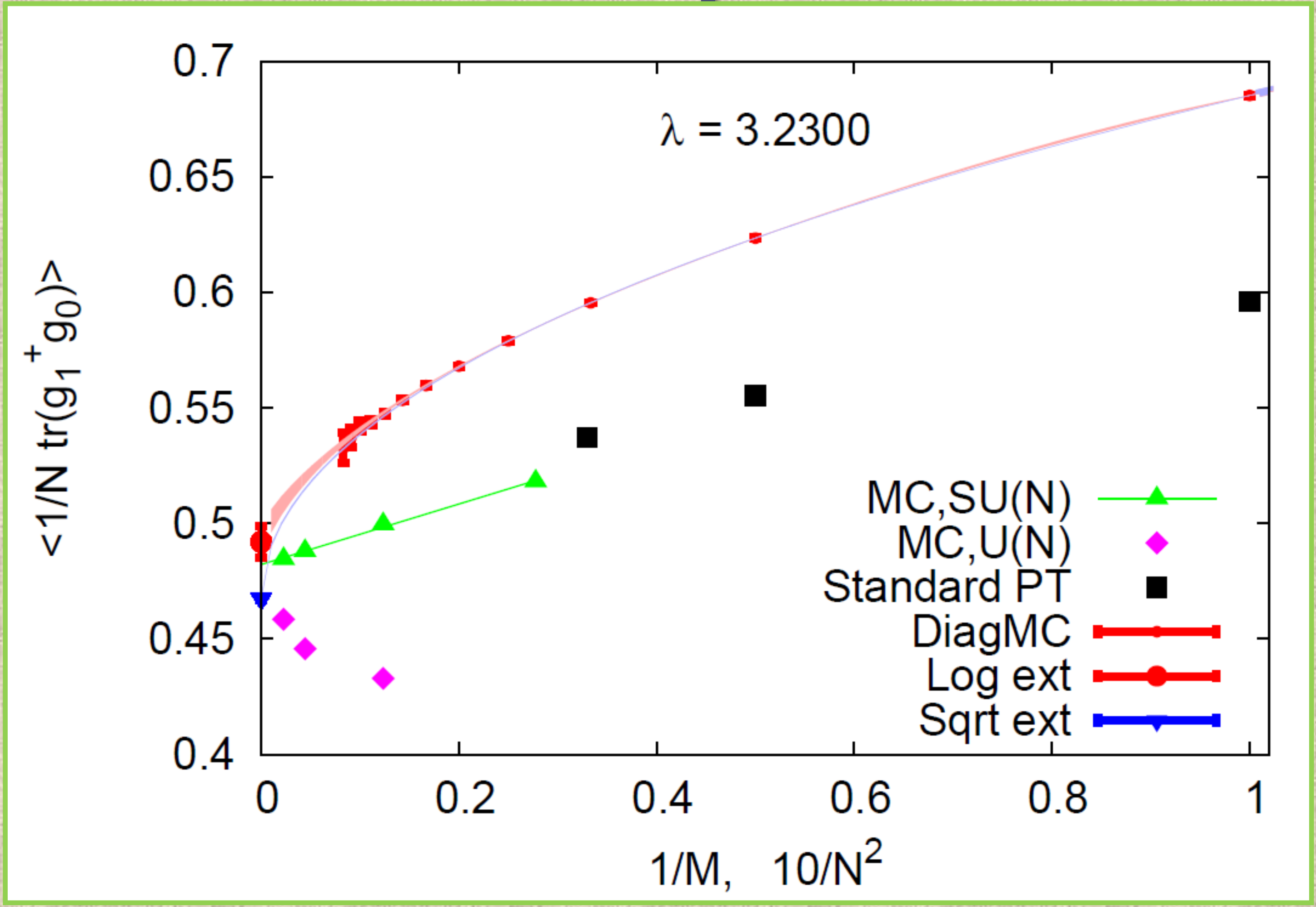
- Mean **sign decays exponentially** with order
- Limits practical simulations to **orders ~ 10**
- Sign problem depends on **spacing, not volume**

Restoration of SU(N)xSU(N) symmetry



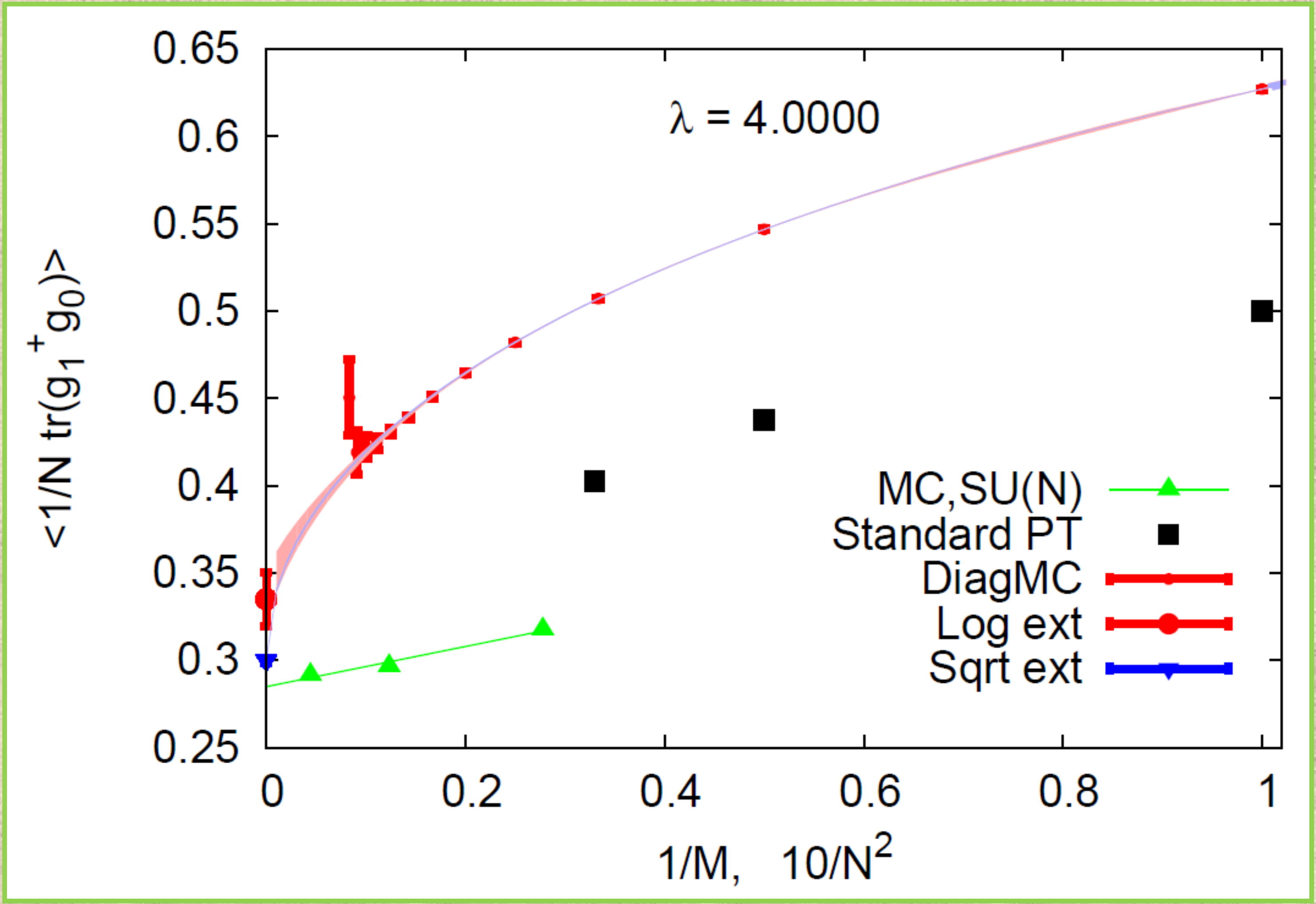
* Perturbative vacuum not SU(N)xSU(N) symm.
* Symmetry seems to be restored at high orders
* Restoration is rather slow

Mean link vs expansion order



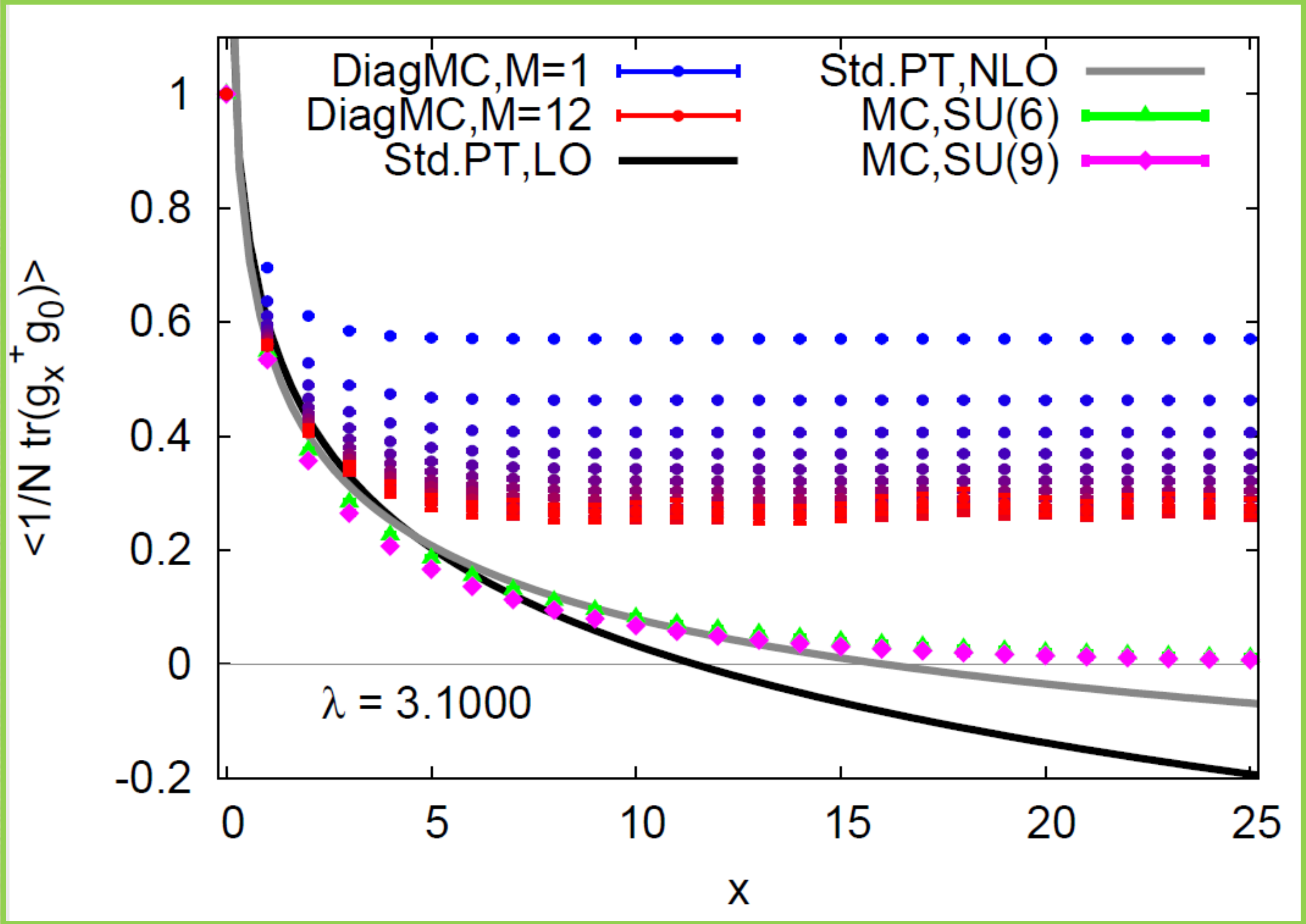
- **Good agreement with *N->∞ extrapolation***
- **Convergence slower than for standard PT**
- **MC Data from [Vicari, Rossi, Campostrini'94-95]**

Mean link vs expansion order



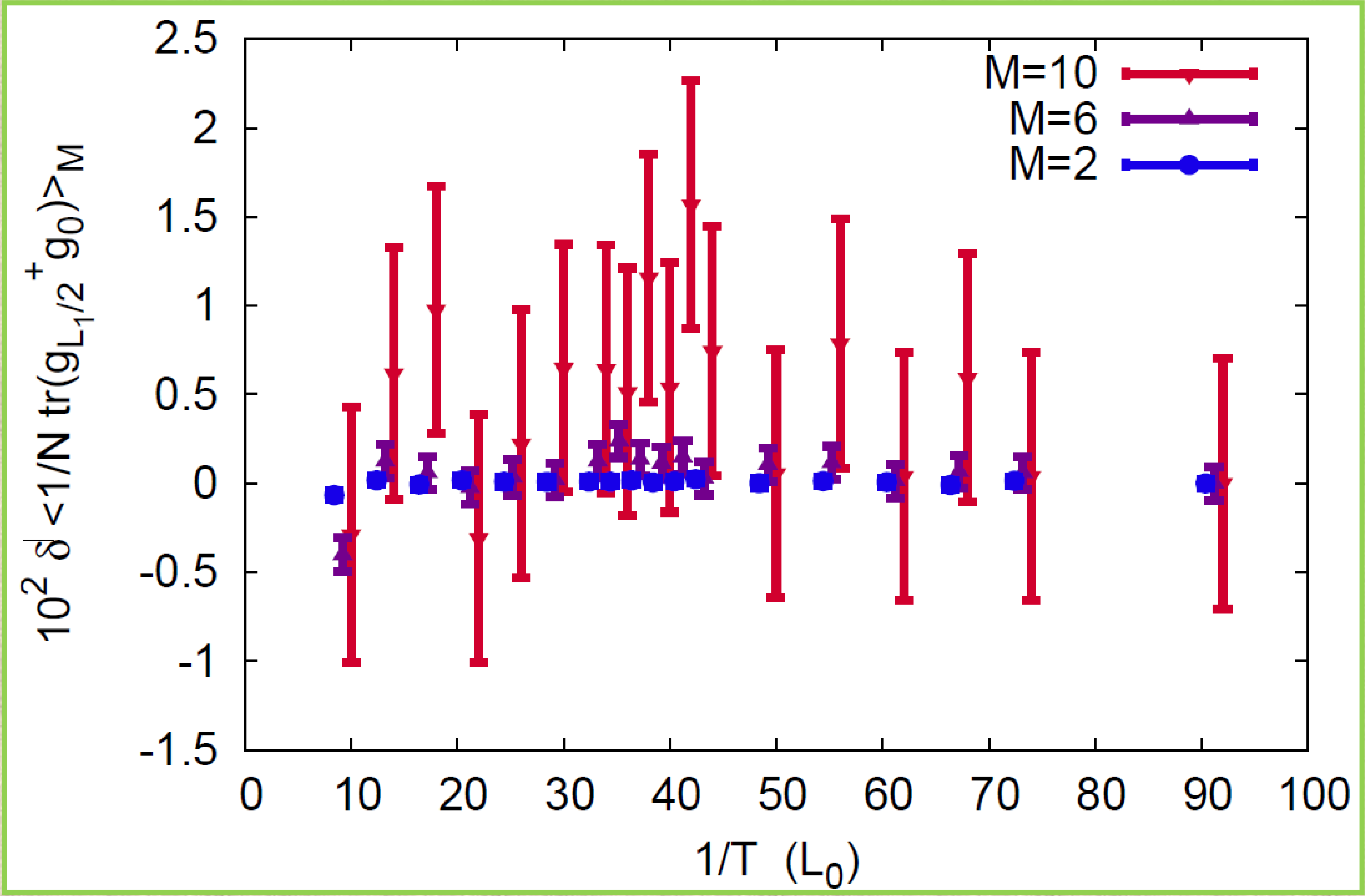
- **Wrong or no convergence after large-N phase transition (*$\lambda > \lambda_c = 3.27$*) [hep-lat/9412102]**

Long-distance correlators



- Constant values at large distances, consistent with ***$\langle 1/N\, \mathrm{tr}\, g_x \rangle > 0$***
- Converges slower than standart PT, but IR-finite

Finite temperature (phase) transition?



Weak enhancement of correlations at L0 ~ 35-40 [P.B.,Valgushev, 1706.08954]

Resume

Weak-coupling DiagMC in the large-N limit:

+ IR-finite, convergent series
+ Capture non-perturbative physics
+ Volume-independent algorithm

- Sign problem vs. Standard MC
- Slower convergence than standard PT
- Starts with symmetry-breaking vacuum
- Convergence not yet exponential, so no polynomial complexity solution

How to proceed?

Outlook

Resummation of logs:

$$\sum_{k,m} c_{km} \log(\lambda)^k \lambda^l \to \exp(-\beta_0/\lambda)$$

Easy in mean-field-approximation (for O(N) sigma-model just one exponent)

DiagMC with mean-field (Bold DiagMC)???

(Hope for much faster convergence)

Outlook

Bold DiagMC… Not easy in non-Abelian case

For O(N) model, the well-known lagrange multiplier trick

$$\mathcal{Z} = \int dg_x \int d\xi_x \exp\left(-\frac{N}{\lambda} \sum_{x \neq y} D_{xy} \mathrm{tr}\, \left(g_x^\dagger g_y \right) - \frac{iN}{\lambda} \sum_x \mathrm{tr}\, \left(\xi_x g_x^\dagger g_x - \xi_x \right) \right) =$$

$$= \int d\xi_x \exp\left(N \mathrm{tr}\, \ln\left(D_{xy} + i \xi_x \delta_{xy} \right) + \frac{iN}{\lambda} \sum_x \mathrm{tr}\, \xi_x \right)$$

… Matrix-valued Lagrange multipliers???

V Starts from the correct vacuum

? Formal expansion scheme possible, but seems not convergent …