Towards the Poles

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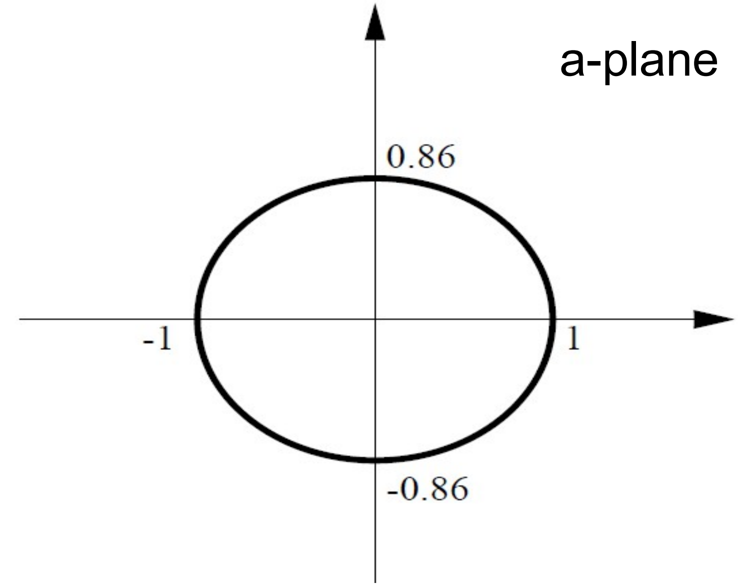
Summary on Seiberg-Witten solution — N=2 SYM

--- Wish to get the low-energy effective action. Example of SU(2) group. SU(2) is broken to U(1) by Higgs mechanism. Order parameter for the vacuum state U=<Trϕ^2>. How the effective action F(a) depends on the order parameter ?

--- Tools — RG flows and electro<-> magnetic duality. Perturbatively at large u all is under control due to asymptotic freedom. Small u — strong coupling, need to perform summation of instanton series(no antiinstantons)

--Naively the mass of W-boson vanishes at the origin since the classical Vev of Higgs vanishes. Wrong! Instead the instanton series which have naive poles at the origin of u plane disappears and there are two singular points connected by the curve of marginal stability where W boson decays into monopole and dyon. No W-boson at strong coupling at all.

Naive picture with
massless W-boson



Exact answer CMS

$$\phi = \tfrac{1}{2} a \sigma_3 .$$

$$\mathcal{F} = i \frac{1}{2\pi} \mathcal{A}^2 \ln \frac{\mathcal{A}^2}{\Lambda^2} + \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\mathcal{A}} \right)^{4k} \mathcal{A}^2$$

Information about low-energy effective action F(a) and spectrum
of BPS particles is encoded in Riemann surface. For SU(N) genus=N-1.

$$Z = a n_e + a_D n_m .$$

$$a_D = \oint_{\gamma_1} \lambda$$

$$a = \oint_{\gamma_2} \lambda .$$

$$\lambda = \frac{\sqrt{2}}{2\pi} \frac{dx \, \sqrt{x-u}}{\sqrt{x^2-1}} :$$

SW solution versus classical integrable systems

SUSY physical theory	Pure gauge SYM theory, gauge group G	SYM theory with adj. matter	SYM theory with fund. matter
4d	inhomogeneous periodic Toda chain for the dual affine $\widehat{\mathfrak{g}}^\vee$ (*non-periodic Toda chain*)	elliptic Calogero model (*trigonometric Calogero model*)	inhomogeneous periodic **XXX** spin chain (*non-periodic chain*)
5d	periodic relativistic Toda chain (*non-periodic chain*)	elliptic Ruijsenaars model (*trigonometric Ruijsenaars*)	periodic **XXZ** spin chain (*non-periodic chain*)
6d	periodic "Elliptic" Toda chain (*non-periodic chain*)	Dell system (*dual to elliptic Ruijsenaars, elliptic-trig.*)	periodic **XYZ** (elliptic) spin chain (*non-periodic chain*)

Krichever,Marshakov,Mironov,Morozov,A.G. 95, Donagi -Witten 95, Martinec-Wrner 95, MarshakovMironov,Morozov ,A.G 96

SW and classical integrable systems

– Seiberg-Witten curve — spectral curve for holomorphic integrable system. Values of all Hamiltonians are fixed

–

$$a_i = \oint_{A_i} dS,$$
$$a_D = \oint_{B_i} dS,$$

SW diferentials are action variables in the Integrable system

--- Moduli space of vacua -space of integrals of motion in integrable system

--- Coordinates X in the integrable system — positions of the defect branes(surface operators) in the internal space

Gukov,Mironov,A.G. 97

SW and Whitham dynamics

Second Hamiltonian system- Whitham hierarchy

Action

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \qquad \frac{\partial \mathcal{F}}{log \Lambda} = const H$$

momentum

coordinate

Time variable

Equation of motion in the Whitham system - Ward identity in the gauge theory

GKMMM 95, Edelstein-Marino-Mas 96

Whitham dynamics goes across the spectrum of the first Hamiltonian system. RG flows. In quantum mechanics the equation of motion in Whitham system ----P\NP relation(for genus one checked)

Nekrasov partition function and NS limit

--- Nekrasov (2004) evaluated the low-energy effective action introducing the Omega-background with two deformation parameters. The prepotenial -weighted sum over the equivariant volumes of the instanton moduli space

$$Z(a,\epsilon_1,\epsilon_2;q)=\exp\left(\frac{\mathcal{F}^{inst}(a,\epsilon_1,\epsilon_2;q)}{\epsilon_1\epsilon_2}\right)$$

$$Z(\epsilon_2 \doteq 0.\text{-}\hbar;q)=\sum_{\vec{\mathrm{k}}} q^{|\mathbf{k}|}\prod_{(l,i)\neq(n,j)}\frac{a_{ln}+\hbar\left(k_{l,i}-k_{n,j}+j-i\right)}{a_{ln}+\hbar\left(j-i\right)}$$

- Nekrasov-Shatashvili limit $\epsilon_2=0.$ relates the prepotential F with the twisted superpotential in 2D theory upon reduction.Similar to the reduction to the lowest Landau level in magnetic field. Discrete set of vacua a=k.. ϵ_1

$$W(\mathrm{a},\hbar)=\frac{1}{\hbar}\mathcal{F}^{pert.}(\mathrm{a},\hbar)+\frac{1}{\hbar}\mathcal{F}^{inst.}(\mathrm{a},\hbar)$$

$$\mathcal{F}(\mathbf{a},\hbar)=\lim_{\varepsilon_2\to 0}\left(-\hbar\varepsilon_2\log Z(\mathbf{a},\hbar,\varepsilon_2)\right)$$

NS limit and quantum integrable systems

– What happens with 2 integrable systems in NS limit? One of them Calogero-Toda-spin chain gets quantized with ...is the Planck constant. The second Whitham system remains classical but its Hamiltonian gets deformed

---Twisted superpotential in 2d = Yang-Yang function in the corresponding quantum integrable system with finite degress of freedom. (NS-2009). BA equations in integrable system coincide with the equation defining ground state

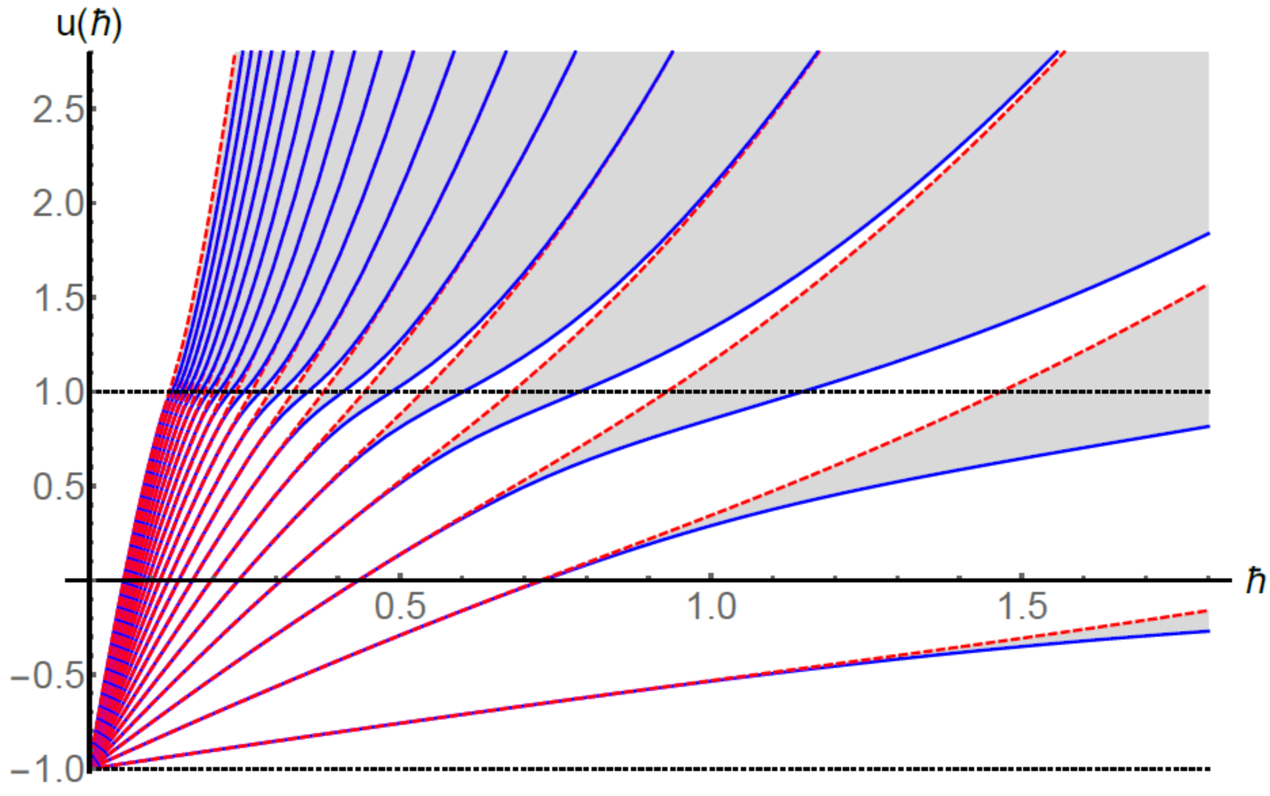
--There are naive poles in the twisted superpotential W(a,Lambda,e) at a=ke — What is the meaning of these poles if any?

--What is the meaning of the QM tunneling on the gauge theory side?

Pure N=2 SYM in NS limit

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\Lambda^2\cos(x)\psi=u\,\psi$$

$$\psi(z+\pi)=e^{i\theta}\,\psi(z),$$



Level splitting at small u and small gaps at large u — nonperturbative phenomena due to instantons-antiinstantons

New tools and questions

--- Nonperturbative exact WKB quantization

Zinn-Justin-Jentschura, Dunne-Unsal, Codesido-Marino-Zakany, Kashani-Poor,
Krefl, Schiappa,Vonk,
Mironov-Morozov, Sakai et al + many others

--Resurgence in general versus complex saddles

Math+Phys--- fractional instantons, Lefshetz thimbles, top strings, matrix models.....

---P/NP relation

Dunne-Unsal-Basar — explicit+ quantum geometry , Marino et al,- Picard-Lefshetz equation,
Milekhin-A.G — equation of motion in Whitham

--- Question concerning level splitting versus poles in the superpotential
Nekrasov talks, Jeong, .Marino et al

Pole structure in SU(2) pure SYM

$$\mathcal{W}(a,\hbar) = \frac{1}{\hbar}\mathcal{F}^{pert.}(a,\hbar) + \frac{1}{\hbar}\sum_{k=1}^{\infty} F_k(s)\hbar^{2-4k}\Lambda^{4k}$$

$$s = \frac{2a}{\hbar}$$

$F_k(s)$ Has poles at s=n $-k \leq n \leq k.$

$$F_k(s) = \frac{c_1^{(k)}}{(1-s)^{2k-1}} + \frac{c_2^{(k)}}{(1-s)^{2k-3}} + \ldots + \frac{c_k^{(k)}}{(1-s)} + F_k^{reg.}(s)$$

$$\mathcal{F}^{sing.} = \hbar^2 \sum_{k=1}^{\infty} g_k^{(1)} \left(\frac{q}{1-s}\right) q^{2k-1}$$

$q = (\Lambda/\hbar)^2.$ Is small parameter

Mathematical reason — no longer isolated fixed points.

Pole structure in SU(2) pure SYM

$$g_1^{(1)}(z) = -z\,{}_3F_2(1/2,1,1;2,2;-4z^2) = -\frac{\sqrt{4z^2+1}-1-\log\frac{1+\sqrt{4z^2+1}}{2}}{z} \qquad z = \frac{q}{1-s}$$

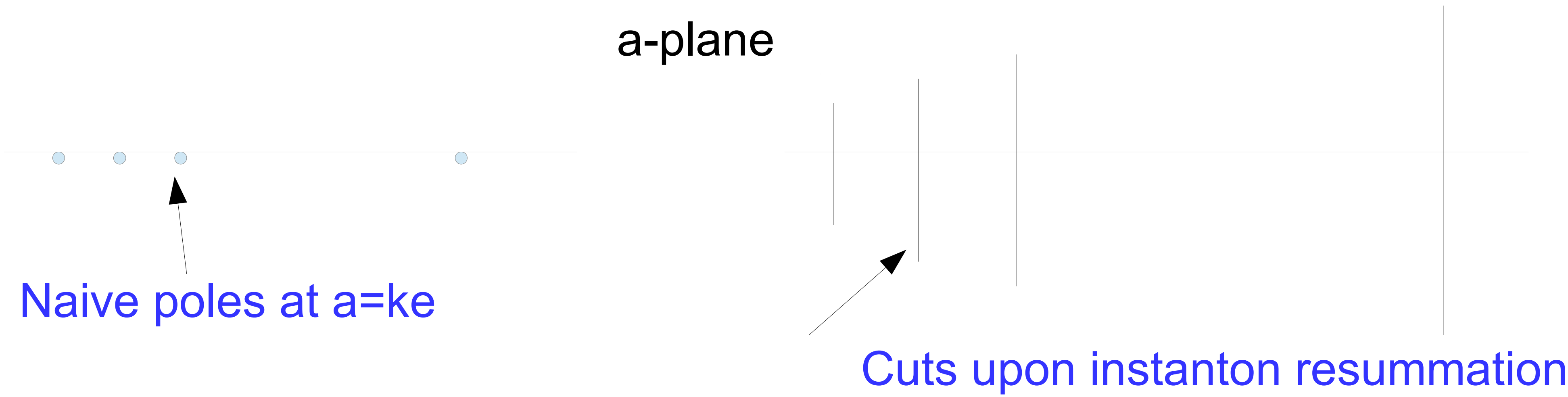
Resummed instanton series near the first pole

$$g_1^{(n)}(z) = -\frac{\sqrt{1+\frac{4z^2}{(n!(n-1)!)^2}} - \log\left(\frac{1+\sqrt{\frac{4z^2}{(n!(n-1)!)^2}}}{2}\right) - 1}{z}$$

Resummed instanton series near the n-th pole. Exact answer at large Planck constant.

Poles disappears upon the summation over all instantons! We get cuts instead. (also Beccaria,16 for special case of one-gap Lame)

At k-th naive pole a=ke, the W-boson with angular momentum k becomes massless. This never happens upon the instanton resummation. Very similar to a=0 point in the SW solution.



Surprise:

at $\Lambda^4 \geq (2a-\hbar)^2\hbar^2$ We obtain Λ^2. terms

No naive interpretation since it is highly quantum region. However these terms resemble contribution of «fractional instanton» or «IR renormalon».

Universality. The function g(z) near the first pole has the same form at large Planck constant for all asymptotically free SU(2) SYM theories with different matter content.

Local 2d models near the poles

$$-\hbar^{-1}\mathcal{W}(x) = q\left(g^{pert.} + g_1\right) = q\sqrt{x^2+4} - xq\,\mathrm{arcsinh}(x/2) - xq\log q.$$

Twisted superpotential with the perturbative contribution added

$$-\hbar^{-1}\mathcal{W} = (qx)Y + e^{-Y} + q^2 e^{Y}.$$

Our twisted superpotential in the mirror description- CP(1) model with twisted mass term.(Hori-Vafa 2000)

2 vacua in the CP(1) model differed by the order parameter .

$$M_{sol.} = \frac{\pi}{2}\frac{\Lambda^2}{\hbar}\left|\sqrt{x^2+4} + \frac{x}{2}\log\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right|.$$

(Dorey,98)

Local model near the poles

2d $\mathcal{N} = (2,2)$ chiral doublet theory coupled to 4d $\mathcal{N} = 2$ SU(2) theory

(Gaiotto-Gukov-Seiberg,13)

Surprizingly similar situation. In their case theory on the 2d defect,in our case on reduced 4d theory in Omega-background

$$\mathcal{W} = -\langle \mathrm{Tr}(m+\Phi)\left(\log(m+\Phi) - 1\right)\rangle_{4d}$$

Superpotential at the defect

$$R(x) = \mathrm{Tr}\,\frac{1}{x+\Phi} = \frac{2x}{\sqrt{(x^2-u)^2 - 4\Lambda^2}}$$

Averaging over 4d theory via the resolvent trick

$$-\partial_m \mathcal{W} \approx \log \Lambda^2 + \mathrm{arccosh}\,\frac{m-a}{\Lambda^2/a}.$$

Twisted mass

$$\mathcal{W} = (m-a)Y + e^{-Y} - \left(\frac{\Lambda}{2a}\right)^2 e^{Y}$$

CP(1) superpotential at the defect

CMS near cuts

$$\{Q_\alpha, Q_\beta^\dagger\}(\gamma^0)_{\beta\gamma} = 2\left[P_\mu\gamma^\mu + Z\,\frac{1-\gamma_5}{2} + Z^\dagger\,\frac{1+\gamma_5}{2}\right]_{\alpha\gamma},$$

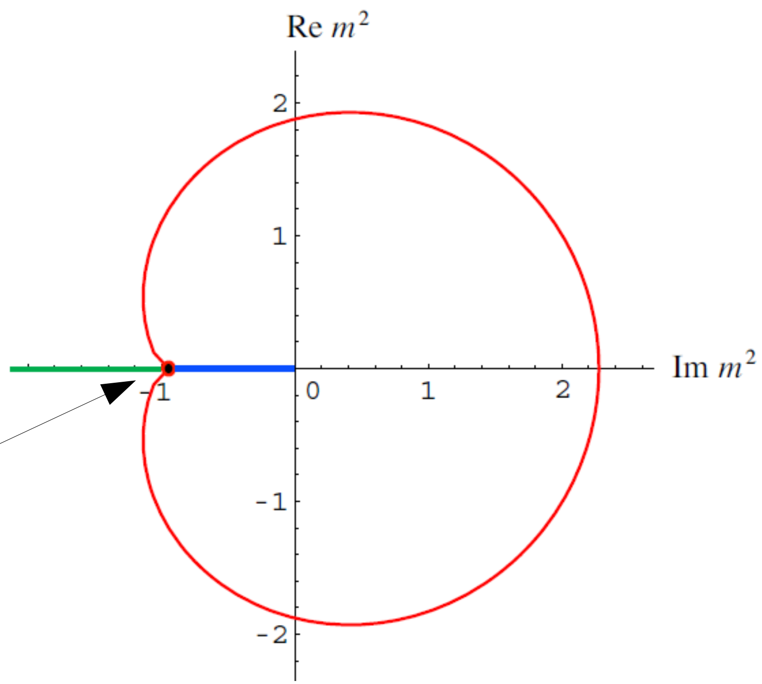
$$\mathcal{J}_\mu^a = G_{i\bar{j}}\left[\phi^{\dagger\bar{j}}T^a i\overleftrightarrow{\partial}_\mu\phi^i + \bar{\psi}^{\bar{j}}T^a\gamma_\mu\left(\psi^i + \Gamma^i_{lk}\phi^l\psi^k\right)\right].$$

Noether global charge

$$Z = \sum_{a=1}^{r} m_a\, q^a - i\int dz\,\partial_z\, O\,.$$

Topological charge

$$O_{\rm canon} = \sum_{a=1}^{r} m_a D^a(\phi, \phi^\dagger)$$



AD point

$$\langle Z\rangle_{q,T} = q\,m + T\,m_D\,,$$

Dorey 98,
Shifman,Vainshtein,Zwicky 06

$$\mathrm{Re}\left\{\log\frac{1+\sqrt{1+4\Lambda^2/m^2}}{1-\sqrt{1+4\Lambda^2/m^2}}-2\sqrt{1+4\Lambda^2/m^2}\right\}=0\,.$$

Equation for CMS. No stable particle without topological charge inside CMS.

$$m^2 = -4\Lambda^2.$$

Argyres-Douglas point in 2d theory

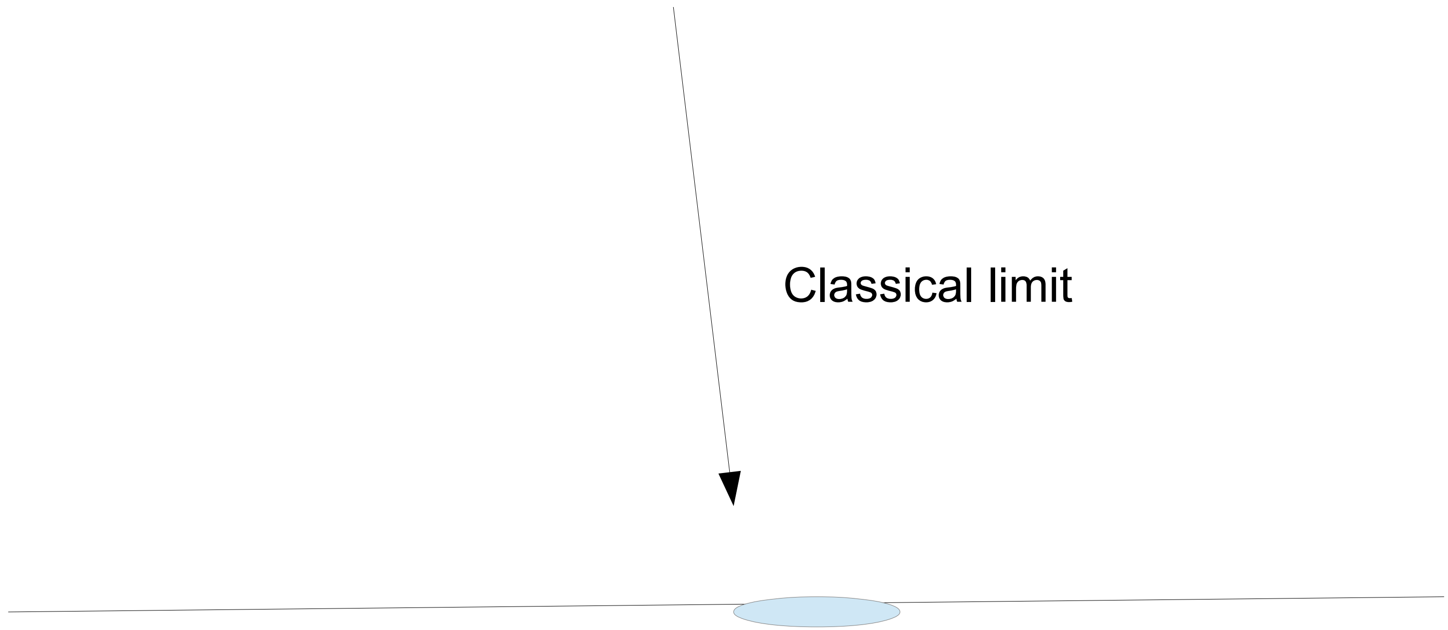
Twisted mass in our case → $\hbar - 2a$

W-bosons decays at the CMS in our local model near the pole!.

At AD point in CP(1) model there is the collision of two vacua. The masses of solitons vanishes. The superconformal theory Is expected.

a-plane

Infinite number of CMS in infinite number of CP(1) models around each naive pole at large Planck constant limit.



Infinite number of CMS in the NS limit somehow have to be glued together in the classical SW limit into the single CMS.

Back to quantum mechanics

$$u = \frac{\hbar^2}{4} q \frac{\partial \mathcal{F}}{\partial q} - \frac{\hbar^2}{48}$$

Quantum Matone relation

$$\text{Type } A: \quad \frac{a_D}{\hbar} = \frac{\partial \mathcal{W}(a,\hbar)}{\partial a} = k, \quad k \in \mathbb{Z}$$

$$\text{Type } B: \quad \frac{a}{\hbar} = k + \frac{\theta}{2\pi}, \quad k \in \mathbb{Z}, \quad \theta \in [0, 2\pi)$$

Two types of quantization in QM implied by the Omega-background (Nekrasov-Witten 2010). Dirichlet or Newmann boundary conditions.

We consider Type B quantization

Back to quantum mechanics

$2a = n\hbar, n \in \mathbb{Z}$

Poles in Nekrasov function correspond to WKB quantization condition

Resummation of all instantons

$$\frac{4\left(u_1^+ - u_1^-\right)}{\hbar^2} = \Lambda \frac{\partial \mathcal{F}_{s=1}^{sing.}}{\partial \Lambda}$$

Width of the 1-th gap

$$\frac{4\left(u_1^+ + u_1^-\right)}{\hbar^2} = \Lambda \frac{\partial\left(\mathcal{F}^{pert.} + \mathcal{F}_{s=1}^{reg.}\right)}{\partial \Lambda} - \frac{1}{6}$$

Middle of the 1-th gap

Tunneling phenomena in QM correspond to the account of solitons in 2D local CP(1) model near the pole!

Schwinger-like process?

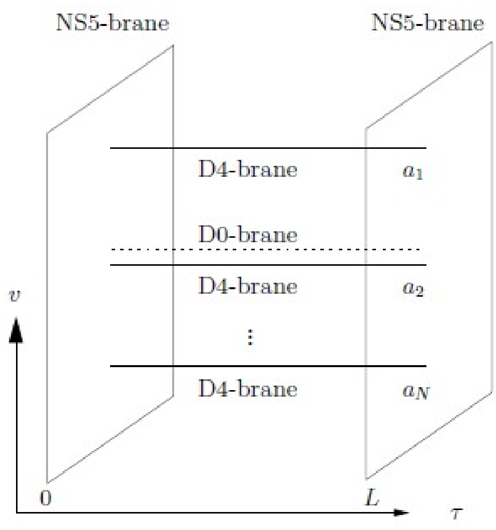
$$\Delta u = \frac{\hbar}{\pi}\frac{\partial u}{\partial a}\arctan\left(e^{-\frac{2\pi}{\hbar}\operatorname{Im} a_D}\right) = \frac{\hbar}{\pi}\frac{\partial u}{\partial a}\sum_{n=1,3,5,\dots}\frac{1}{n}e^{-\frac{2\pi n}{\hbar}\operatorname{Im} a_D}$$

Using the quantum Matone relation we get

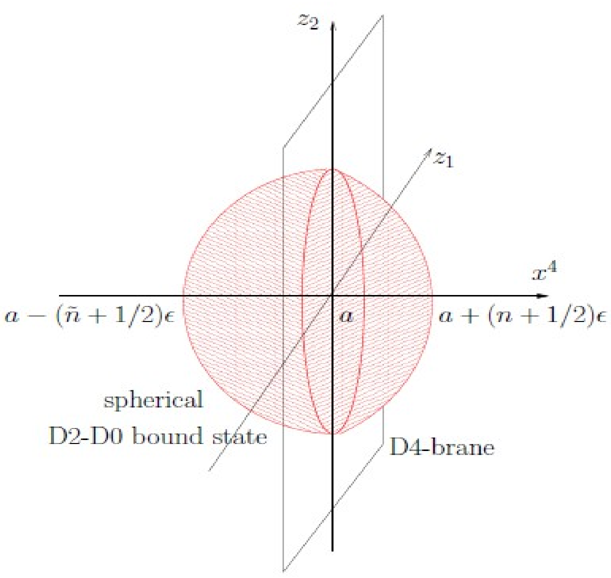
$$\Delta F = \frac{1}{16\pi^2}\sum_{n=1,3,5,\dots}\frac{1}{n^2}e^{-\frac{2\pi n}{\hbar}\operatorname{Im} a_D}$$

It looks like a monopole-pair creation in the Omega-background.
Where is comes from? Schwinger process corresponds to the account of instanton-antiinstanton pairs while we consider Nekrasov partition function for the instantons only
Possible relation with Schwinger mentioned in (Dunne-Basar,15)

Conjecture



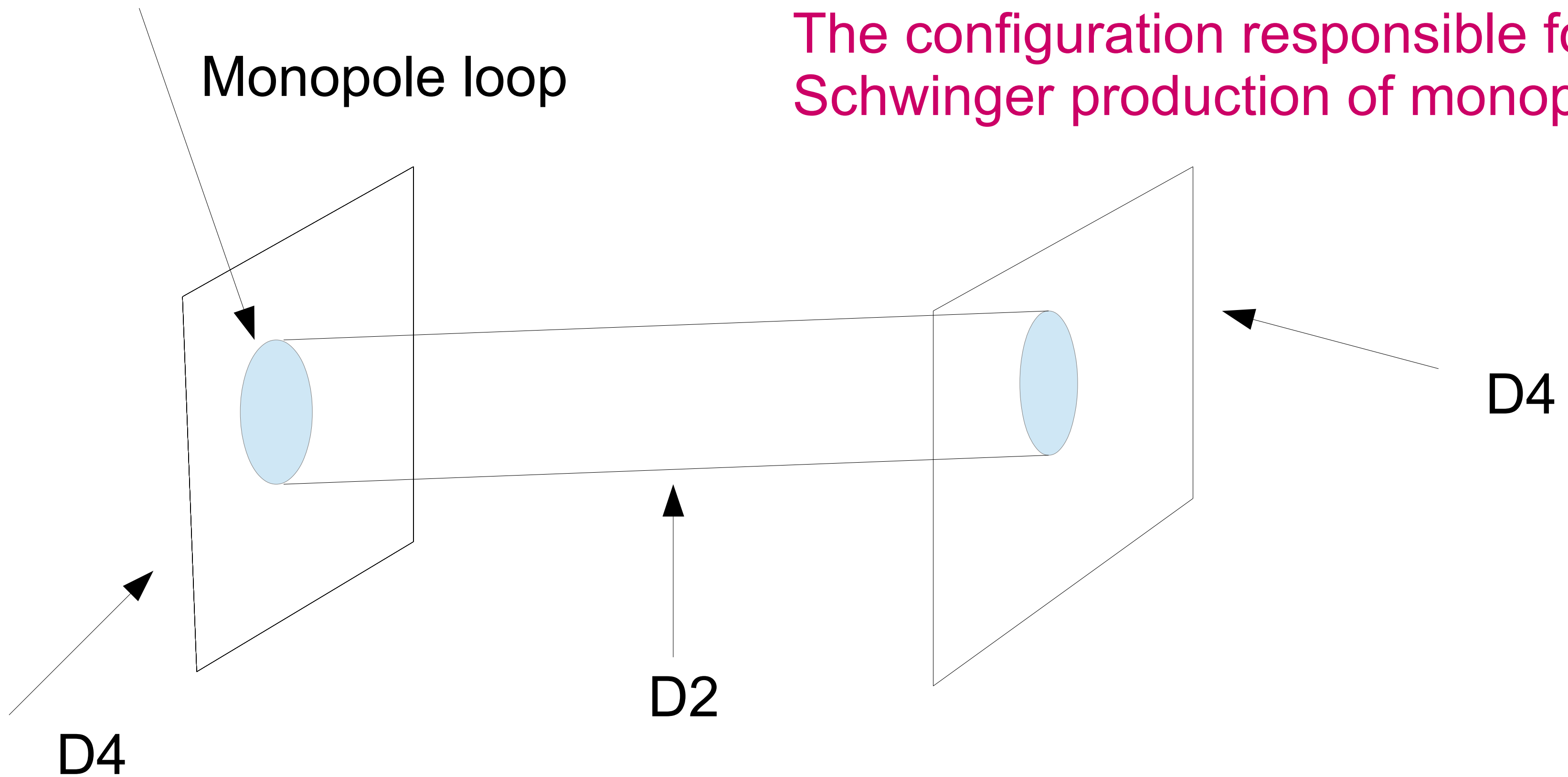
Whithout deformation



With selfdual Omega-deformation
Spherical D2 branes emerge
Due to the Myers effect!

(Matsuura,08)

The configuration responsible for the Schwinger production of monopole pair



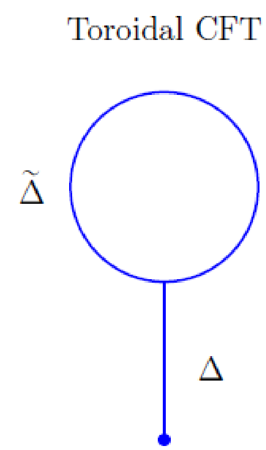
Very similar situation happens in selfdual Omega-background
At positions of the poles of Nekrasov function! At
a=ne spherical D2 brane from on D4 brane touches the
neibohr D4 brane.

This needs to proper identication in NS limit.

Poles in Classical Liouville theory

According to AGT NS limit corresponds to the semiclassical limit In Liouville theory $c \to -\infty$

$$\mathcal{F}_{c,\Delta}(\Lambda) = Z_{nek}(\Lambda, a, \epsilon_1, \epsilon_2)$$



At the limit $\epsilon_2 \to 0$

$$Z_{nek}(\Lambda, a, \epsilon_1, \epsilon_2) \to \exp((\epsilon_2^{-1} W(\Lambda, a, \epsilon_1))$$

$$\mathcal{F}_{c,\Delta}(\Lambda) \to \exp(\frac{1}{b^2} f_\delta(\frac{\Lambda}{\epsilon_1})), \qquad \Delta = b^{-2}(\frac{1}{4} - \frac{a^2}{\epsilon_1}^2)$$

Twisted superpotential = classical conformal block.

Poles in Classical Liouville Theory

Twisted superpotential in pure N=2 corresponds to the irregular conformal block. Norm of the coherent Gaiotto state.

$$\mathcal{F}_{c,\Delta}(\Lambda) =< \Delta, \Lambda^2|\Delta, \Lambda^2 >$$

$$< \Delta_1, \Lambda^2|V_+(z)|\Delta_2, \Lambda^2 > \to z^{\Delta_1-\Delta_+-\Delta_2}\phi(\frac{\Lambda}{\epsilon}, z)exp(\frac{1}{b^2}f_\delta(\frac{\Lambda}{\epsilon_1}))$$

Solution to the Matheau eq.
Is the semiclassical Limit of decoupling equation

$$E = 4r^2 - \Lambda\frac{\partial}{\partial\Lambda}f_\delta(\frac{\Lambda}{\epsilon})$$

Fateev,Litvinov 09, Litvinov,Nekrasov,Lukyanov, Zamolodchikov 11

There are naive poles in the conformal blocks at fixed values of the intermediate classical conformal dimensions. They are artefact of inproper expansion and disappear upon the «instanton resummation».

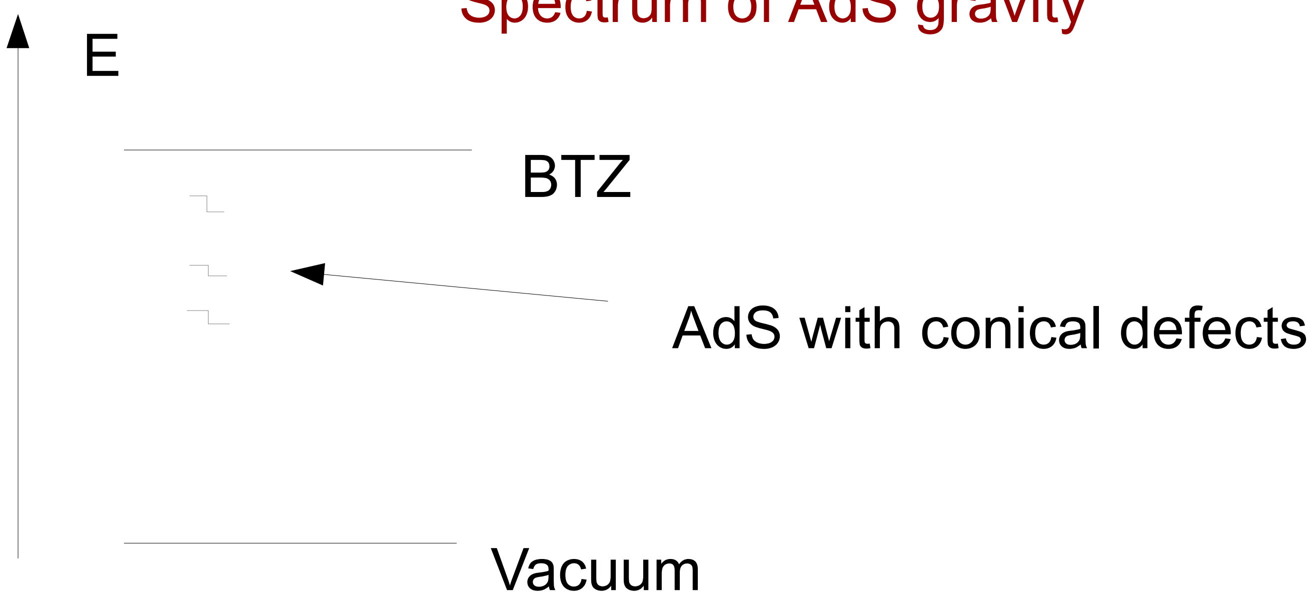
Holography. Poles as AdS3 modes

According to holography the semiclassical limit of Liouville theory
Is described via AdS3 gravity.

Heavy operators (dimension proportional to c) at the boundary yield the defects like BTZ black holes in the bulk.Light operators correspond to the particle moving along geodesics.

Spectrum of AdS gravity

Fitzpatrick,Kaplan, Belavin, Alkalaev,
+ many others



Holography.Poles in AdS3

---There are spectrum of linearized excitations around the BTZ Black hole.

---The poles in the Liouville conformal blocks correspond to the quasinormal modes around the BTZ black hole.

Verlinde,Tnsuri16, Kaplan et al 16

--- The conformal block in our case corresposponding to the pure N=2 superpotential corresponds to the classical conformal dimension above the BH threshold. Hence we are near the BH geometry. The poles correspond to the quasinormal modes therefore.

– Hence we conjecture from the our analysis that the naive quasinormal mode gets modified upon the resummation of the nonperturbative effects. There is a kind of splitting of quisinormal mode?

Conclusion

- A life is simplier - no naive poles and massless W-bosons. It is harder- rich nonperturbative structure in NS limit instead, parallel to QM.
- Appearence of the «fractional instantons» upon resummation of the ordinary ones
- Tunneling phenomena in QM get mapped into the QFT nonperturbative physics highly nontrivially. Solitons in CP1, CMS near cuts etc
- Very intriguing relation to the «nonperturbative physics» of the quasinormal BTZ BH modes

Open questions

- Generalization for higher rank. «Higher genus» QM systems.Only a few examples yet.
- Follow the transition from the large Planck constant to the «classical» Seiberg-Witten limit. How all CMS around the cuts condence into the single CMS in the classical limit?
- Understand better the nonperturbative phenomena around BTZ BH
- To prove the Schwinger mechanism for level splitting

Open questions

- Generalization to 5 d and 6d theories.
- How the dualities between the quantum integrable systems act on the near-pole structure?
- There is the instanton-torus knot duality in 5d SQCD(Milekhin,Sopenko,A.G.15-16). The sum over instantons get transformed into the sum over the torus knots. How this structure is seen in the local model?