Trans-series & hydrodynamics far from equilibrium

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many works, but see

1610.02023 [hep-th] lecture notes

1707.02282 [hep-ph] review with Florkowski and Spalinski

Introduction

Motivation

experiment (2000 ++):

ultrarelativistic heavy-ion collisions at RHIC & LHC

pheno:

hydrodynamic description of $\langle T^{\mu\nu} \rangle$

microscopics:

QCD $\approx$ holography ($\mathcal{N}$=4 SYM)

What is hydrodynamics and when does it work?

Textbook definition of relativistic hydrodynamics

hydrodynamics is an EFT of the slow (?) evolution of conserved currents in collective media close to equilibrium (?)

DOFs: always local energy density ϵ and local flow velocity u^μ $(u_\nu u^\nu = -1)$

EOMs: conservation eqns $\nabla_\mu \langle T^{\mu\nu} \rangle = 0$ for $\langle T^{\mu\nu} \rangle$ expanded in gradients

$$\langle T^{\mu\nu} \rangle = \epsilon\, u^\mu u^\nu + P(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\} \overbrace{- \eta(\epsilon)\, \sigma^{\mu\nu} - \zeta(\epsilon)\{\, g^{\mu\nu} + u^\mu u^\nu \,\}(\nabla \cdot u) \boxed{+ \ldots}}^{\pi^{\mu\nu}}$$

microscopic input:

EoS
$\left(P(\epsilon) = \frac{1}{3}\epsilon \text{ for CFTs}\right)$

shear viscosity contribution

bulk viscosity
(vanishes for CFTs)

This talk: behaviour of the gradient expansion at large orders in the number of ∇

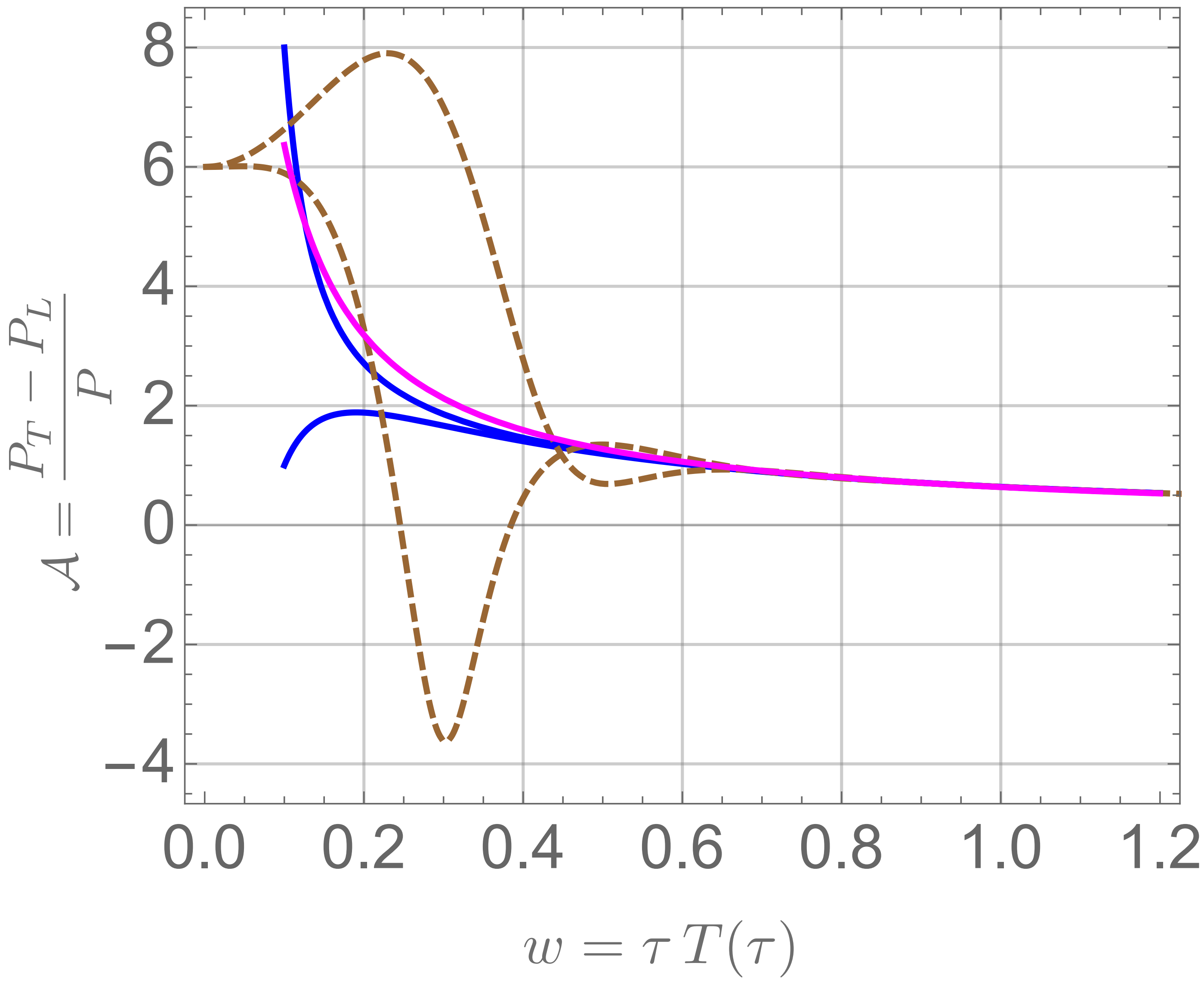
In practical applications one encapsulates part of this info in an EOM for $\pi^{\mu\nu}$, e.g.

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha \pi^{\mu\nu} + \lambda_1 \pi^{\langle\mu}{}_\alpha \sigma^{\nu\rangle\alpha} + \lambda_2 \pi^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} + \lambda_3 \Omega^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha}$$

Baier-Romatschke-Son-Starinets-Stephanov 0712.2451

Hydrodynamics far from equilibrium

0906.4426, 1011.3562 by Chesler & Yaffe; **1103.3452** with Janik & Witaszczyk



N=4 SYM

BRSSS

$$-\eta\,\sigma^{\mu\nu} \longrightarrow \frac{\Delta\mathcal{P}}{\mathcal{E}/3} = \frac{2}{\pi} w^{-1}$$

Viscous hydrodynamics works despite huge anisotropy in the system:

hydrodynamization $\neq$ local thermalization

Hydrodynamic & transient modes

Modes in BRSSS theory

Mode = solution of linearized equations of finite-T theory without any sources

Technical issue: tensor perturbs. ⟶ channels (**here everywhere sound channel**):

Assuming momentum along x^3 direction $e^{-i\,\omega\,x^0 + i\,k\,x^3}$: δT, δu^3 & $\delta\pi^{33}$

conservation

$+$

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha \pi^{\mu\nu} + \lambda_1 \pi^{\langle\mu}{}_\alpha \pi^{\nu\rangle\alpha} + \lambda_2 \pi^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} + \lambda_3 \Omega^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha}$$

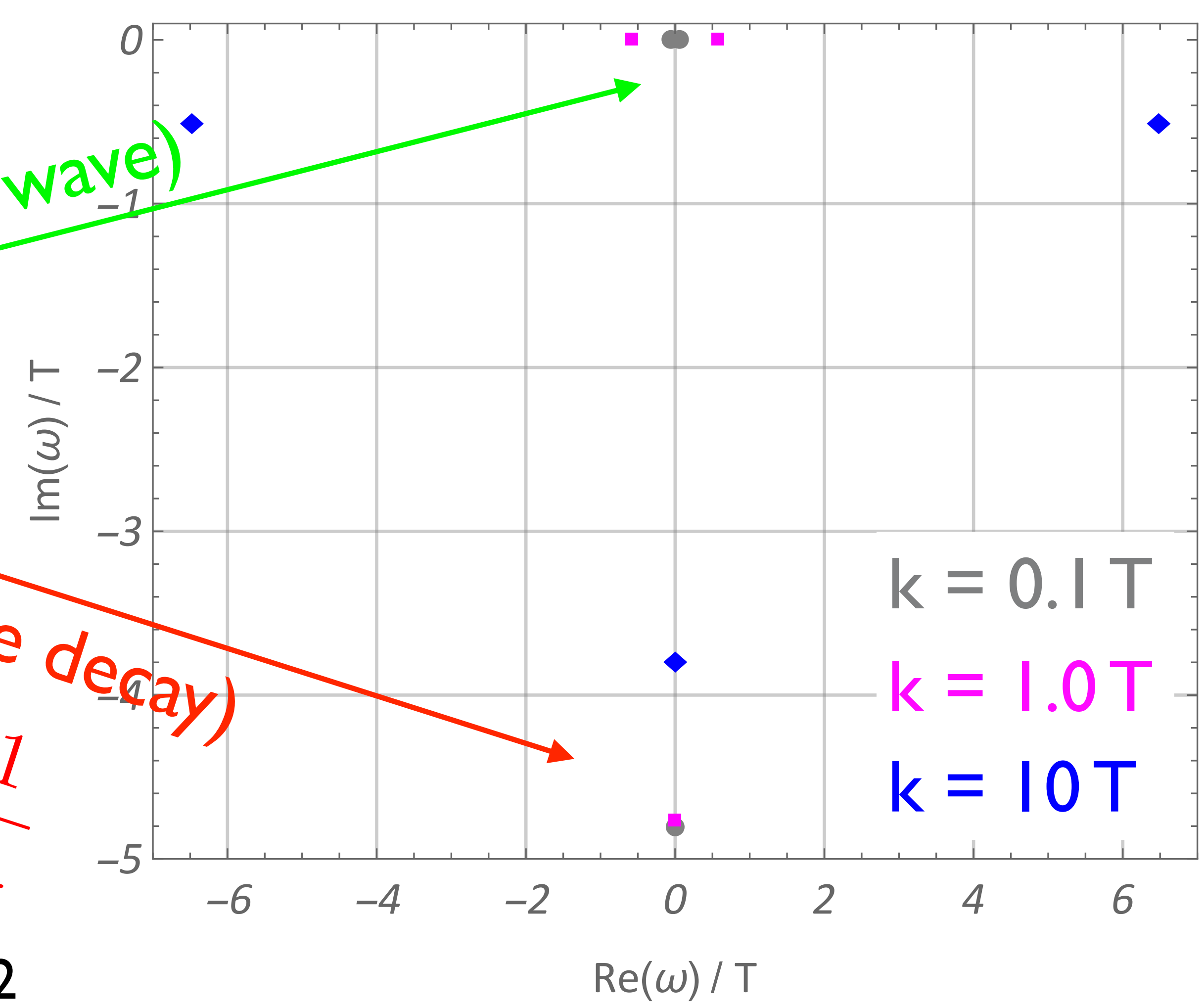
$$\omega^3 + (\ldots)\,\omega^2 + (\ldots)\omega + (\ldots) = 0$$

two modes:

hydro (sound wave)

transient (pure decay)

$$\omega\Big|_{k=0} = \frac{1}{\tau_\pi}$$

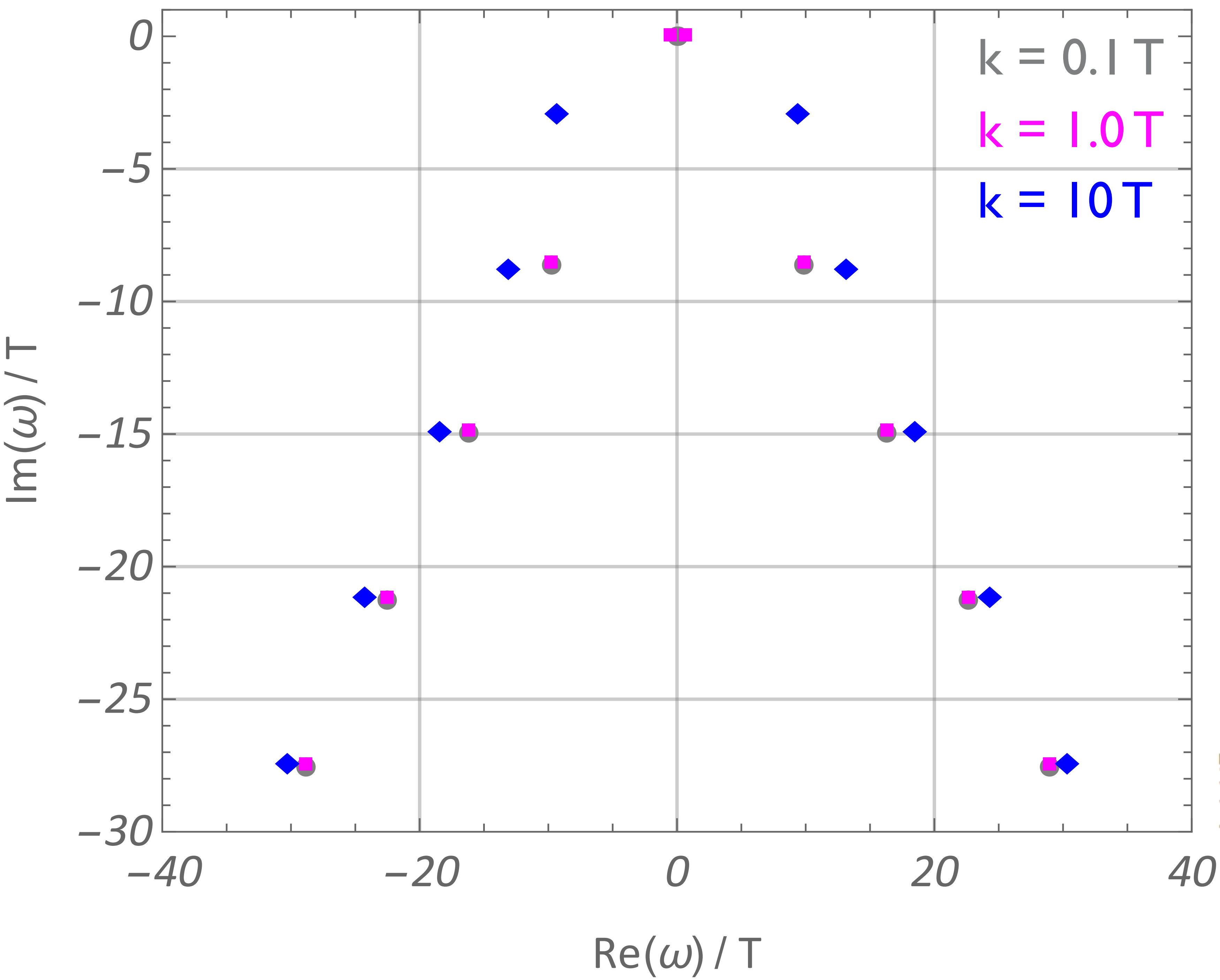


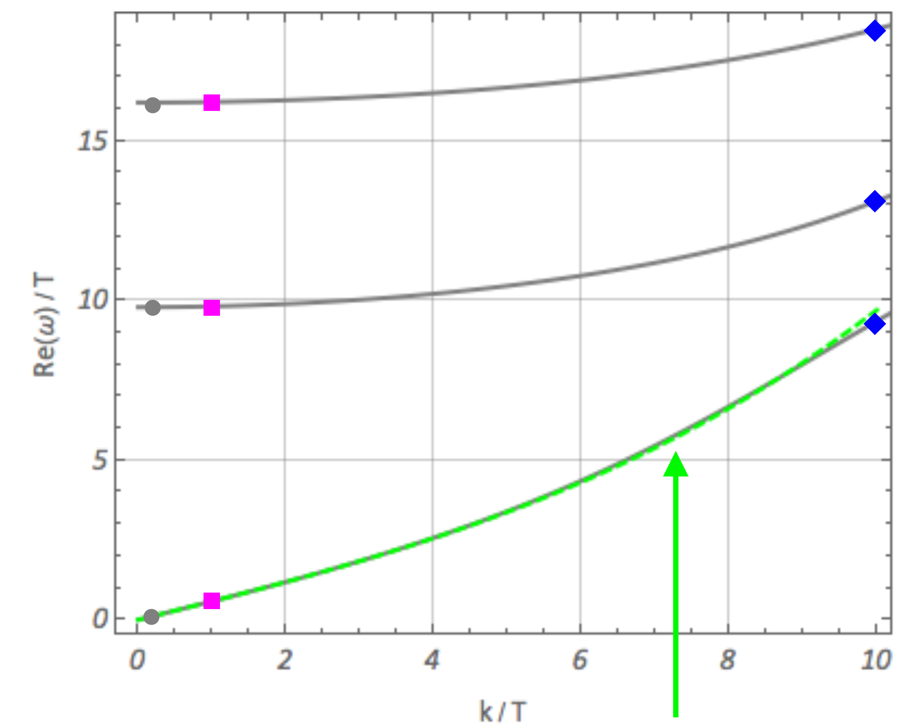
Modes in Einstein-Hilbert holography = QNMs

vanishes at the boundary

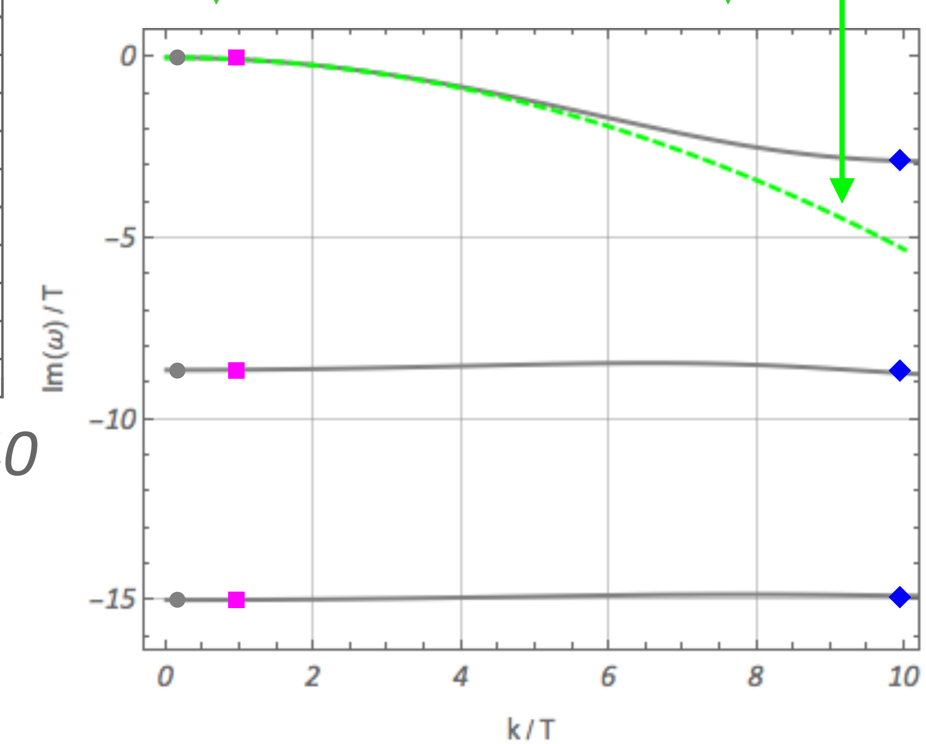
$$ds^2 = \frac{L^2}{u^2}\left\{-2dx^0du - \left(1-\pi^4T^4u^4\right)\,\left(x^0\right)^2 + d\vec{x}^2\right\} + \delta g_{ab}(u)\, e^{-i\,\omega\, x^0 + i\, k\, x^3}$$

ingoing (regular) at the horizon





$$\omega/T \approx \pm\frac{1}{\sqrt{3}}\,k/T - i\,\frac{2}{3}\,\frac{1}{4\pi}\,(k/T)^2 \pm \frac{3-2\log 2}{24\sqrt{3}\,\pi^2}\,(k/T)^3$$



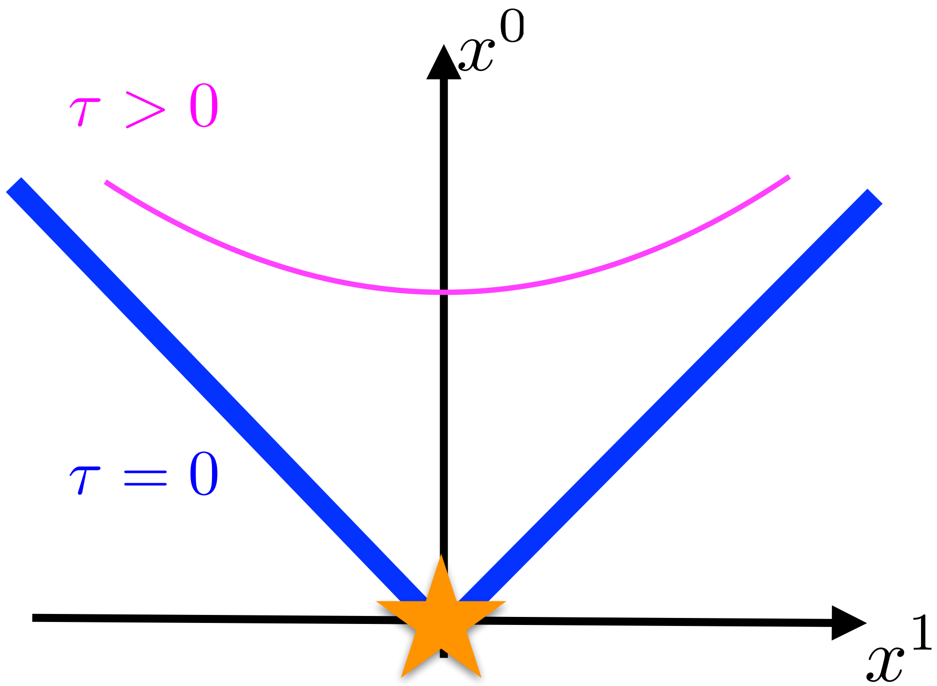
hep-th/0506184 by Kovtun & Starinets

Hydrodynamics & trans-series

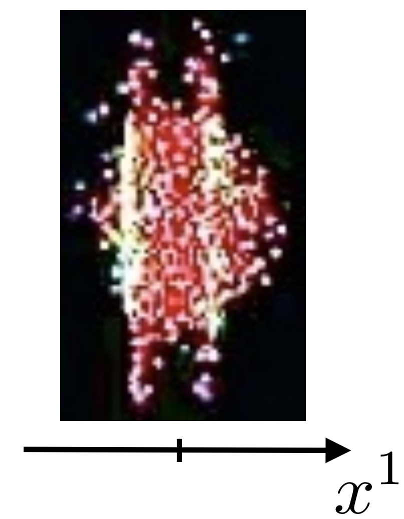
1503.07514 with Spalinski (see also **1509.05046** by Basar & Dunne)

1302.0697 with Janik & Witaszczyk

Boost-invariant flow [Bjorken 1982]



const x^0 slice:



Boost-invariance: in $\left(\tau \equiv \sqrt{x_0^2 - x_1^2},\quad y \equiv \operatorname{arctanh}\frac{x_1}{x_0},\, x_2,\, x_3\right)$ coords no y-dep

In a CFT: $\langle T^{\mu}_{\ \nu}\rangle = \operatorname{diag}\left\{-\mathcal{E}(\tau), -\mathcal{E} - \tau\,\dot{\mathcal{E}},\, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}},\, \mathcal{E} + \frac{1}{2}\tau\,\dot{\mathcal{E}}\right\}$

$$\langle T^2{}_2\rangle - \langle T^y{}_y\rangle \equiv \Delta\mathcal{P} \qquad \left(\frac{\mathcal{E}(\tau)}{\frac{3}{8}\pi^2 N_c^2}\right)^{1/4} \equiv T$$

and via scale-invariance $\frac{\Delta\mathcal{P}}{\mathcal{E}/3} \equiv \boxed{\mathcal{A}}$ is a function of $\boxed{w} \equiv \tau\, T$

Gradient expansion: series in $\frac{1}{w}$. **1103.3452** with Janik & Witaszczyk

Large order gradient expansion: BRSSS 1503.07514 with Spalinski

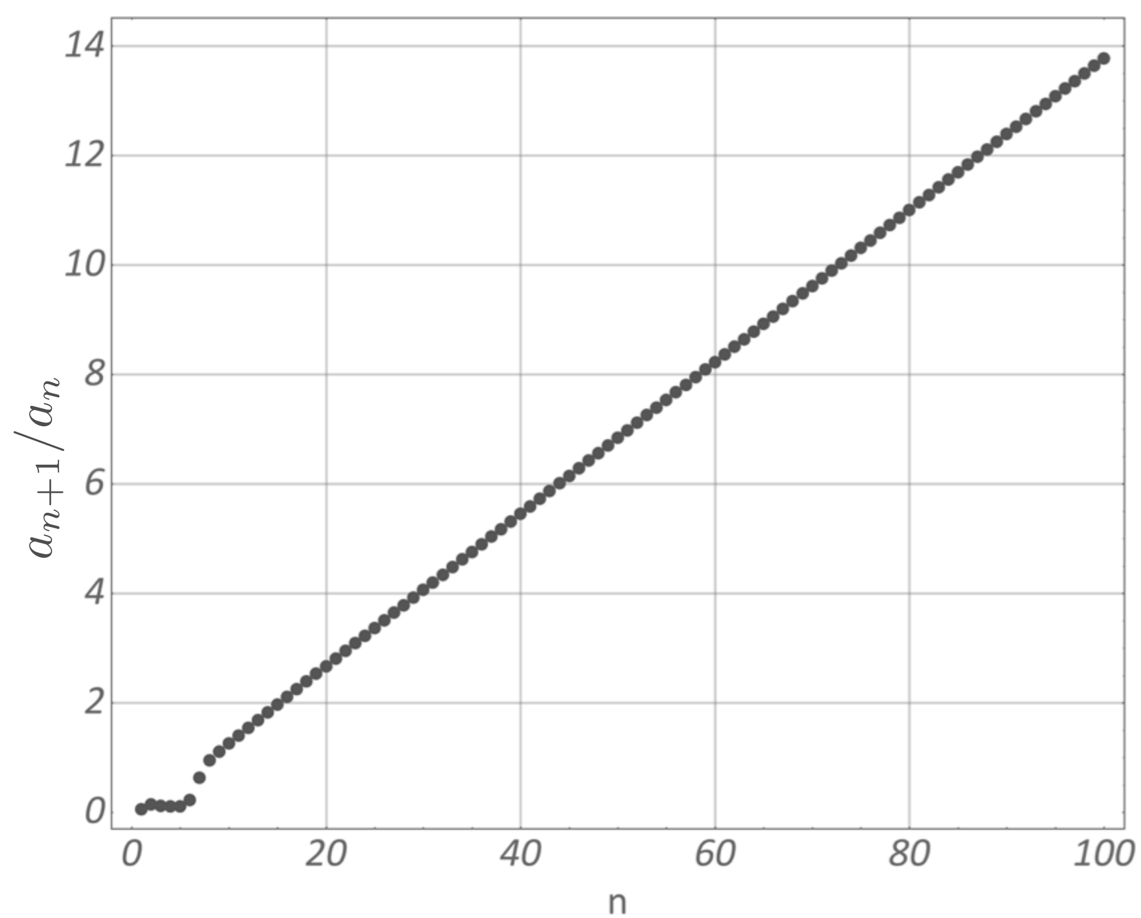
conservation (always the same) $\longrightarrow$ $$\frac{\tau}{w}\frac{dw}{d\tau} = \frac{2}{3} + \frac{1}{18}\mathcal{A}$$

$$\pi^{\mu\nu} = -\eta\sigma^{\mu\nu} - \tau_\pi\, u^\alpha \mathcal{D}_\alpha \pi^{\mu\nu}$$
$$+ \lambda_1 \pi^{\langle\mu}{}_\alpha \pi^{\nu\rangle\alpha} + \lambda_2 \pi^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} + \lambda_3 \Omega^{\langle\mu}{}_\alpha \Omega^{\nu\rangle\alpha} \longrightarrow \boxed{C_{\tau_\pi} w\,(1+\frac{1}{12}\mathcal{A})\mathcal{A}' + \left(\frac{1}{3}C_{\tau_\pi} + \frac{1}{8}\frac{C_{\lambda_1}}{C_\eta} w\right)\mathcal{A}^2 + \frac{3}{2}\, w\,\mathcal{A} - 12\, C_\eta = 0}$$

$$\left(\eta = \underbrace{C_\eta}_{=\frac{1}{4\pi}}\,\mathcal{S}, \qquad \tau_\pi = \frac{\overbrace{C_{\tau_\pi}}^{=\frac{2-\log 2}{2\pi}}}{T}, \qquad \lambda_1 = \underbrace{C_{\lambda_1}}_{=\frac{1}{2\pi}}\frac{\eta}{T}\right)$$

$$\mathcal{A}(w) \approx \sum_{n=1}^{\infty}\frac{a_n}{w^n} = 8\,C_\eta\,\frac{1}{w} + \frac{16}{3}\,C_\eta\,(C_{\tau_\pi} - C_{\lambda_1})\,\frac{1}{w^2}\boxed{+\ldots} \longrightarrow$$

(note that a_n do not depend on ini. cond.)



Hydrodynamic gradient expansion is a divergent series: $a_n \sim \Gamma(n+\beta)$

Hydrodynamics & transient modes: BRSSS

Key observations:

- $\sum_{n=1}^{\infty} \frac{r_n}{w^n}$ does not make sense without a resummation
- there must be sth else that cares about ini. cond.

resurgence

1503.07514 with Spalinski

Linearization of $C_{\tau_\pi}\, w\,(1+\frac{1}{12}\mathcal{A})\,\mathcal{A}' + \left(\frac{1}{3}C_{\tau_\pi} + \frac{1}{8}\frac{C_{\lambda_1}}{C_\eta}w\right)\mathcal{A}^2 + \frac{3}{2}\,w\,\mathcal{A} - 12\,C_\eta = 0$ around $\sum_{n=1}^{\infty}\frac{a_n}{w^n}$ gives:

integration const. (ini. cond.)

further hydro dressing (another div. series)

$$\delta\mathcal{A} = \sigma\, e^{-\frac{3}{2}\,\frac{1}{C_{\tau_\pi}}\,w}\, w^{\frac{C_\eta - 2\,C_{\lambda_1}}{C_{\tau_\pi}}} \left\{1 + \sum_{j=1}^{\infty}\frac{a_j^{(1)}}{w^j}\right\}$$

In equilibrium one has $e^{-\frac{1}{C_{\tau_\pi}}\,T\,t}$

It is still true here, but only at a given instance: $e^{-\frac{1}{C_{\tau_\pi}}\int_{\tau_i}^{\tau} T(\tau')\,d\tau'}$

Using $T = \frac{\Lambda}{(\Lambda\tau)^{1/3}}\left(1 - C_\eta\frac{1}{(\Lambda\,\tau)^{2/3}} + \ldots\right)$ one gets $e^{-\frac{3}{2}\,\frac{1}{C_{\tau_\pi}}\,w}\, w^{\frac{C_\eta}{C_{\tau_\pi}}}\ldots$

To wrap-up, we have just seen the hydro-dressed transient mode of BRSSS at k =0

see also **hep-th/0606149** by Janik & Peschanski

Transseries and resurgence

1503.07514 with Spalinski
see also 1509.05046 by Basar & Dunne

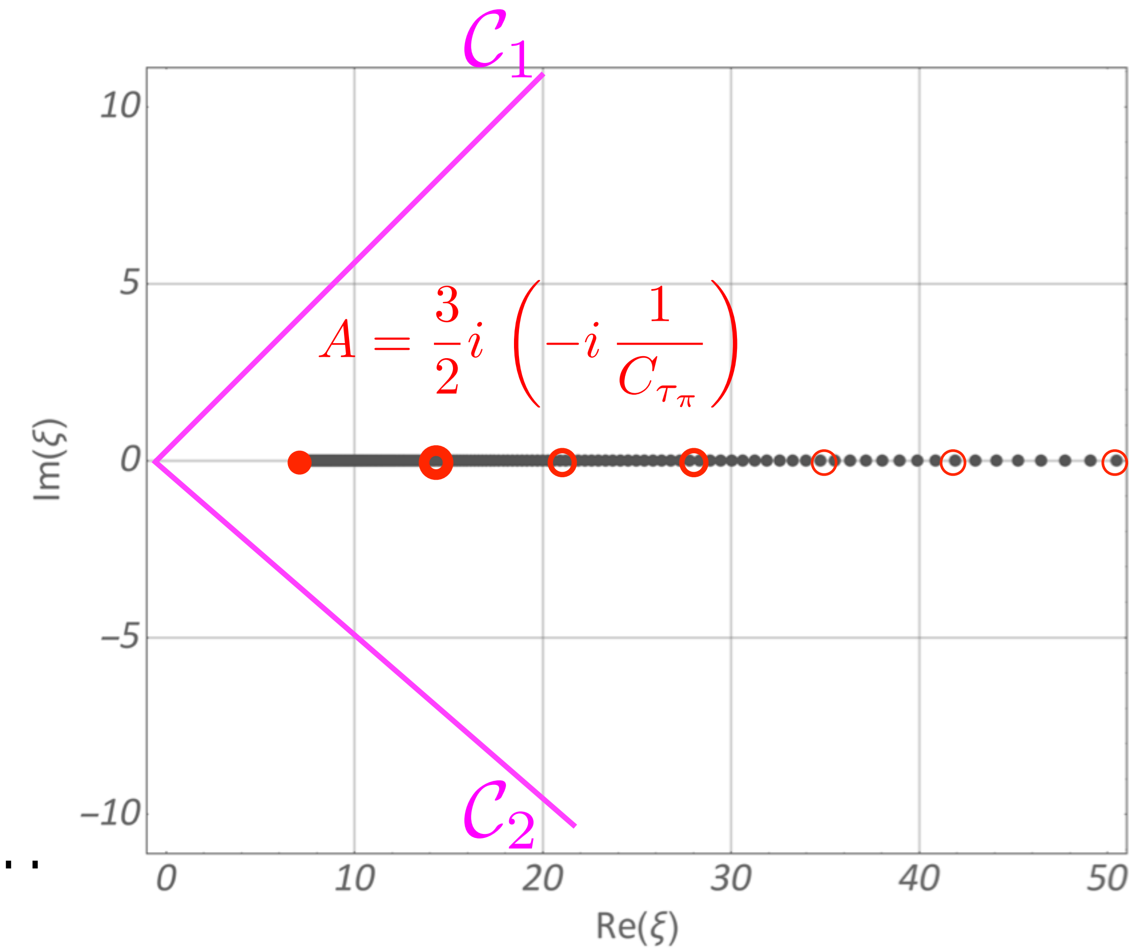
approx. analytic cont.

$$\mathcal{A} \approx \sum_{n=1}^{\infty} \frac{a_n}{w^n} \xrightarrow[\text{Borel trafo.}]{} B\mathcal{A}(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{\Gamma(n+\beta)}\, \xi^n \approx \frac{b_0 + \ldots + b_{100}\xi^{100}}{c_0 + \ldots + c_{100}\xi^{100}}$$

Borel (re)summation

$$\left(\int_{\mathcal{C}_1} d\xi - \int_{\mathcal{C}_2} d\xi\right) w^{\beta}\, \xi^{\beta-1} e^{-w\,\xi}\, B\mathcal{A}(\xi)$$

$$\sim e^{-\underbrace{\frac{3}{2}\,\frac{1}{C_{\tau_\pi}}}_{\equiv A} w}\; w^{\underbrace{\frac{C_\eta - 2\,C_{\lambda_1}}{C_{\tau_\pi}}}_{\equiv \beta}} \ldots$$



Ambiguity in resummation

$B\mathcal{A}(\xi) = \text{reg.} + (A - \xi)^{\beta}\, \text{reg.} + \ldots$ $\sim$ transient mode + …

nonlinear effects

Trans-series: $\mathcal{A}(w) = \sum_{j=0}^{\infty} \sigma^j\, e^{-j\,A\,w}\, w^{j\,\beta}\, \Phi_{(j)}(w)$

~ 1/w expansions

~ resum. ambig. + ini. cond.

Resurgence: trans-series yields an unambiguous answer up to 1 real int. const.

Hydrodynamics & transient modes: holography

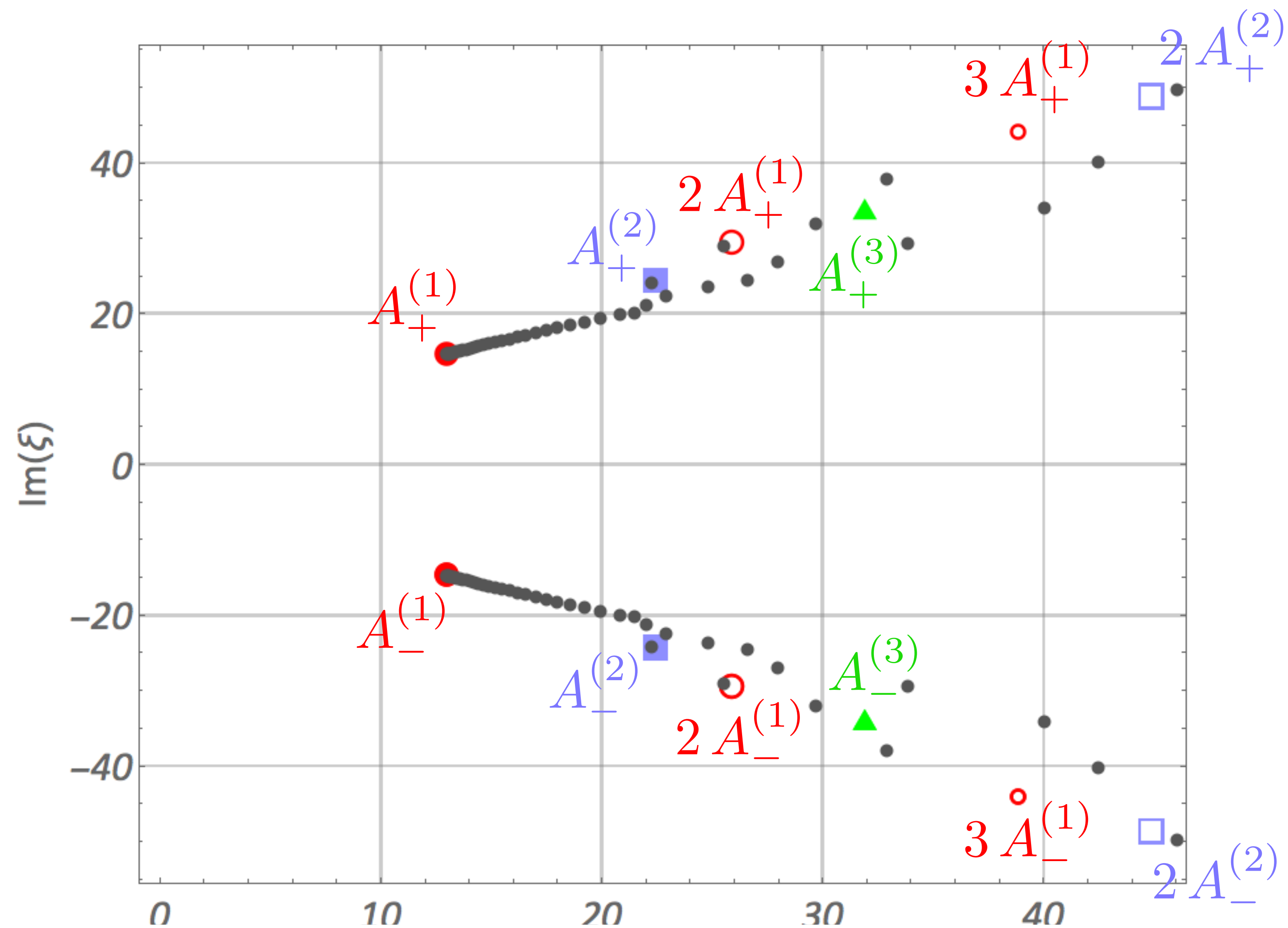
1302.0697 with Janik & Witaszczyk

see also **1511.06358** by Aniceto & Spalinski as well as **1708.01921** by Spalinski

$$\mathcal{A} \approx \sum_{n=1}^{\infty} \frac{a_n}{w^n}$$

$$\downarrow$$

$$B\mathcal{A}(\xi) = \sum_{n=1}^{\infty} \frac{a_n}{n!}\, \xi^n \approx \frac{b_0 + \ldots + b_{120}\xi^{120}}{c_0 + \ldots + c_{120}\xi^{120}}$$



$$\mathcal{A}(w) = \sum_{n_\pm^{(1)}, n_\pm^{(2)}, \ldots = 0}^{\infty} \Phi_{\left(n_+^{(1)} | n_-^{(1)} | n_+^{(2)} | n_-^{(2)} | \ldots\right)}(w) \times$$

$$\times \prod_{j=1}^{\infty} \left(\sigma_+^{(j)}\right)^{n_+^{(j)}} \left(\sigma_-^{(j)}\right)^{n_-^{(j)}} \mathrm{e}^{-\left(n_+^{(j)} A_+^{(j)} + n_-^{(j)} A_-^{(j)}\right) w}\, w^{n_+^{(j)} \beta_+^{(j)} + n_-^{(j)} \beta_-^{(j)}}$$

Infinitely many transient QNMs ⟶ infinitely many parameters in the transseries

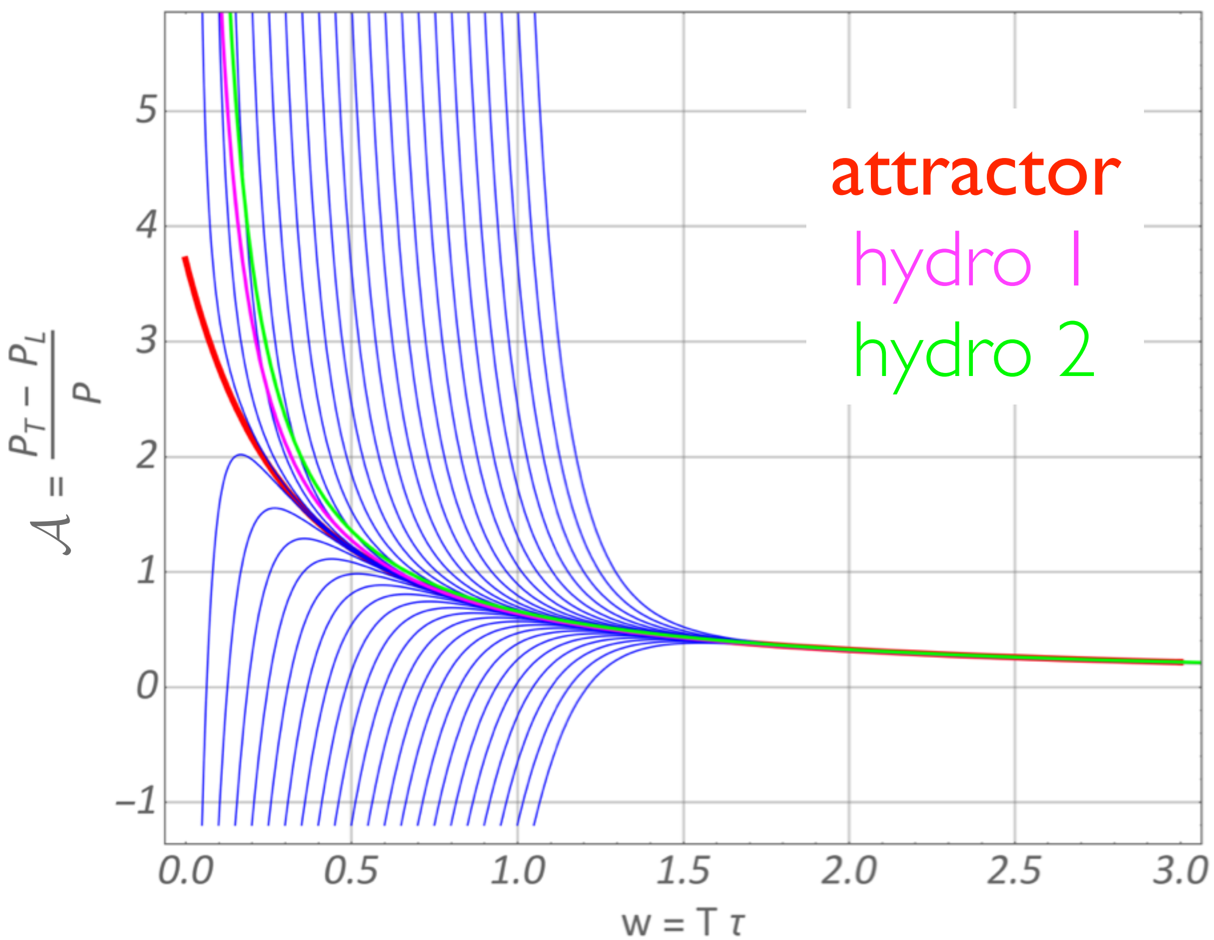
Hydrodynamics far from equilibrium

1503.07514 with Spalinski

see also **1704.08699** by Romatschke

Hydrodynamics far from equilibrium = attractors

1503.07514 with Spalinski



BRSSS:

$$\cancel{C_{\tau_\pi}\, w \,(1 + \frac{1}{12}\mathcal{A})\,\mathcal{A}'} + \left(\frac{1}{3}C_{\tau_\pi} + \frac{1}{8}\frac{C_{\lambda_1}}{C_\eta} w\right)\mathcal{A}^2 + \frac{3}{2}\,w\,\mathcal{A} - 12\,C_\eta = 0$$

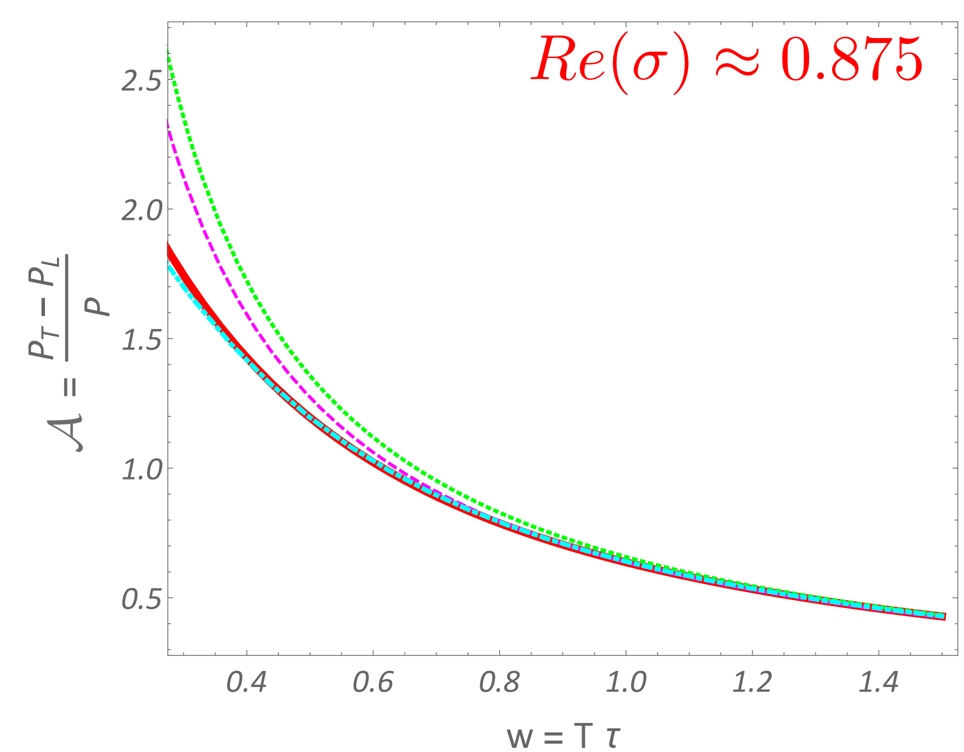
≈ attractor solution

Recently Romatschke in 1704.08699 found such attractors in kinetic theory & holography

One can also approx. resum transseries:

$$\mathcal{A}(w) \approx \sum_{j=0}^{2} \sigma^j\, e^{-j\,A\,w}\, w^{j\,\beta}\, \Phi_{(j)}(w)$$

Requires 3 Borel summations



Summary

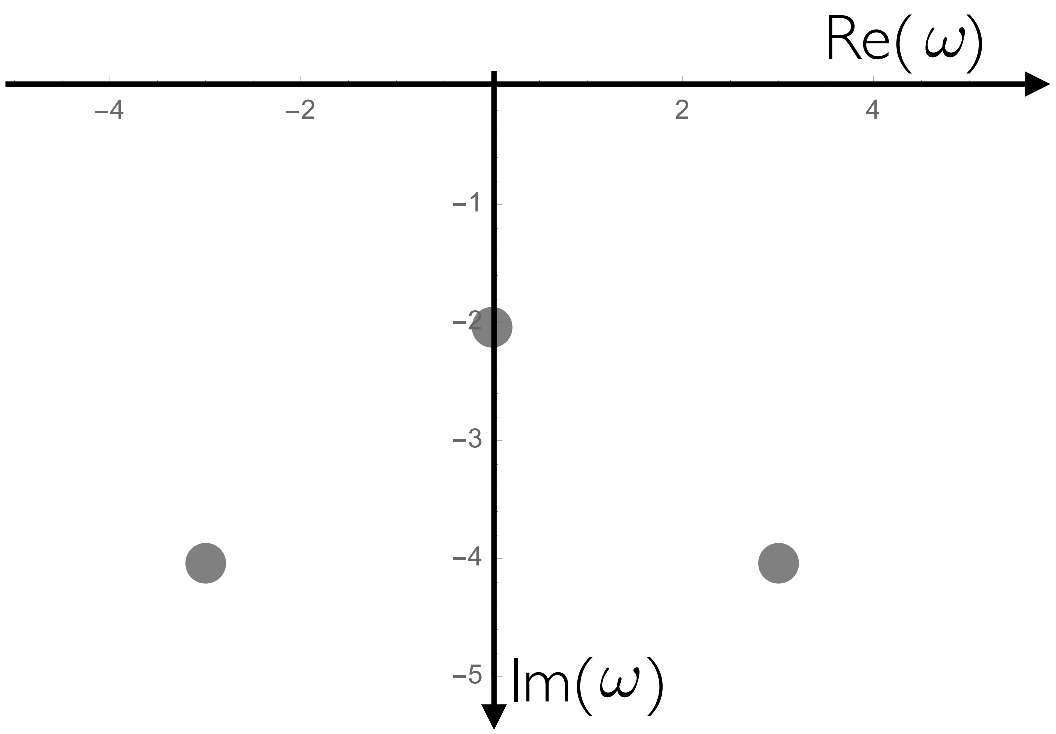
many works, but see

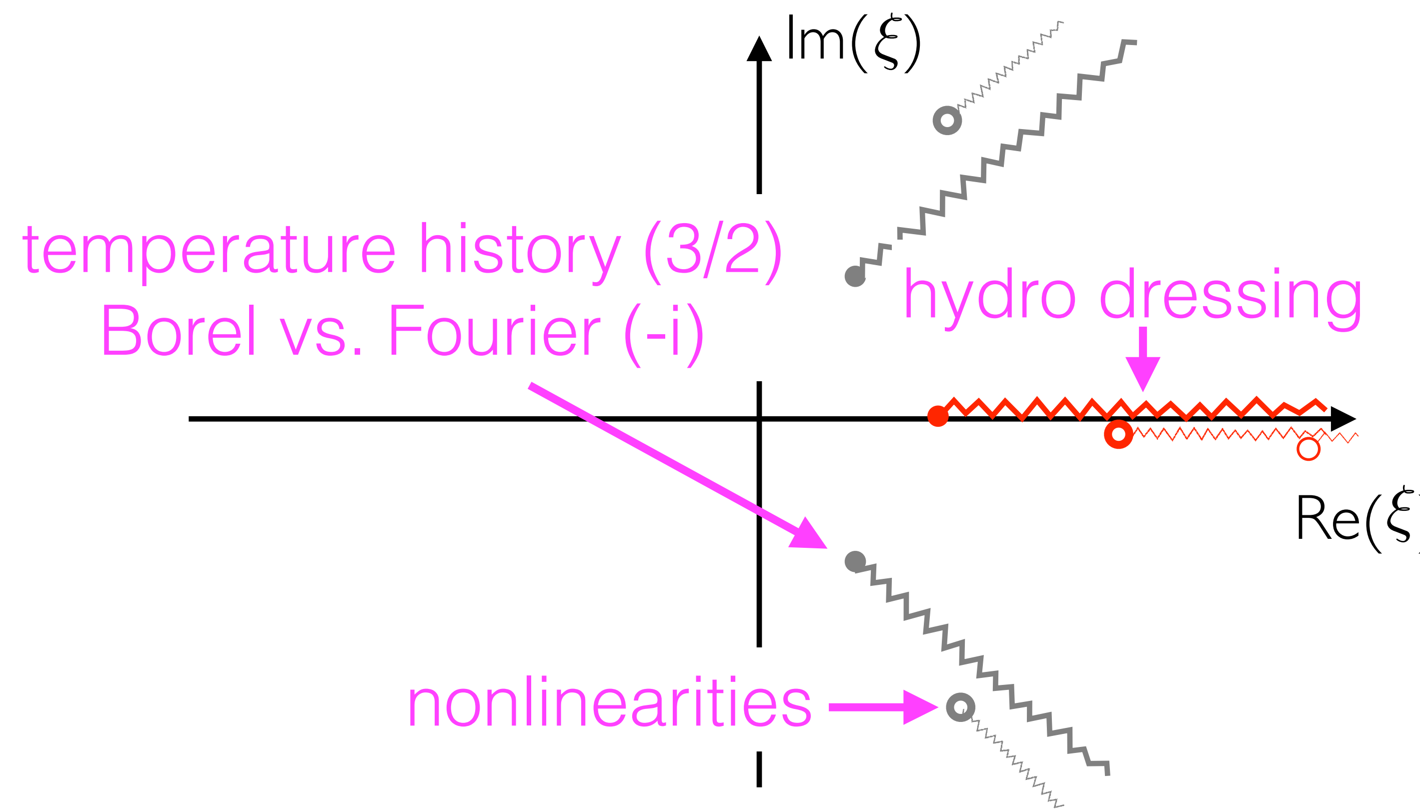
1610.02023 [hep-th] lecture notes

1707.02282 [hep-ph] review with Florkowski and Spalinski hydrodynamization

trans-series ⟷ hydrodynamics far from equilibrium

Transient modes at $k=0$ vs. singularities of Borel transform of hydro





Appealing analogy with quantum mechanics:

non-equilibrium physics	QM with $V = -\frac{1}{2}x^2\left(1-\sqrt{g}\,x\right)^2$
gradient expansion in $\frac{1}{w}$	perturbative series in g
transient QNMs $e^{-i\,\frac{3}{2}\,\Omega_{\pm}\,w}\,(\ldots)$	instanton $e^{-1/(3g)}(\ldots)$

Support

Lesson from cosmology

1603.05344 with Buchel & Noronha

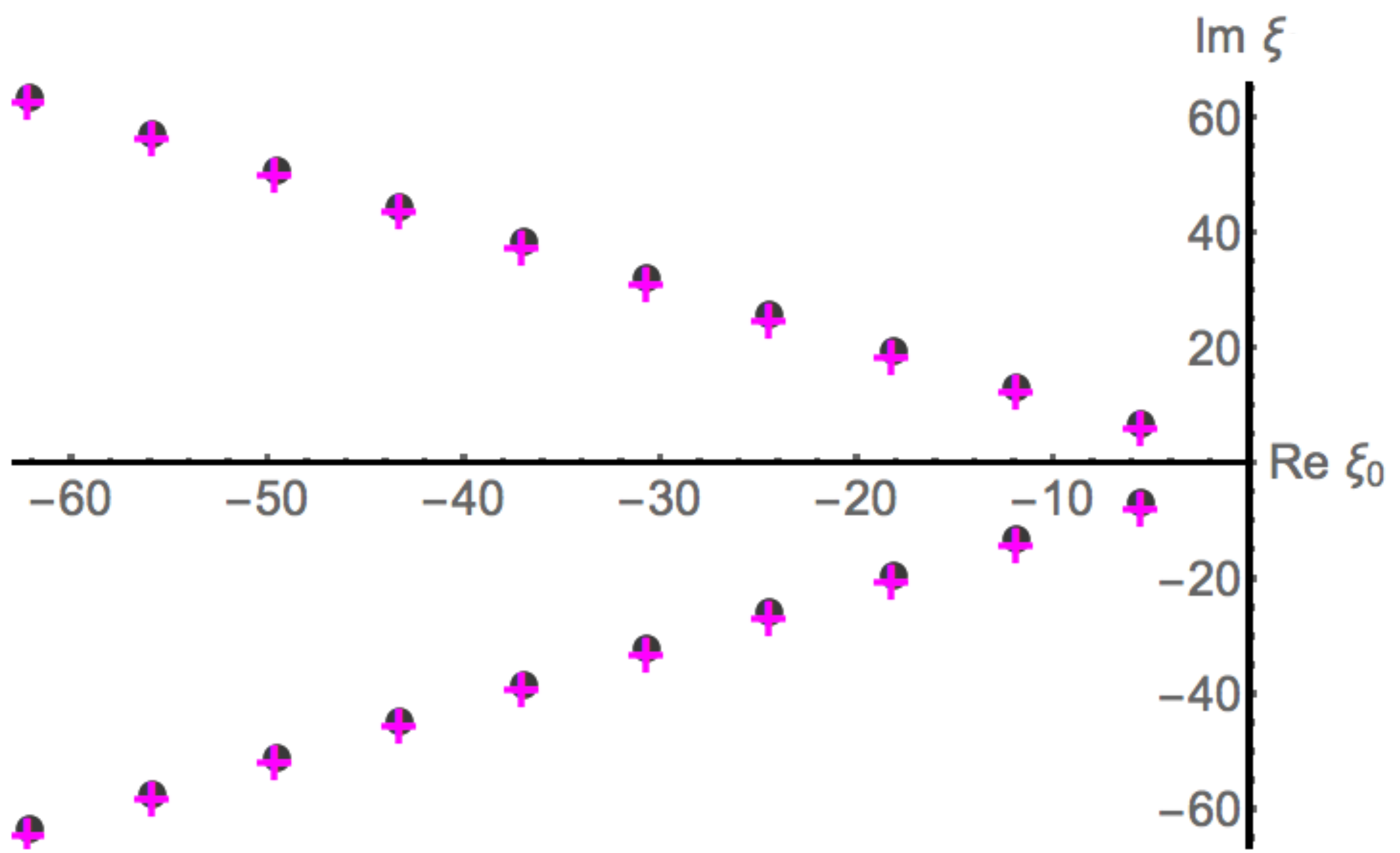
$$\frac{d\,\mathrm{Entropy}}{dt} = V \times \left(\sum_{n=0}^{\infty} c_n \xi^n\right)^2 + \ldots$$ with $\xi = \frac{H}{T}$ for a h~~CFT~~ in $-dt^2 + e^{2Ht}\,d\vec{x}^2$

$$T \sim e^{-H\,t} \longrightarrow e^{-i\,\Omega_\pm \int_{t_i}^{t} T(t')dt'} \sim e^{-i\,\Omega_\pm \cdot \left(-\frac{T(t)}{H}\right)}$$

$$\sum_{n=0}^{300} \frac{c_n}{n!}\xi^n \approx \frac{\sum_{m=0}^{150} d_m\,\xi^m}{\sum_{l=0}^{150} e_l\,\xi^l}$$

● singularities of Borel trafo

\+ 10 lowest transient QNM $\hat{\omega}$'s

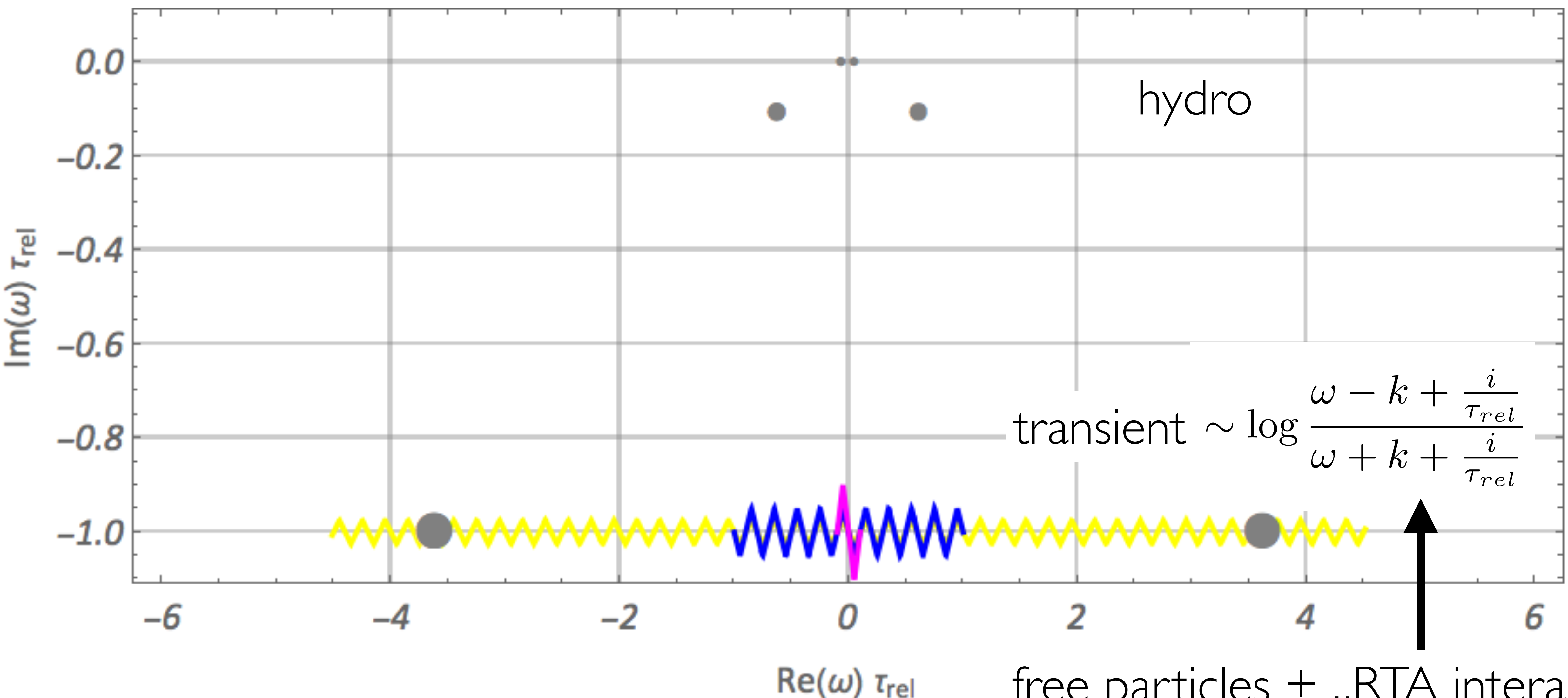


Hydrodynamic gradient expansion knowns about all transient QNMs

Modes in RTA kinetic theory

1512.02641 by Romatschke
1707.02282 with Florkowski & Spalinski

Sound channel at k τ_{rel} = 0.1, 1.0 & 4.531



free particles + „RTA interactions"

Very different from holography: one hydro mode and $\boxed{\text{one branch-cut at } k \neq 0}$

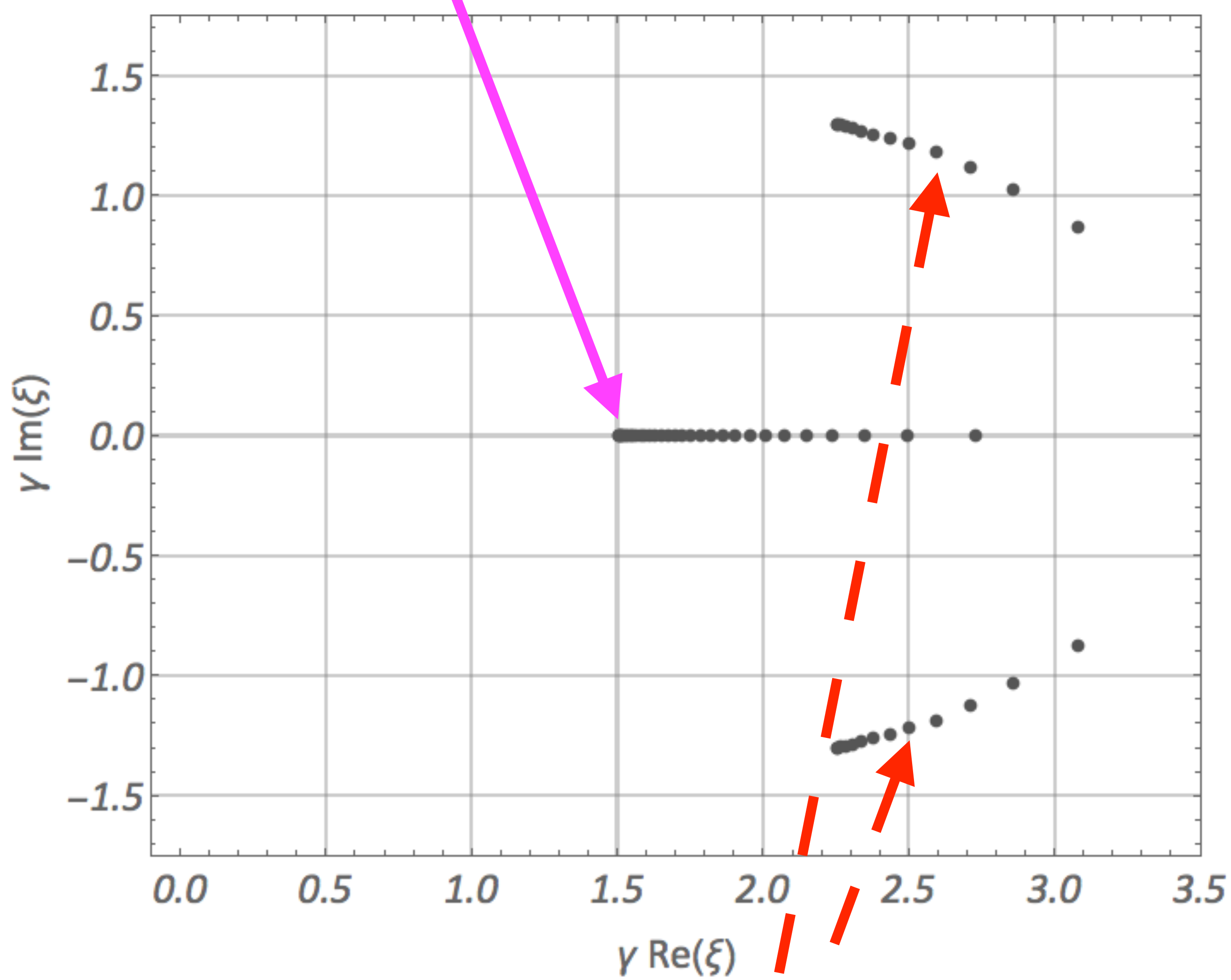
$\downarrow k \to 0$

single pole at $\omega = -i\,\dfrac{1}{\tau_{rel}}$

QNM in kinetic theory?

1609.04803 with Kurkela & Spalinski
work in progress with Svensson

$\xi_{sing} = \frac{3}{2\gamma}$ → assuming sing. $\sim \left(\xi - \frac{3}{2\gamma}\right)^{\beta}$ → $\delta\mathcal{A} \sim \exp\left(-\frac{3}{2\,\gamma}\right) w^{-1.43} (\ldots)$



$$\delta\mathcal{A} \sim \exp\left(-\frac{2.25}{\gamma} \pm \frac{1.3}{\gamma}\, i\right)$$???