Conformal Blocks, Nekrasov Partition Function and WKB

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Based on joint work with Jan Troost.

Motivation

Conformal blocks $\xleftrightarrow{\text{AGT}}$ Nekrasov partition fcn of $N=2$ gauge theory

$$Z_{Nek} \sim e^{\sum F_g g_s^{2g-2}}$$

↑ 4d limit

$$Z_{top} \sim e^{\sum F_g g_s^{2g-2}}$$

← genus of worldsheet

target space | worldsheet

Outline

1. Review $N = 2$ gauge theory, Z_{Nek}, and AGT $\rightarrow$ WKB
2. Computing using WKB
3. Quantum geometry
4. Exact WKB and gauge theory

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$\mathcal{N} = 2$ SUSY gauge theories come in families

$\mathcal{N} = 2$ vector multiplet:

$$\phi \quad \begin{matrix} \psi \\ \chi \end{matrix} \quad A^\mu$$

By $\mathcal{N} = 2$ supersymmetry, flat directions in potential for ϕ

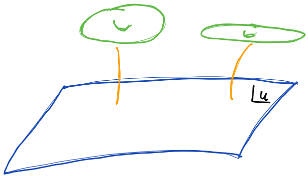
$\Rightarrow$ families parametrized locally by $a = \langle \phi \rangle$.

The two derivative $\mathcal{N} = 2$ Lagrangian

- Dependence of Lagrangian on the modulus a captured by complex function: the prepotential $F_0(a)$.
- In the effective action, $F_0(a)$ receives instanton corrections.

The Seiberg-Witten formalism

Family of QFTs maps to family of tori which encode prepotential.



Global coordinate on moduli space is u.

$\lambda(u)$ meromorphic one form satisfying $a = \oint_A \lambda\,, \quad a_D = \oint_B \lambda$.

Prepotential determined via $a_D = \frac{dF_0}{da}$.

Some examples

- Pure $SU(2)$: a single vector multiplet

$$y^2 = (x-u)(x^2-\Lambda^4), \quad \lambda \sim \frac{\sqrt{x-u}}{\sqrt{x^2-\Lambda^4}}$$

- $N = 2^*$: an adjoint matter hypermultiplet of mass m

$$y^2 = m^2 \wp(z|\tau) - u\,, \quad \lambda = \sqrt{m^2 \wp(z|\tau) - u}$$

- $N_f = 4$: four fundamental matter hypermultiplets of mass m_i

$$y^2 = \sum_{i=0}^{3} m_i^2 \wp(z + \underset{\substack{\uparrow \\ \text{2-torsion points of torus}}}{\omega_i}|\tau) - u\,, \quad \lambda = \sqrt{\sum_{i=0}^{4} m_i^2 \wp(z+\omega_i|\tau) - u}$$

The Nekrasov partition function

Compute instanton contributions via localization.

$$Z_{Nek}(\epsilon_1, \epsilon_2) = \sum_n \underset{\uparrow}{q^n} \int_{\underset{\uparrow}{\tilde{\mathcal{M}}_n}} \overset{\downarrow}{\omega}$$

appropriate equivariant class

instanton counting parameter

compactified n-instanton moduli space for gauge group $U(2)$

Recover prepotential $F_0 = F^{0,0}$ as leading term in asymptotic expansion.

$$Z_{Nek}(\epsilon_1, \epsilon_2) \sim \exp \sum_{n,g} F^{n,g}(q) \overset{(\epsilon_1+\epsilon_2)^2}{\overset{\downarrow}{s}}{}^n (\underset{\underset{\epsilon_1\epsilon_2}{\uparrow}}{g_s^2})^{g-1}$$

From formal to analytic?

$$Z_{Nek}(\epsilon_1, \epsilon_2) \underset{\substack{\uparrow \\ \text{elucidate}}}{\sim} \exp \sum_{n,g} F^{n,g}(q) s^n g_s^{2g-2}$$

- in gauge theory: LHS convergent for large a
- in topological string theory: both sides are formal power series, in q and (g_s, s) respectively

WKB for $\mathcal{N} = 2$ gauge theory via AGT

gauge theory	CFT
Z_{Nek}	conformal block on $\Sigma_{g,n}$
ϵ_1, ϵ_2	$b^2 = \frac{\epsilon_2}{\epsilon_1}$, $Q = b + 1/b$,
	$c = 1 + 6Q^2$
coupling constants	complex structure parameters of $\Sigma_{g,n}$
a	exchanged momentum $h = \frac{Q^2}{4} - \frac{a^2}{\epsilon_1\epsilon_2}$
masses m_i	weights of insertions $h_{m_i} = \frac{Q^2}{4} - \frac{{m_i}^2}{\epsilon_1\epsilon_2}$
Z_{Nek} with surface operator	conformal block with degenerate insertion

AGT with surface operator insertion

gauge theory	CFT
Z_{Nek} with surface operator	conformal block with degenerate insertion

$$\Psi\big(\underset{\uparrow}{z}\big|\tau\big)$$

modulus of surface operator

Null vector decoupling $\rightarrow$ WKB

$\Psi(z|\tau)$ satisfies null vector decoupling equation.

Boundary condition to project onto conformal block: monodromy in z

$$\Psi(z|\tau) \rightarrow e^{\frac{a}{\epsilon_1^2}} \Psi(z|\tau)$$

Null vector decoupling equations for our examples

- pure $SU(2)$

$$\left(\epsilon_1^2\partial_z^2 + \Lambda^2(e^z + e^{-z}) + \epsilon_1\epsilon_2\frac{\Lambda}{4}\partial_\Lambda\right)\ \Psi(z|\Lambda) = 0\,.$$

- $N = 2^*$

$$\left(\epsilon_1^2\partial_z^2 + (\frac{\epsilon_1^2}{4} - m^2)\wp(z) + \epsilon_1\epsilon_2 2\pi i\partial_\tau\right)\Psi(z|\tau) = 0\,.$$

- $N_f = 4$

$$\left(\epsilon_1^2\partial_z^2 + 4\sum_{i=0}^{3}(\frac{\epsilon_1^2}{16} - m_i^2)\wp(z+\omega_i) + \epsilon_1\epsilon_2 4\pi i\partial_\tau\right)\Psi(z|\tau) = 0\,.$$

Semi-classical limit

Factorize effect of degenerate insertion via limit

$$\epsilon_2 \to 0\,.$$

Ansatz:

$$\Psi(z|\tau) = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\tau) + \frac{1}{\epsilon_1}\mathcal{W}(z|\tau) + O(\epsilon_2)\right]\,.$$

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Note that

$$g_s^2 = \epsilon_1\epsilon_2 \xrightarrow{\epsilon_2\to 0} 0\,.$$

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WKB ansatz: $N = 2^*$

$$\left(\epsilon_1^2\partial_z^2 + \left(\frac{\epsilon_1^2}{4} - m^2\right)\wp(z) + \epsilon_1\epsilon_2 2\pi i\partial_\tau\right)\underset{\uparrow}{\Psi}(z|\tau) = 0\,.$$

$$\Psi(z|\tau)=\exp\left[\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\tau)+\frac{1}{\epsilon_1}\mathcal{W}(z|\tau)+O(\epsilon_2)\right],$$

gives rise to

$$-\frac{1}{\epsilon_1}\mathcal{W}''(z|\tau) - \frac{1}{\epsilon_1^2}\mathcal{W}'(z|\tau)^2 + \left(\frac{1}{\epsilon_1^2}m^2 - \frac{1}{4}\right)\wp(z) = (2\pi i)^2\frac{1}{\epsilon_1^2}q\partial_q\mathcal{F}(\tau)\,.$$

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$$\Psi(z|\tau) = \exp\left[\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\tau) + \frac{1}{\epsilon_1}\mathcal{W}(z|\tau) + O(\epsilon_2)\right],$$

gives rise to

$$-\frac{1}{\epsilon_1}\mathcal{W}''(z|\tau) - \frac{1}{\epsilon_1^2}\mathcal{W}'(z|\tau)^2 + \left(\frac{1}{\epsilon_1^2}m^2 - \frac{1}{4}\right)\wp(z) = (2\pi i)^2 \frac{1}{\epsilon_1^2} q\partial_q \mathcal{F}(\tau)\,.$$

Ansatz: $\mathcal{F}(\tau) = \sum_{n=0}^{\infty} \mathcal{F}_n(\tau)\epsilon_1^n\,, \quad \mathcal{W}(z|\tau) = \sum_{n=0}^{\infty} \mathcal{W}_n(z|\tau)\epsilon_1^n\,.$

Solving for WKB series

$$
\begin{aligned}
-\mathcal{W}_0'^2 + m^2 \wp &= (2\pi i)^2 q \partial_q \mathcal{F}_0 \,, \\
-\mathcal{W}_0'' - 2\mathcal{W}_0'\mathcal{W}_1' &= (2\pi i)^2 q \partial_q \mathcal{F}_1 \,, \\
-\mathcal{W}_1'' - \mathcal{W}_1'^2 - 2\mathcal{W}_0'\mathcal{W}_2' - \frac{1}{4}\wp(z) &= (2\pi i)^2 q \partial_q \mathcal{F}_2 \,, \\
-\mathcal{W}_n'' - \sum_{i=0}^{n+1} \mathcal{W}_i'\mathcal{W}_{n+1-i}' &= (2\pi i)^2 q \partial_q \mathcal{F}_{n+1} \quad \text{for} \quad n \geq 2 \,,
\end{aligned}
$$

with boundary conditions

$$
\oint_A \mathcal{W}_0' = \pm 2\pi i a \,, \quad \oint_A \mathcal{W}_i' = 0 \quad \text{for} \quad i > 0 \,.
$$

Modularity (massless case)

$$-\mathcal{W}_0'^2 = (2\pi i)^2 q\partial_q \mathcal{F}_0 \quad \Rightarrow \quad \mathcal{W}_0' \quad z\text{-independent}$$

Solve via boundary condition:

$$\oint_A \mathcal{W}_0' = \mathcal{W}_0' \oint_A 1 = \pm 2\pi i a \quad \Rightarrow \quad \mathcal{W}_0' = 2\pi i a \quad \Rightarrow \quad q\partial_q \mathcal{F}_0 = -a^2$$

Modularity arises

$$\oint_A \left(-2\mathcal{W}_0'\mathcal{W}_2' - \frac{1}{4}\wp(z)\right) = (2\pi i)^2 q\partial_q \mathcal{F}_2 \oint_A 1$$

$$\Rightarrow \quad (2\pi i)^2 q\partial_q \mathcal{F}_2 = -\frac{1}{4}\oint_A \wp(z) \quad \Rightarrow \quad \mathcal{W}_2' = \frac{1}{4}\frac{\oint_A \wp(z) - \wp(z)}{4\pi i a}\,.$$

$$\oint_A \wp^n(z) \in \mathbb{C}[E_2, \underset{\substack{\uparrow \\ \text{ring of quasi-modular forms}}}{E_4}, E_6]$$

$$\mathcal{W}_{2n}' = \frac{p_{2n}^e(\wp)}{a^{2n-1}}\,, \quad \mathcal{W}_{2n+1}' = \frac{\wp'\, p_{2(n-1)}^o(\wp)}{a^{2n}}\,.$$

with p_n^e, p_n^o homogeneous of weight n,

$$p_{2n}^e(\wp)\,, p_{2n}^o(\wp) \in \mathbb{C}[E_2, E_4, E_6][\wp]\,.$$

First several orders for $\mathcal{W}_n'$ and $\mathcal{F}_n$

$$\partial_\tau \mathcal{F}_2 = -i\frac{\pi}{24}E_2\,, \quad \mathcal{W}_2' = i\frac{\pi^2 E_2 + 3\wp}{48\pi a}\,,$$

$$\partial_\tau \mathcal{F}_3 = 0\,, \quad \mathcal{W}_3' = -\frac{\wp'}{64\pi^2 a^2}\,,$$

$$\partial_\tau \mathcal{F}_4 = i\frac{\pi(E_2^2 - E_4)}{4608a^2}\,, \quad \mathcal{W}_4' = -i\frac{(2E_2^2 - 25E_4)\pi^4 + 6\pi^2 E_2\wp + 225\wp^2}{9216\pi^3 a^3}\,,$$

$$\partial_\tau \mathcal{F}_6 = -i\frac{\pi(5E_2^3 + 21E_2E_4 - 26E_6)}{1105920a^4}\,,$$

$$\partial_\tau \mathcal{F}_8 = i\frac{\pi(35E_2^4 + 329E_2^2E_4 - 1402E_4^2 + 1038E_2E_6)}{297271296a^6}\,.$$

From $\partial_\tau \mathcal{F}_n$ to $\mathcal{F}_n$

$$\int d\tau : \mathbb{C}[E_2, E_4, E_6] \nrightarrow \mathbb{C}[E_2, E_4, E_6]$$

Algebraic constraints on coefficients via Ramanujan relations

$$q\partial_q E_2 = \frac{E_2^2 - E_4}{12}\,, \qquad q\partial_q E_4 = \frac{E_2 E_4 - E_6}{3}\,, \qquad q\partial_q E_6 = \frac{E_2 E_6 - E_4^2}{2}\,.$$

And yet

$$\mathcal{F}_4 = \frac{E_2}{768a^2}\,, \quad \mathcal{F}_6 = -\frac{5E_2^2 + 13E_4}{368640a^4}\,, \quad \mathcal{F}_8 = \frac{175E_2^4 + 1092E_2E_4 + 3323E_6}{743178240a^6}\,.$$

Too many trees

Quasi-modularity of $\mathcal{F}_n$ hard to see from

$$\begin{aligned} -\mathcal{W}_0'^2 + m^2 \wp &= (2\pi i)^2 q \partial_q \mathcal{F}_0 \,, \\ -\mathcal{W}_0'' - 2\mathcal{W}_0'\mathcal{W}_1' &= (2\pi i)^2 q \partial_q \mathcal{F}_1 \,, \\ -\mathcal{W}_1'' - \mathcal{W}_1'^2 - 2\mathcal{W}_0'\mathcal{W}_2' - \frac{1}{4}\wp(z) &= (2\pi i)^2 q \partial_q \mathcal{F}_2 \,, \\ -\mathcal{W}_n'' - \sum_{i=0}^{n+1} \mathcal{W}_i' \mathcal{W}_{n+1-i}' &= (2\pi i)^2 q \partial_q \mathcal{F}_{n+1} \quad \text{for} \quad n \geq 2 \,. \end{aligned}$$

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Quantum geometry

Does Seiberg-Witten framework lift to $\mathcal{F}_n$?

I.e., can we introduce $\lambda(\epsilon)$ such that

$$a = \oint_A \lambda(\epsilon_1)\,, \quad a_D = \oint_B \lambda(\epsilon_1)$$

where

$$a_D = \frac{\partial \mathcal{F}(\epsilon_1)}{\partial a} \quad ?$$

ϵ-deformed Seiberg-Witten differential

Natural candidate for λ

$$-\frac{1}{\epsilon_1}\mathcal{W}''(z|\tau) - \frac{1}{\epsilon_1^2}\mathcal{W}'(z|\tau)^2 + \left(\frac{1}{\epsilon_1^2}m^2 - \frac{1}{4}\right)\wp(z) = (2\pi i)^2\frac{1}{\epsilon_1^2}q\partial_q\mathcal{F}(\tau)$$

$$\Rightarrow \lambda(\epsilon_1) = \mathcal{W}'(z|\tau) = \sqrt{m^2\wp - (2\pi i)^2 q\partial_q\mathcal{F} - \epsilon_1\mathcal{W}'' - \epsilon_1^2\frac{\wp}{4}}$$

Note that $\mathcal{W}''$ appears on the RHS: definition of λ as formal power series in ϵ_1.

Relate $\mathcal{F}_n$ to $a_D(\epsilon_1)$ via Riemann bilinear identity

Define

$$U = 2\pi i \partial_\tau \mathcal{F} \,.$$

a, ϵ_1 dependence of U fixed by

$$\oint_A \lambda(\epsilon_1) = 2\pi i a \,.$$

We show for $\mathcal{N} = 2^*$

$$2\pi i \frac{\partial a_D}{\partial \tau} = -\frac{1}{4\pi i} \frac{\partial U}{\partial a} \,.$$

Proof of quasi-modularity

This provides a proof of the quasi-modularity of $\mathcal{F}$, via

$$\frac{\partial \mathcal{F}}{\partial a} = \underbrace{-2 \oint_{B_+} \sqrt{\wp - 2\pi i \partial_\tau \mathcal{F} - \epsilon_1 \mathcal{W}'' - \epsilon_1^2 \frac{\wp}{4}}\, dz}_{\text{manifestly quasi-modular}} \,.$$

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Non-convergence of ϵ_i expansion

$$\log Z_{top} = \frac{1}{\epsilon_1 \epsilon_2} \sum_{r,s} \underbrace{\left(\sum_n \psi_n^{rs} q^n \right)}_{\text{sum of quasi-modular forms, hence absolutely convergent}} \epsilon_1^r \epsilon_2^s \,.$$

If sum over ϵ_1, ϵ_2 is convergent, can invert order of summation.

But for $|\epsilon_1|, |\epsilon_2| < \delta_n$,

$$\sum_{rs} \psi_n^{rs} \, \epsilon_1^r \epsilon_2^s = \frac{p_n(\epsilon_1, \epsilon_2)}{q_n(\epsilon_1, \epsilon_2)} \,, \quad p_n, q_n \text{ polynomials}$$

such that coefficients of ϵ_1, ϵ_2 increase with n without bound

$$\Rightarrow \lim_{n \to \infty} \delta_n = 0 \,.$$

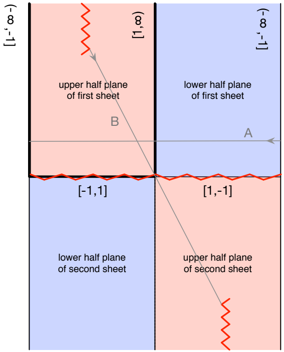
Beyond formal power series

Can we make sense of $\mathcal{F}$ beyond a formal expansion in ϵ_1?

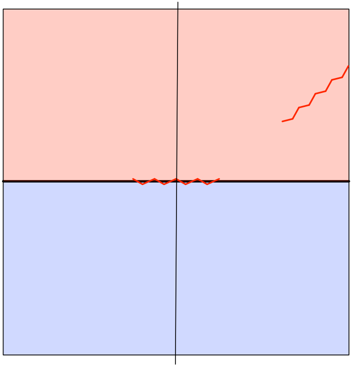
Case study: pure SU(2) (Mathieu equation)

$$[\epsilon^2 \partial_q^2 - (\cos q - u)]\Psi(q) = 0\,.$$

q-plane



x-plane



Floquet theory

$$[\epsilon^2\partial_q^2 - (\cos q - u)]\Psi(q) = 0\,.$$

Differential equation with periodic potential $\quad\rightarrow\quad$ Floquet theory

Two independent solutions

$$\psi_\pm(q) = e^{i\nu_\pm q/2}\underset{\substack{\uparrow\\ \text{periodic in } q}}{\chi_\pm(q)}$$

Characteristic exponents $\nu_\pm : \nu_+ + \nu_- = 0$ determine monodromy.

$\Rightarrow\quad$ Identify $\frac{1}{2}\nu$ with a_{exact}

Exact WKB: Strategy

- Determine monodromy matrix M of differential equation via study of Borel resummed WKB solutions

$$\operatorname{tr} M = e^{a_{exact}} + e^{-a_{exact}}$$

- Invert $a_{exact}(u, \epsilon)$ to obtain $u_{exact}(a, \epsilon)$.
- Obtain $\mathcal{F}_{exact}$ via

$$u_{exact} = \Lambda \partial_\Lambda \mathcal{F}_{exact}$$

Borel transform

- Borel transform

$$\psi(\epsilon) = \sum_{k=0}^{\infty} \psi_k \epsilon^k \quad \rightarrow \quad \psi_B(y) = \sum_{k=1}^{\infty} \psi_k \frac{y^{n-1}}{(n-1)!}\,.$$

- Inverse Borel transform

$$\psi_B(y) \quad \rightarrow \quad \mathcal{S}_\theta[\psi](\epsilon) = \psi_0 + \int_{\underset{\substack{\uparrow \\ \text{ray in direction } \theta}}{\ell_\theta}} e^{-\frac{y}{\epsilon}} \psi_B(y)\, dy\,.$$

Stokes lines

- If integral in direction θ exists, $S_\theta[\psi] \sim \psi(\epsilon)$.
- If integral is singular along $\theta = \theta_{sing}$, but exists along $\theta_\pm = \theta \pm \epsilon$,

$$S_{\theta_+}[\psi] \sim \psi(\epsilon) \sim S_{\theta_-}[\psi]\,,$$

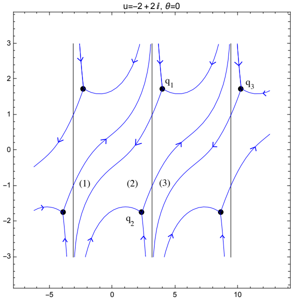
  hence $S_{\theta_+}[\psi]$ and $S_{\theta_-}[\psi]$ differ by exponentially suppressed contributions in ϵ.

Rays in the direction of such θ_{sing} subdivide the y-plane into Stokes sectors.

Stokes regions

In case of q dependence: $\psi(q,\epsilon) \quad \rightarrow \quad \theta_{sing}(q)$

Keeping θ fixed but varying q divides q-plane into Stokes regions.



The Stokes lines emanate from turning points q_i, i.e. zeros of the potential $V = \cos q - u$.

The WKB solution

$$\psi(q,\epsilon) = \exp\left(\int^q S\,dq\right), \qquad S = \frac{1}{\epsilon}S_{-1} + S_0 + \epsilon S_1 + \dots .$$

$$\begin{aligned} S_{-1}^{\pm} &= \pm\sqrt{\cos q - u}\,, \\ S_0^{\pm} &= -\frac{1}{2} d\log S_{-1}/dq = \frac{1}{4}\frac{\sin q}{\cos q - u}\,, \\ S_1^{\pm} &= \pm\frac{\cos 2q + 8u\cos q - 9}{64(\cos q - u)^{5/2}}\,, \end{aligned}$$

The WKB solution in Floquet form

$$S_{odd} = \frac{1}{2}(S^+ - S^-)\,, \quad S_{even} = \frac{1}{2}(S^+ + S^-)\,,$$

$$\Rightarrow \quad S_{even} = -\frac{1}{2}\frac{d\log S_{odd}}{dq}\,.$$

$$\Rightarrow \quad \psi_\pm(q,\epsilon) = \frac{1}{\sqrt{S_{odd}(q,\epsilon)}} \exp\left(\pm \int^q S_{odd}(q,\epsilon)\, dq\right)\,.$$

Normalization determined by lower bound of integration.

Basis of WKB solutions depends on Stokes region

Resumming the two formal WKB solutions in region I, analytically continuing to region II

$$\neq$$

Resumming the two formal WKB solutions in region II

Connection matrices

Bases in two different Stokes regions are connected via so-called connection matrices.

Connection matrices are simple, e.g.

$$S_{dom} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix},$$

when WKB solutions are normalized at the turning point from which the Stokes line being crossed emanates:

$$\Rightarrow \quad \psi_\pm(q,\epsilon) = \frac{1}{\sqrt{S_{odd}(q,\epsilon)}} \exp\left(\pm \int_{\underset{\substack{\uparrow \\ \text{turning point of potential}}}{q_i}}^{q} S_{odd}(q,\epsilon)\, dq \right).$$

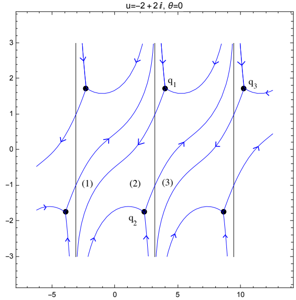
Changing normalization

$$\psi_\pm(q,\epsilon) = \frac{1}{\sqrt{S_{odd}(q,\epsilon)}} \exp\left(\pm \int_{\underset{\text{turning point of potential}}{\underset{\uparrow}{q_i}}}^{q} S_{odd}(q,\epsilon)\, dq \right).$$

Changing normalization between two turning points q_1 and q_2 via Voros multiplier:

$$\underset{\text{normalized at } q_1}{\underset{\uparrow}{(q_1)}}{}^{(i)}_{\pm} = \left(\exp\left[\pm \int_{q_1}^{q_2} S_{odd} \right] \right)_{\underset{\text{sector of resummation}}{\underset{\uparrow}{s}}} (q_2)^{\overset{\text{Borel region of resummation}}{\overset{\downarrow}{(i)}}}_{\pm}.$$

Moving in the q-plane



$$\begin{pmatrix} (q_1+2\pi)_+^{(3)} \\ (q_1+2\pi)_-^{(3)} \end{pmatrix}(q+2\pi) =$$

$$N_{2\to 3} S^{-1}_{(2)\to(3)} N_{1\to 2} S^{-1}_{(1)\to(2)} \begin{pmatrix} (q_1)_+^{(1)} \\ (q_1)_-^{(1)} \end{pmatrix}(q+2\pi)\,.$$

Monodromy

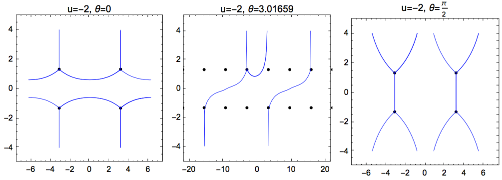
By periodicity of differential equation,

$$\left(\begin{array}{c} (q_1+2\pi)_+^{(3)} \\ (q_1+2\pi)_-^{(3)} \end{array} \right)(q+2\pi) = \left(\begin{array}{c} (q_1)_+^{(1)} \\ (q_1)_-^{(1)} \end{array} \right)(q)\,,$$

$$\Rightarrow \left(\begin{array}{c} (q_1)_+^{(1)} \\ (q_1)_-^{(1)} \end{array} \right)(q) = \underbrace{N_{2\to 3} S^{-1}_{(2)\to(3)} N_{1\to 2} S^{-1}_{(1)\to(2)}}_{\text{monodromy matrix}} \left(\begin{array}{c} (q_1)_+^{(1)} \\ (q_1)_-^{(1)} \end{array} \right)(q+2\pi)\,.$$

Evaluation of monodromy matrix: sector dependence

$$\operatorname{tr} M = \left(2\cosh\frac{a}{\epsilon} + e^{\mp\frac{1}{\epsilon}(a(1+2n)+2a_D)} \right)_{\underset{\text{sector}}{\underset{\uparrow}{(n,\pm)}}}$$



Sector dependence compensated by jumping behavior of Voros multipliers

$$(e^{\frac{a_\gamma}{\epsilon}})_- = (e^{\frac{a_\gamma}{\epsilon}})_+ (1 + (e^{\frac{a_{\gamma_0}}{\epsilon}})_+)^{-(\overset{\overset{\text{cycle circling double Stokes line}}{\downarrow}}{\gamma_0},\gamma)} .$$

Determining $a_{exact}(u, \epsilon)$

Need to solve

$$e^{\frac{a_{exact}}{\epsilon}} + e^{-\frac{a_{exact}}{\epsilon}} = \left(e^{\frac{a}{\epsilon}} + e^{-\frac{a}{\epsilon}} + e^{\mp \frac{1}{\epsilon}(a(1+2n)+2a_D)} \right)_{(n,\pm)}$$

Transseries ansatz

$$a = a_{exact} + \epsilon \sum_{m,n=1}^{\infty} c_{mn} e^{-\frac{1}{\epsilon}(2m\, a_{exact} + 2n\, a_D(a_{exact}, \epsilon))} .$$

Determining $\mathcal{F}$

$$\mathcal{F} = \frac{1}{\pi^2} a^2 \log \Lambda + \frac{i}{2\pi} \epsilon_1^2 \exp\left(-\frac{2a + 2a_{\mathrm{D}}(a/(\sqrt{2}\Lambda))}{\epsilon_1} \right) + \dots .$$

Conclusions

- Exact WKB takes us someways towards controlling the semi-classical expansion, but computations with transseries require better control.
- WKB offers systematic approach to quantum geometry, in the NS limit.
- Fun to be had along the way (e.g. with quasi-modularity).