Semiclassical decoding

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In Physics, we usually have two different approaches to a theory:

A non-perturbative definition: it often relies heavily on numerics (numerical diagonalization, lattice gauge theory), so it is not very illuminating for human brains. Not always available! (string theory)

A perturbative/semiclassical series: intuitive, analytic to a large extent, but often incomplete due to the existence of non-perturbative effects

In resurgence, we enlarge perturbative series to *trans-series*, to try to account for the exact non-perturbative answer. This leads to the following working definition:

> A non-perturbative function in quantum theory can be *semiclassically decoded* if it can be written as the Borel-Ecalle resummation of a trans-series

This seems to be the case in many examples in Quantum Mechanics, and in some simple low-dimensional/topological/supersymmetric theories. It might not be true in Yang-Mills theory (in infinite volume).

Semiclassical decoding in string theory

In order to see whether semiclassical decoding is at work in string theory, one needs first a non-perturbative definition.

For example, in non-critical (super)string theory, non-perturbative definitions involve nonlinear ODEs of the Painlevé type, which can be semiclassically decoded [Ecalle, Costin]

In this talk I will discuss topological strings whose target is a *toric (non-compact) Calabi-Yau* (CY) threefold. They provide the next layer of complexity beyond non-critical (super)strings. They are also useful for AdS/CFT (i.e. in ABJM theory).

As is well-known, (topological) string theory is defined by a formal, divergent genus expansion

$$F(\lambda, g_s) \sim \sum_{g \geq 0} F_g(\lambda) g_s^{2g-2} \qquad F_g(\lambda) \sim (2g)!$$

↑ modulus

The free energies $F_g(\lambda)$ can be computed genus by genus by using many different techniques (enumerative geometry, topological vertex, mirror symmetry/BKMP conjecture…).

The most efficient technique is the BCOV holomorphic anomaly equation, combined with modularity [Klemm et al.], which gives ~100 orders of the genus expansion

Question: is there a *well-defined, computable function* of the moduli λ and the string coupling constant which has the genus expansion as an asymptotic expansion?

This is the problem of formulating (topological) string theory non-perturbatively.

A non-perturbative definition

As we learned in the previous talk, given a toric CY X, we can associate to it a positive, trace class operator ρ_X . Its *spectral traces* are all finite,

$$Z_\ell = \mathrm{Tr}\, \rho_X^\ell = \sum_{n \geq 0} \mathrm{e}^{-\ell E_n}, \qquad \ell = 1, 2, \cdots$$

and its *Fredholm determinant* is well defined

$$\Xi_X(\kappa) = \det(1 + \kappa \rho_X) = \exp\left\{ \sum_{\ell=1}^{\infty} \frac{(-1)^\ell}{\ell} Z_\ell \kappa^\ell \right\}$$

$$= 1 + \sum_{N=1}^{\infty} Z_X(N, \hbar) \kappa^N$$

⟵ *"fermionic" spectral traces*

We define the non-perturbative topological string free energy as

$$F_X(N, \hbar) = \log Z_X(N, \hbar)$$

In the asymptotic, 't Hooft-like limit

$$N \to \infty \qquad \frac{N}{\hbar} = \lambda \quad \text{fixed}$$
$$\hbar \to \infty$$

we recover (conjecturally) the genus expansion of the topological string!

$$F_X(N, \hbar) = \log Z_X(N, \hbar) \sim \sum_{g \geq 0} F_g(\lambda) \hbar^{2-2g}$$

$$\hbar \sim g_s^{-1}$$

This is the *asymptotic* version of the conjecture in [Grassi-Hatsuda-M.M.]. There is an *exact* version of this conjecture which makes it possible to define the fermionic spectral trace for *arbitrary* N, and not just for positive integer N.

(there are *many* high-precision tests of the above conjectures and no single counterexample)

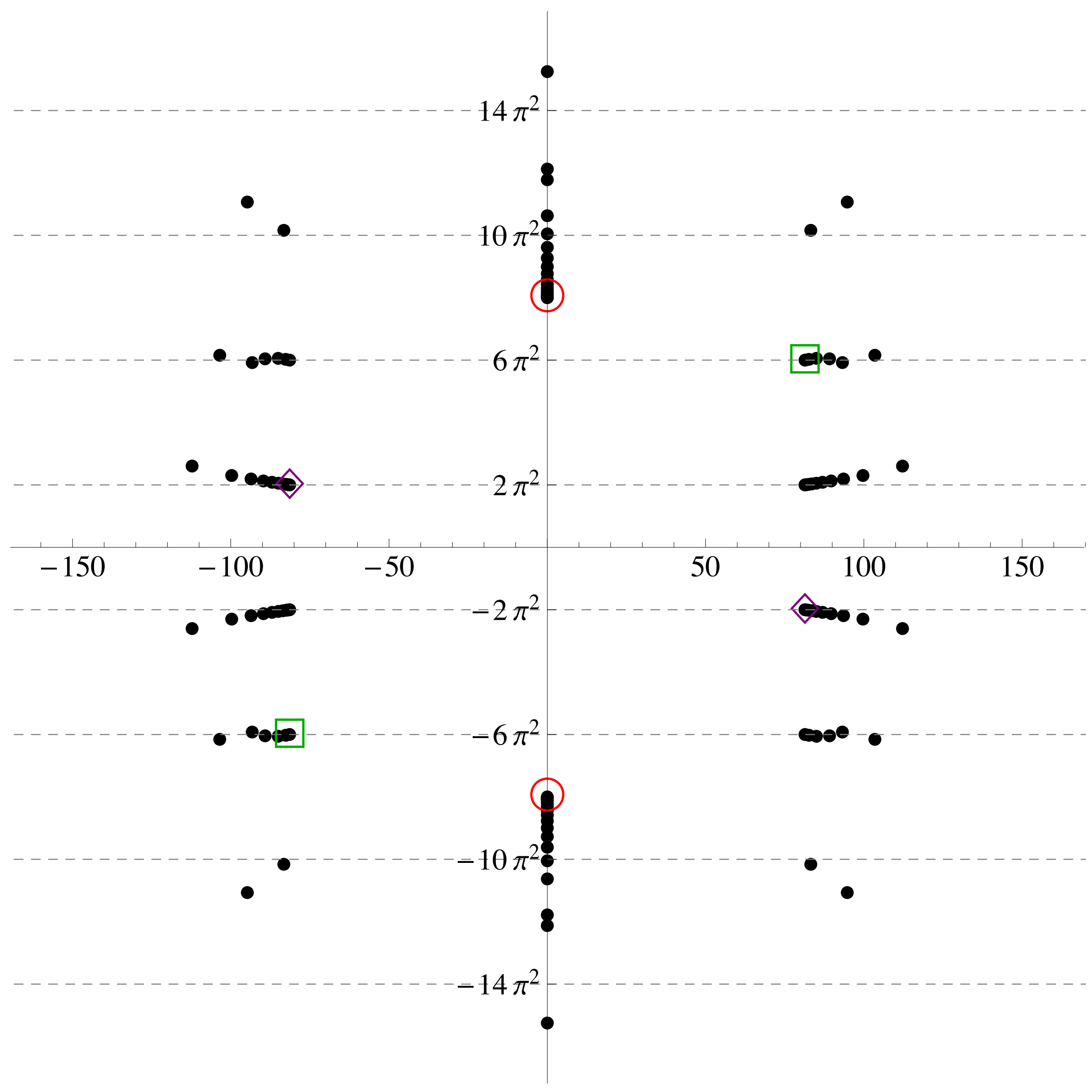
We can now ask the following question [Couso-M.M.-Schiappa]:

Can we decode semiclassically the non-perturbative definition I have just given for the topological string?

In order to answer this question we should first study the Borel summability of the perturbative, genus expansion. This is feasible in since we can generate many terms of this expansion

In the following, I will focus on a concrete model, in which the CY is local $\mathbb{P}^2$.

It turns out that the perturbative genus expansion *is Borel summable* for almost all real, positive λ



singularities in the Borel plane for

$$\lambda = \frac{1}{\pi}$$

We can then do standard Borel resummation of the perturbative series and get numerical answers.

For N=2 and $\hbar = 4\pi$ we obtain

$$F_{\rm Borel}(N=2, \hbar = 4\pi) = -\underline{9.049\,862\,103}\,051\,21\ldots$$

Our non-perturbative definition gives

$$F_{\mathbb{P}^2}(N=2, \hbar = 4\pi) = \log\left(\frac{5}{324} - \frac{1}{12\sqrt{3}\pi}\right)$$
$$= -\underline{9.049\,862\,102}\,738\,02\ldots$$

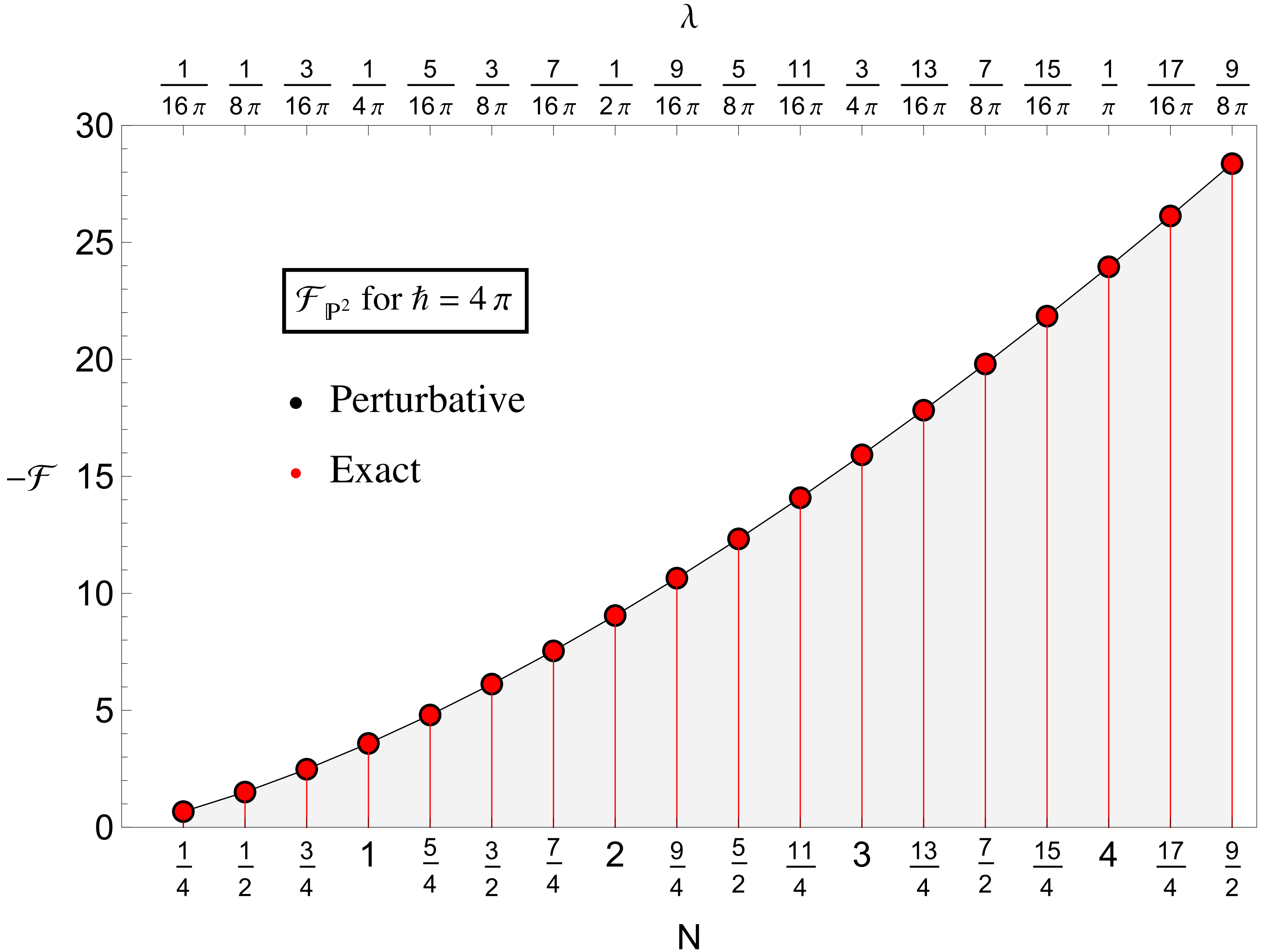
This is *not* the same number!

We conclude that *the perturbative series is Borel summable but its resummation does not agree with the non-perturbative definition*

The same thing happens to the $1/N$ expansion of the ABJM matrix model [Grassi-M.M.-Zakany] and in the semiclassical expansion of the "quantum volume" appearing in WKB quantization [Codesido-M.M.-Schiappa]

This might well be the true novelty of "stringy", doubly-factorially divergent expansions, and is in contrast to standard Borel summable perturbative expansions where Borel resummations typically agree with the exact answer (e.g. the quartic oscillator in QM)

Note however that the numbers are *very close*, for many values of λ



The difference is not visible to the naked eye!

This is in fact evidence for our conjecture, since the difference between $F_X(N,\hbar)$ and the Borel resummation should be *exponentially small*, i.e. a non-perturbative effect.

Can we compute this NP effect explicitly? Can we decode our non-perturbative definition in terms of a trans-series of the form

$$\sum_{g\geq 0} g_s^{2g-2}\, F_g(\lambda) + C \mathrm{e}^{-A/g_s} \sum_{g\geq 0} g_s^{g-1}\, F_g^{(1)}(\lambda) + \cdots \qquad ?$$

↑ 1-instanton correction

The perturbative free energies of topological string theory can be promoted to non-holomorphic objects which satisfy the holomorphic anomaly equation of BCOV

As shown by [Couso et al.], there are trans-series solutions to this equation, with the right resurgent properties. The instanton action turns out to be a particular period of the CY, in agreement with previous proposals [Balian-Parisi-Voros, Drukker-M.M.-Putrov]

We can now consider the Borel resummation of the trans-series obtained in this way and compare it to our non-perturbative definition. We have included the one-instanton correction with a natural appropriate parameter *C*.

We get a remarkable agreement for a large range of values!

$$F_{\mathbb{P}^2}(N=2, \hbar = 4\pi) = \log\left(\frac{5}{324} - \frac{1}{12\sqrt{3}\pi}\right)$$

$$= -\underline{9.049\,862\,102\,738\,02}042\ldots$$

$$F_{\text{Borel}} + C F^{(1)}_{\text{Borel}} = -\underline{9.049\,862\,102\,738\,02}\ldots$$

↑ 1-instanton correction

$$F_{\text{Borel}} = -\underline{9.049\,862\,10}3\,051\,21\ldots$$

This gives evidence that our non-perturbative completion can be "semiclassically decoded" in terms of the above trans-series

Conclusions

We have given a *rigorous* and *concrete* non-perturbative definition of topological string theory on toric CYs, in the spirit of large *N* dualities.

We have shown that this definition can be decoded in terms of a natural trans-series coming from the holomorphic anomaly equation. In particular, it is exponentially close to the Borel-resummed perturbative series, as required by a *bona fide* completion.

$Z_X(N,\hbar)$ can be written as a matrix model [M.M.-Zakany].
Can we compute the trans-series directly in this context?

What is the geometric and physical meaning of the trans-series we have obtained? Do they correspond to non-perturbative objects in topological string theory (e.g. topological D-branes)?

Thank you for your attention!