Resurgence in Quantum Mechanics

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Table of Content

References:

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Borel nonsummable series in QM

sine-Gordon Quantum mechanics

$$S_{\mathrm{E}} = \int d\tau \left[\frac{1}{4g^2} \left(\frac{d\theta}{d\tau} \right)^2 + \frac{m^2}{4g^2} \sin^2 \theta \right]$$

Ground state energy has Divergent Perturbation Series in g^2

$$E_{\mathrm{pert}}(g^2) = \lim_{\beta \to \infty} \frac{-1}{\beta} \log \int D\theta(t) e^{-S_{\mathrm{E}}} \sim -\frac{2m}{\pi} \sum_{K=0}^{\infty} K! \left(\frac{g^2}{2m} \right)^K$$

Pole on positive real axis of Borel plane : non-summable

$$BE_{\mathrm{pert}}(g^2 t) \sim -\frac{2m}{\pi} \sum_{K=0}^{\infty} \left(\frac{g^2 t}{2m} \right)^K = -\frac{2m}{\pi} \frac{1}{1 - \frac{g^2}{2m} t}$$

Resurgence : Bions

Borel resummation has imaginary ambiguity for $g^2 > 0$

$$\mathbb{E}_{\rm pert}(g^2 \pm i0) = \int_0^\infty dt e^{-t} BE_{\rm pert}(tg^2 \pm i0)$$

$$= -\frac{2m}{\pi}\int_0^\infty dt e^{-t}\frac{\mathcal{P}}{1-\frac{g^2}{2m}t} \mp i\frac{4m^2}{g^2}e^{-\frac{2m}{g^2}}$$

Search for Nonperturbative Saddle points

Instantons as nonperturbative saddle points with $S_I = \frac{m}{g^2}$

Bion : A pair of Instanton and Anti-instanton with $S_{\rm Bion} = \frac{2m}{g^2}$

Interaction of instanton and anti-instanton at separation τ_r

$$V(\tau_r) = -\frac{4m}{g^2}e^{-m\tau_r} + \epsilon m\tau_r$$

Large τ_r is regularized by ϵ species of Fermion (zero modes)

Well-defined integral when $g^2 < 0 \to$ analytic cont.to $g^2 > 0$

(Bogomolny PL**B91** 431 (1980), Zinn-Justin NP**B192** 125 (1981)); $\cdots$

Exact Saddles of Complexified Theory

$\rightarrow$ Bion imaginary ambiguity cancels perturbative imaginary ambiguity

Bion is not an Exact Solution

To find saddle points, Complexify the Theory

(Witten, [arXiv:1001.2933 [hep-th]]; $\cdots$)

Bions as Exact Saddle Points in Complexified Theory

Fermionic Deformation

Double-Well Potential with ϵ number of fermions (SUSY at $\epsilon = 1$)

Complexify the theory

Exact Bion Solutions : Real Bion and Complex Bion

(Behtash et al, PRL**116**, 011601 (2016))

Cancellation of Real Bions and Complex Bions (sine-Gordon QM)

$\rightarrow E_{\text{ground}} = E_{\text{real bion}} + E_{\text{complex bion}} = 0$

Explicit evaluation of 1-loop determinant

(for sine-Gordon, $\mathbb{C}P^{N-1}$ QM) (Fujimori et al, PRD**94**, 105002 (2016))

Ubiquitous and Hidden Resurgence

At SUSY point ($\epsilon = 1$) : resurgence structure cannot be seen

$$E_{\text{ground}} = 0$$

Hidden topological angles (Behtash et al, PRL**115**, 041601 (2015))
At highly symmetric points, Resurgence structure is often hidden
(Kozcaz et al, [arXiv:1609.06198 [hep-th]])
Resurgence is Ubiquitous
Deviation away from the symmetric point reveals Resurgence

- Imaginay ambuguities of Borel resummed perturbation series
- Infinitely many powers of nonperturbative exponentials
- Resurgence of Transseries : Imaginary part of Borel resummed perturbation series on p-Bion background is cancelled by $(p+1)$-Bion semi-classical contribution

(Fujimori et al, PRD95(2017)105001)

$\mathbb{C}P^1$ QM near SUSY

(Lorentzian) $\mathbb{C}P^1$ QM with fermions

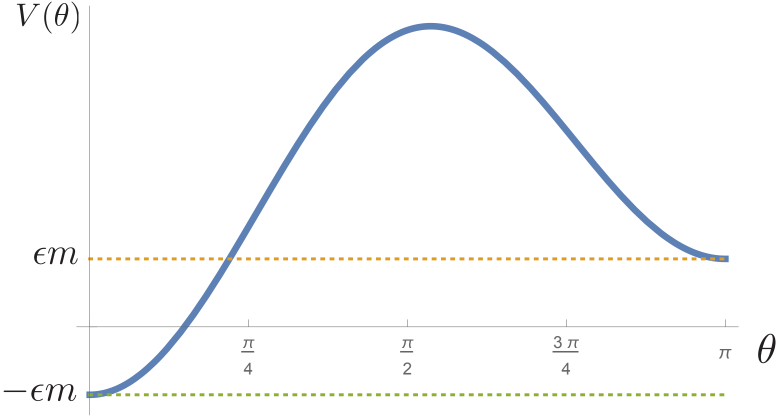
$$L = \frac{1}{g^2}\left[G\Big(|\partial_t\varphi|^2 - |m\varphi|^2 + i\bar{\psi}\mathcal{D}_t\psi\Big) - \epsilon\,\frac{\partial^2\mu}{\partial\varphi\partial\bar{\varphi}}\,\psi\bar{\psi}\right]$$

$$G = \frac{\partial^2}{\partial\varphi\partial\bar{\varphi}}\log(1+\varphi\bar{\varphi}), \quad \mathcal{D}_t = \partial_t + \partial_t\varphi\,\frac{\partial}{\partial\varphi}\log G$$

Moment Map : $\mu = m|\varphi|^2/(1+|\varphi|^2)$
SUSY ($\mathcal{N} = (2,0)$) when $\epsilon = 1$
States are classified by Fermion number $F \equiv G\psi\bar{\psi} = 0, 1$
Bosonic Lagangian for $F = 0$ sector (containing ground state)

$$L = \frac{|\partial_t\varphi|^2}{g^2(1+|\varphi|^2)^2} - V\,, \quad V = \frac{1}{g^2}\frac{m^2|\varphi|^2}{(1+|\varphi|^2)^2} - \epsilon m\frac{1-|\varphi|^2}{1+|\varphi|^2}$$

Potential with fermion deformation



Exact Ground state Energy around SUSY

Hamiltonian

$$H = -g^2(1+\varphi\bar{\varphi})^2 \frac{\partial}{\partial\varphi}\frac{\partial}{\partial\bar{\varphi}} + V$$

SUSY ($\epsilon = 1$) ground state $H_{\epsilon=1}\Psi_0 = 0$:

$$\Psi_0 = \langle\varphi|0\rangle = \exp\left(\frac{m}{2g^2}\frac{1-\varphi\bar{\varphi}}{1+\varphi\bar{\varphi}}\right) = \exp\left(-\frac{\mu}{g^2}\right)$$

We are interested in deformations around $\epsilon = 1$

Expansion around SUSY: rich and exact resurgence structure

$$\delta\epsilon \equiv \epsilon - 1, \quad \delta H = H - H_{\epsilon=1}$$

$$E = \delta\epsilon\, E^{(1)} + \delta\epsilon^2\, E^{(2)} + \cdots ,$$

First Order in SUSY Breaking

$$E^{(1)} = \frac{\langle 0|\delta H|0\rangle}{\langle 0|0\rangle} = g^2 - m\coth\frac{m}{g^2} = g^2 - m - \sum_{p=1}^{\infty} 2me^{-\frac{2pm}{g^2}}$$

Convergent power series in nonperturbative exponential e^{-2m/g^2}

$$E^{(i)} = \sum_{p=0}^{\infty} E_p^{(i)}, \quad E_p^{(i)} \propto e^{-\frac{2mp}{g^2}}$$

1st order in SUSY breaking

$$E_0^{(1)} = -m + g^2, \quad E_p^{(1)} = -2me^{-\frac{2mp}{g^2}}, \ (p \geq 1)$$

Perturbation series is terminated at finite orders.
Nonperturbative exponential does not accompany perturbation series on that background.
No resurgence structure is seen.

Second Order in SUSY Breaking

With first order correction $\delta\Psi$ to wave function, we obtain

$$E^{(2)} = -\frac{\langle \delta\Psi | H_{\epsilon=1} | \delta\Psi \rangle}{\langle 0|0 \rangle}$$

$$= g^2 - \frac{m \coth \frac{m}{g^2}}{2 \sinh^3 \frac{m}{g^2}} \left[E_i \left(-\frac{2m}{g^2} \right) + \bar{E}_i \left(\frac{2m}{g^2} \right) - 2\gamma - 2 \log \frac{2m}{g^2} \right]$$

Exponential integral functions are defined as $(x > 0)$

$$E_i(-x) = -\int_x^\infty dt e^{-t} \frac{1}{t}, \quad \bar{E}_i(x) = -\int_{-x}^\infty dt e^{-t} \frac{\mathcal{P}}{t}$$

Real and Symmetric under $m/g^2 \to -m/g^2$

Divergent Perturbation Series

Convergent series of Nonperturbative Exponentials

$$\frac{m \coth m/g^2}{2 \sinh^3 m/g^2} = 4m \sum_{k=1}^{\infty} k^2 e^{-\frac{2mk}{g^2}}$$

Divergent asymptotic series of perturbation

$$E_i\left(-\frac{2m}{g^2}\right) \sim e^{-\frac{2m}{g^2}} \sum_{n=1}^{\infty} (n-1)! \left(\frac{-g^2}{2m}\right)^n$$

Borel summable divergent series

$$\bar{E}_i\left(\frac{2m}{g^2}\right) \sim -e^{-\frac{2m}{g^2}} \sum_{n=1}^{\infty} (n-1)! \left(\frac{g^2}{2m}\right)^n$$

Borel nonsummable divergent series

Borel Resummation

Divergent perturb.series in $g^2 \to$ Borel resummation gives

$$E^{(2)} = \sum_{p=0}^{\infty} E_p^{(2)}$$

$$E_0^{(2)} = g^2 + 2m \int_0^{\infty} dt e^{-t} \frac{1}{t - \frac{2m}{g^2 \pm i0}}$$

$$E_p^{(2)} = \left[2m \int_0^{\infty} dt\, e^{-t} \left(\frac{(p+1)^2}{t - \frac{2m}{g^2 \pm i0}} + \frac{(p-1)^2}{t + \frac{2m}{g^2}} \right) \right.$$
$$\left. +4mp^2 \left(\gamma + \log \frac{2m}{g^2} \pm \frac{\pi i}{2} \right) \right] e^{-\frac{2mp}{g^2}}, \quad (p \geq 1)$$

Perturbation on p-Bion background has Borel nonsummable series, giving maginary ambiguity which is cancelled by leading semi-classical contribution of $(p+1)$-Bion.

Resurgence to all orders of nonperturbative exponentials

Multi-Bion Solutions in Complexified Theory

Complexification : $\varphi \equiv \varphi_R^{\mathbb{C}} + i\varphi_I^{\mathbb{C}}$ and $\tilde{\varphi} \equiv \varphi_R^{\mathbb{C}} - i\varphi_I^{\mathbb{C}}$ (independent)

$$S_E = \int_0^\beta d\tau \left[\frac{\partial_\tau \varphi \partial_\tau \tilde{\varphi}}{g^2(1+\varphi\tilde{\varphi})^2} + V(\varphi\tilde{\varphi}) \right]$$

Saddle point solutions in finite interval : $\varphi(\tau+\beta) = \varphi(\tau)$
elliptic function cs with periods $2K(k)$ and $4iK'(k)$, moduli (τ_c, ϕ_c)

$$\varphi = e^{i\phi_c} \frac{f(\tau-\tau_c)}{\sin\alpha}, \quad \tilde{\varphi} = e^{-i\phi_c} \frac{f(\tau-\tau_c)}{\sin\alpha}, \quad f(\tau) = \mathrm{cs}(\Omega\tau, k)$$

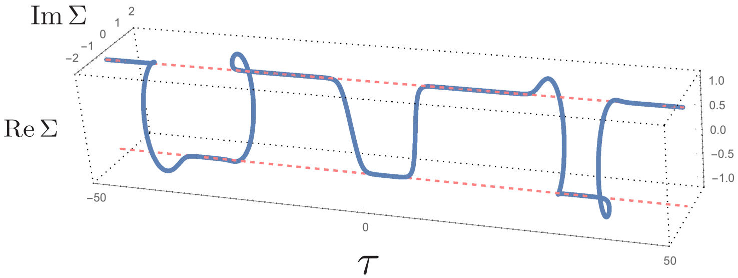
Period $\beta = (2pK + 4iqK')/\Omega$: solution labeled by $\sigma = (p, q)$
Asymptotic forms for large β

$$S \approx pS_{\text{bion}} + 2\pi i\epsilon l, \quad S_{\text{bion}} = \frac{2m}{g^2} + 2\epsilon \log\frac{\omega+m}{\omega-m}$$

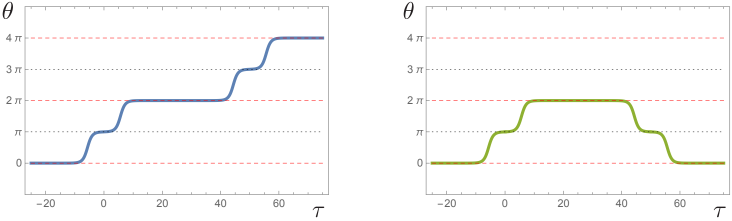
Multi-Bion in complex space

Position of n-th instanton and antiinstaton

$$\tau_n^{\pm} = \tau_c + \frac{n-1}{\omega p}(\omega\beta - 2\pi i q) \pm \frac{1}{2\omega}\log\frac{4\omega^2}{\omega^2 - m^2}$$



Multi-bion solution $\Sigma(\tau) = (1-\varphi\tilde{\varphi})/(1+\varphi\tilde{\varphi})$ for $(p,q) = (3,1)$



Multi-bion solution: $\theta = -2\arctan|\varphi|$ for $(p, q) = (2, 0)$ (left) and for $(p, q) = (2, 1)$ (right) $\Sigma(\tau) = (1 - \varphi\tilde{\varphi})/(1 + \varphi\tilde{\varphi})$ for $(p, q) = (3, 1)$

One-Loop Determinant for Massive Modes

Integrating over Fluctuations around p-Bion Saddle points
One-Loop Determinant for non-zero modes $\det''\Delta$
$\approx$ product of determinant of constituent (anti-)instantons
Quasi-moduli : relative position τ_r and relative phase ϕ_r

$$\frac{Z_p}{Z_0} = \frac{1}{p} \int \prod_{i=1}^{2p} \left[d\tau_i \wedge d\phi_i \, \frac{2m^2}{\pi g^2} \exp\left(-\frac{m}{g^2} - V_i \right) \right]$$

$$V_i = m\epsilon_i(\tau_i - \tau_{i-1}) - \frac{4m}{g^2} e^{-m(\tau_i - \tau_{i-1})} \cos(\phi_i - \phi_{i-1})$$

Lagrange multiplier σ to impose the periodicity

$$2\pi\delta\big(\sum_i \tau_i - \beta\big) = m \int d\sigma \exp\big[im\sigma(\sum_i \tau_i - \beta)\big]$$

Quasi-Moduli integral

Deform τ_r, ϕ_r in complex plane
Determine integration paths (thimbles) and their weight
(by intersection of dual thimbles with the original path)
Integral for each Quasi-Moduli (Prototype)

$$I = \int_{\mathcal{C}} dy \exp\left[-V(y)\right], \quad V(y) \equiv a e^{-y} + by, \quad \mathrm{Re}\, b > 0$$

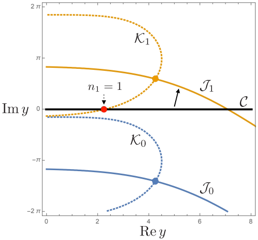
Instanton-instanton : $a > 0$, Instanton-Antiinstanton : $a < 0$
Gradient flow equation

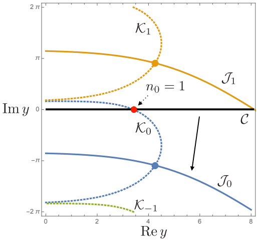
$$\frac{\partial y}{\partial t} = \overline{\frac{\partial V}{\partial y}} = -\bar{a} e^{-\bar{y}} + \bar{b}$$

$\partial y / \partial t = 0$: Saddle point y_s
Thimble $y(t)$ (steepest descent contour): $\lim_{t \to -\infty} y(t) = y_s$
Dual Thimble $y(t)$ (steepest ascent direction): $\lim_{t \to +\infty} y(t) = y_s$

Stokes Phenomena

If the dual thimble intersects with the original contour
$\rightarrow$ integration contour can be deformed to the thimble





Stokes phenomenon The original integration contour $\mathcal{C}$ intersects with $\mathcal{K}_1$ ($\mathcal{K}_0$) for $\theta > 0$ ($\theta < 0$) and hence $\mathcal{C}$ is deformed to $\mathcal{J}_1$ ($\mathcal{J}_0$)

Nonperturbative Exponentials $\rightarrow$ Exact Results

$$E = E_0 - \lim_{\beta\to\infty}\frac{1}{\beta}\log\Big(1+\sum_{p=1}^{\infty} Z_p/Z_0\Big)$$

$$E_p^{(1)} = -\lim_{\epsilon\to 1}\lim_{\beta\to\infty}\frac{1}{\beta}\frac{\partial}{\partial\epsilon}\frac{Z_p}{Z_0} = -2me^{\frac{-2pm}{g^2}}$$

$$E_p^{(2)} = -\frac{1}{2}\lim_{\epsilon\to 1}\lim_{\beta\to\infty}\frac{1}{\beta}\left[\partial_\epsilon^2\frac{Z_p}{Z_0} - \sum_{i=1}^{p-1}\partial_\epsilon\frac{Z_{p-i}}{Z_0}\partial_\epsilon\frac{Z_i}{Z_0}\right]$$

$$= 4mp^2\left(\gamma + \log\frac{2m}{g^2} \pm \frac{\pi i}{2}\right)e^{\frac{-2pm}{g^2}}$$

$\rightarrow$ Borel nonsummable perturbative series on $(p-1)$-Bion backgr.

Borel summable part is obtained by invariance $m/g \rightarrow -m/g$

Trans-series is obtained completely from multi-Bion contributions

Towards Asymptotically-Free QFT

IR Renormalon in 2D $\mathbb{C}P^{N-1}$ QFT

Bion in 2D $\mathbb{C}P^{N-1}$ QFT on $R \times S^1$ with size L

Bions in $\mathbb{C}P^{N-1}$ QM gives Bions in $L \to 0$ limit

Exact Bion solution in QM is still an Exact solution of QFT.

A new feature : Kaluza-Klein modes of fluctuations

We found 1-loop determinant of KK modes to give renormalized running coupling

$$g_{2d}^2(L) = \frac{4\pi}{N \log(\frac{1}{\Lambda^2 L^2})}$$

Bion Pole on Borel plane moves with L

$$\int \frac{e^{-t}}{\frac{4\pi}{N g_{2d}^2(L)} - t} \sim e^{-\frac{4\pi}{N g_{2d}^2(L)}} = \Lambda^2 L^2$$

Conclusions

1. QM : useful arena to explore resurgence.
2. Resurgence structure is ubiquitous, hidden at SUSY point.
3. Exact result of ground state energy of near SUSY $\mathbb{C}P^1$ QM
    - Infinite powers of nonperturbative exponentials
    - Associated divergent perturbation series with logarithms
4. Infinite tower of exact multi-bion solutions are found
5. Semi-classical contributions of multi-bions give nonperturbative contributions exactly
6. Full exact results are recovered from multi-bion amplitudes thanks to resurgence
7. Lefschetz thimble analysis is vital for quasi-moduli integrals
8. Generalizations obtained : sine-Gordon QM, $\mathbb{C}P^{N-1}$ QM, etc.
9. Extension to 2d $\mathbb{C}P^{N-1}$ sigma model is under way. Understanding IR as well as UV renormalon ?