Colour Glass, Froissart bound, and all that

What can we say about high energy scattering in QCD from first principles?

- Deep inelastic scattering at small x
- The Colour Glass Condensate
- Gluon Saturation
- Froissart bound

E. Iancu
Saclay

What is the high energy limit of QCD scattering?

- Can this be addressed in perturbation theory?
  - $\alpha_s(Q^2)$ and not $\alpha_s(s)$! $[s = (E_{C.M.})^2]$
  - Froissart bound: $\sigma(s) \leq \frac{\pi}{m_\pi^2} \ln^2 s$
  - BFKL's failure to describe the high energy limit
    - $\sigma_{BFKL}(s) \sim s^{\omega \alpha_s}$, $\omega = 4(\ln 2) N_c/\pi$
    - "Infrared diffusion": $Q^2 \to 0$ as $s \to \infty$
- Can we use perturb. theory to study quantum evolution with s ?
- What are the relevant degrees of freedom?

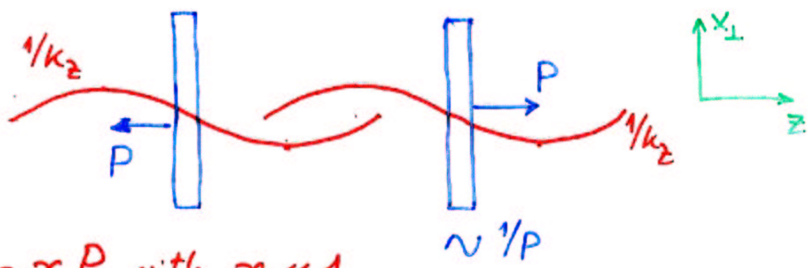
"Small-x" partons (mostly gluons) in a state of high density

"Colour Glass Condensate"

= the matter made of the small-x gluons

High-energy Scattering in QCD

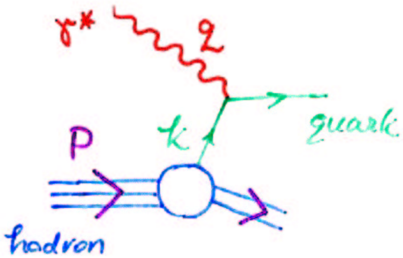
- Hadron-Hadron Collision (center-of-mass frame)



$k_z = x P$ with $x \ll 1$

High energy → Small-x tail of the hadron wavefunction

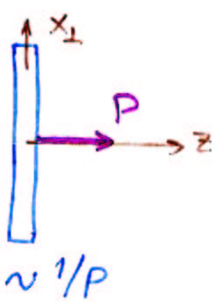
- Deep Inelastic Scattering (Bjorken frame)



$$Q^2 \equiv -q^\mu q_\mu > 0$$

$$x_{Bj} \equiv \frac{Q^2}{2P\cdot q} \sim \frac{Q^2}{s} \ll 1$$

Bjorken frame: $P^\mu \simeq (P, 0, 0, P)$ and $q^\mu \simeq (0, 0, 0, q_z)$



Feynman x: $x = \dfrac{k_z}{P}$

Kinematics ⇒ $x = x_{Bj} \ll 1$

or $k_z = \dfrac{q_z}{2}$

World's data on F_2 at $Q^2 = 15\ \mathrm{GeV}^2$

$$F_2(x, Q^2) = \sum_f e_f^2 \left[x q_f(x, Q^2) + x \bar{q}_f(x, Q^2) \right]$$

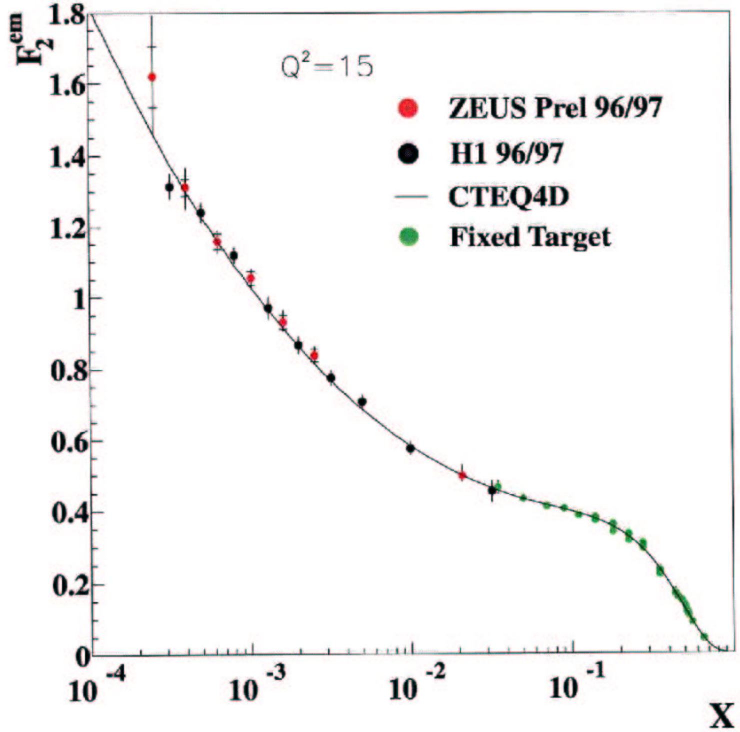


Figure 1: World's data on F_2 at $Q^2 = 15\ \mathrm{GeV}^2$ as a function of x. The solid line is a DGLAP fit by the CTEQ group [?].

$F_2 \sim \dfrac{1}{x^\lambda}$ with $\lambda = 0.4 \div 0.1$ (depending upon Q^2)

1

$$xG(x,Q^2) \equiv \frac{dN_{gluons}}{d\ln\frac{1}{x}} \propto \underbrace{\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2}}_{\text{for sufficiently high } Q^2}$$

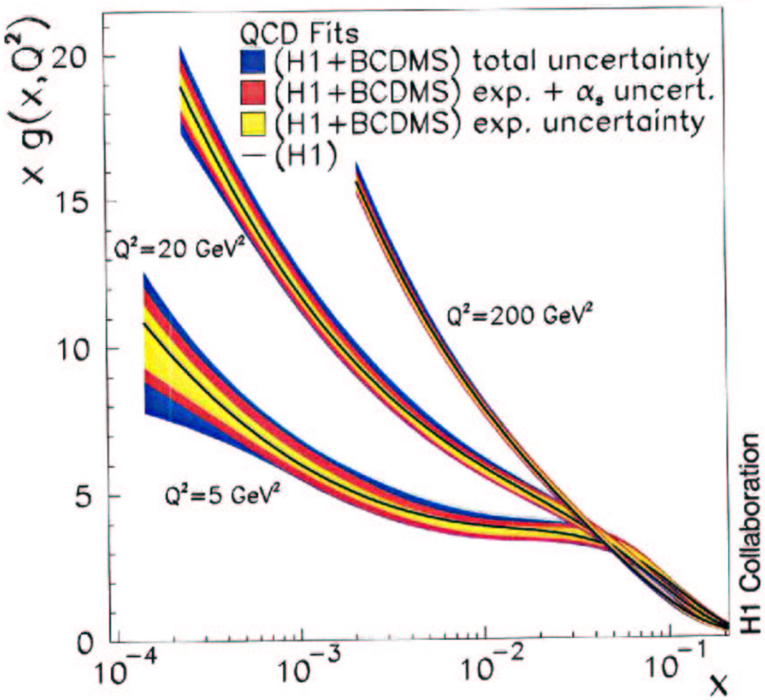
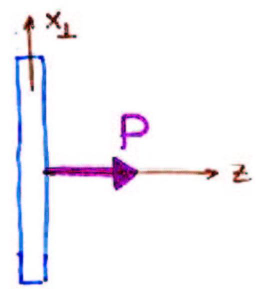


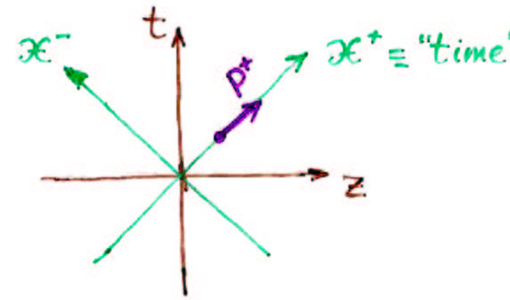
Figure 23: *Gluon distribution resulting from the NLO DGLAP QCD fit to H1 ep and BCDMS μp cross section data in the massive heavy flavour scheme. The innermost error bands represent the experimental error for fixed $\alpha_s(M_Z^2)$ =0.1150. The middle error bands include in addition the contribution due to the simultaneous fit of α_s. The outer error bands also include the uncertainties related to the QCD model and data range. The solid lines inside the error band represent the gluon distribution obtained in the fit to the H1 data alone.*

56

Light-Cone Kinematics

$$v^+ = \frac{1}{\sqrt{2}}(v^0+v^3)\,; \quad v^- = \frac{1}{\sqrt{2}}(v^0-v^3)\,; \quad v_\perp = (v^1, v^2)$$





$$z \simeq t \longleftrightarrow x^- \simeq 0$$

$$P^\mu \simeq (P,0,0,P) \longleftrightarrow P^\mu \simeq (P^+, 0^-, 0_\perp)\,; \quad P^+ = \sqrt{2}P$$

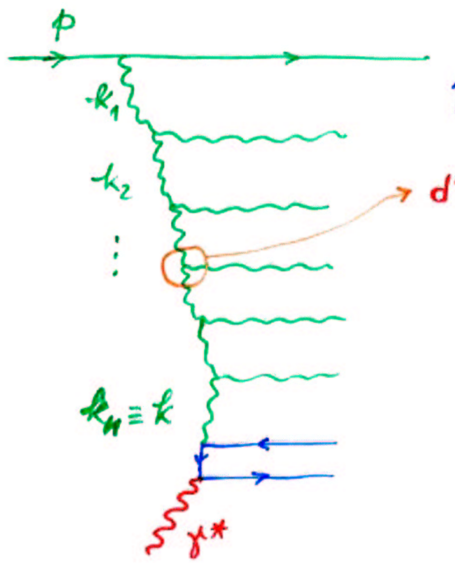
$$k\cdot x = \underbrace{k^-}_{\text{energy}}\underbrace{x^+}_{\text{time}} + \underbrace{k^+x^-}_{\text{longit.}} - \underbrace{k_\perp\cdot x_\perp}_{\text{transverse}}$$

Feyman's x: $\quad x \equiv \frac{k^+}{P^+}$ (boost invariant)

Rapidity: $\quad \tau \equiv \ln\frac{1}{x} = \ln\frac{P^+}{k^+}$

For the struck parton: $\quad \tau \sim \ln s$

The BFKL cascade



$$k^+ = x P^+ , \quad x \ll 1$$

$$p^+ \gg k_1^+ \gg k_2^+ \gg \ldots \gg k_n^+ \equiv k^+$$

$$d\mathcal{P}_\ell \sim \alpha_s N_c \frac{dk_\ell^+}{k_\ell^+} \rightarrow \ln\frac{1}{x}$$

For the whole cascade:

$$\mathcal{P}_n \sim \frac{1}{n!}\left(\alpha_s N_c \ln\frac{1}{x}\right)^n$$

Gluon distribution:

$$\frac{dN}{d\ln\frac{1}{x}} \equiv x G(x, Q^2)$$

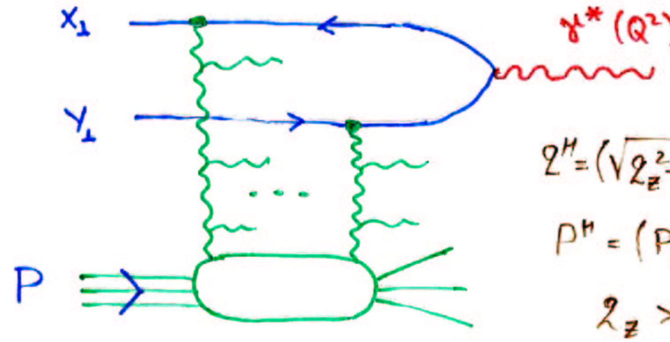
= # of gluons with transverse size $\Delta x_\perp \sim \frac{1}{Q}$ per unit $\underbrace{\text{"rapidity"}}$

$$\tau \equiv \ln\frac{1}{x} \sim \ln s$$

$$\left.\frac{dN}{d\tau}\right|_{BFKL} \sim \alpha_s N_c \sum_{n=0}^{\infty} \mathcal{P}_n = \alpha_s N_c \underbrace{e^{\omega \alpha_s N_c \tau}}_{s^{\omega \alpha_s N_c}}$$

$$\omega = \frac{4\ln 2}{\pi}$$

DIS in the dipole frame



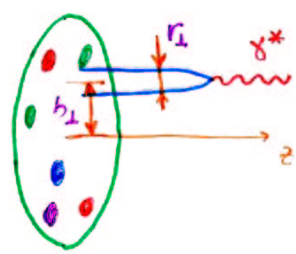
$$q^\mu = \left(\sqrt{q_z^2 - Q^2}, 0, 0, -q_z\right)$$

$$P^\mu = (P, 0, 0, P)$$

$$q_z \gg Q$$

$$\sigma_{\gamma^* p}(\tau, Q^2) = \int d^2 r_\perp \, \Phi(\tau, r_\perp^2 Q^2) \, \sigma_{dipole}(\tau, r_\perp)$$

$$\sigma_{dipole}(\tau, r_\perp) = 2\int d^2 b_\perp \underbrace{\mathcal{N}_\tau(r_\perp, b_\perp)}_{\text{Scattering amplitude}}$$



$$\mathcal{N}_\tau(r_\perp, b_\perp) = 1 - \frac{\mathrm{tr}}{N_c}\left\langle V_{x_\perp}^\dagger V_{y_\perp}\right\rangle_\tau$$

$$V_{x_\perp}^\dagger = P \exp\left\{ig\int dx^- A^+(x^-, x_\perp)\right\}$$

$\left\langle \mathrm{tr}\, V_{x_\perp}^\dagger V_{y_\perp}\right\rangle_\tau$ = average over the hadron wave function "evolved" up to rapidity τ

"

S-matrix element

Assume:

i) Two-point function only: $\langle A_a^+(x) A_b^+(y) \rangle_\tau$

ii) Small-dipole: $r_\perp \ll$ correlation length of A^+

$$ig t^a \left(A_a^+(x_\perp) - A_a^+(y_\perp)\right) \simeq ig t^a r_\perp^i \partial^i A_a^+(x_\perp)$$

$$\mathcal{N}_\tau(r_\perp, b_\perp) \simeq 1 - \exp\left\{-\alpha_s r_\perp^2 \, xG\left(x, \frac{1}{r_\perp^2}, b_\perp\right)\right\}$$

$xG(x, Q^2, b)$ = Wigner transform of $\langle \partial^i A^+ \partial^i A^+ \rangle$

= Fock space, local, gluon distribution

- Low density / very small dipole ⇒ single scattering

$$\mathcal{N}_\tau(r_\perp, b_\perp) \simeq \alpha_s r_\perp^2 \underbrace{xG\left(x, \frac{1}{r_\perp^2}, b_\perp\right)}_{\sim s^{\omega \alpha_s N_c} \quad (\text{BFKL})}$$

"leading twist" $\left(r_\perp^2 \sim \frac{1}{Q^2}\right)$

- The scattering amplitude at fixed $b_\perp$ rises as a power of s

⇒ Violation of unitarity $(\Rightarrow \mathcal{N}_\tau \leq 1)$

Unitarization at fixed $b_\perp$

- Coherent multiple scattering

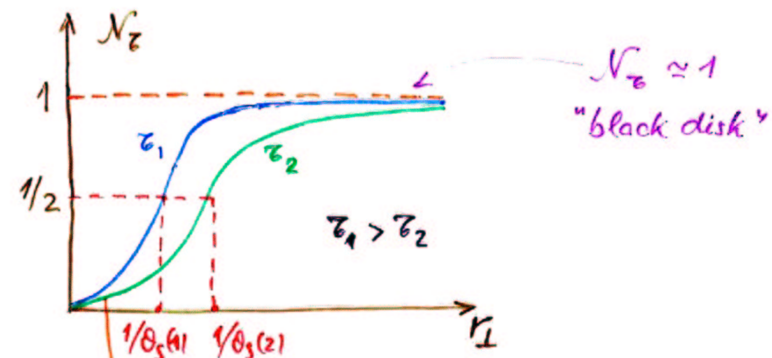
$$\mathcal{N}_\tau(r_\perp, b_\perp) = 1 - \underbrace{\exp\left\{-\alpha_s r_\perp^2 \, xG\left(x, \frac{1}{r_\perp^2}, b_\perp\right)\right\}}_{\text{"all twists"}} \leq 1$$

- Multiple scattering becomes important when

$$r_\perp^2 \gtrsim \frac{1}{Q_s^2(\tau, b)} \equiv \text{"(saturation length)}^2\text{"}$$

$$\boxed{Q_s^2(\tau, b_\perp) \sim \frac{\alpha_s N_c}{N_c^2 - 1} \, xG\left(x, Q_s^2, b_\perp\right)}$$

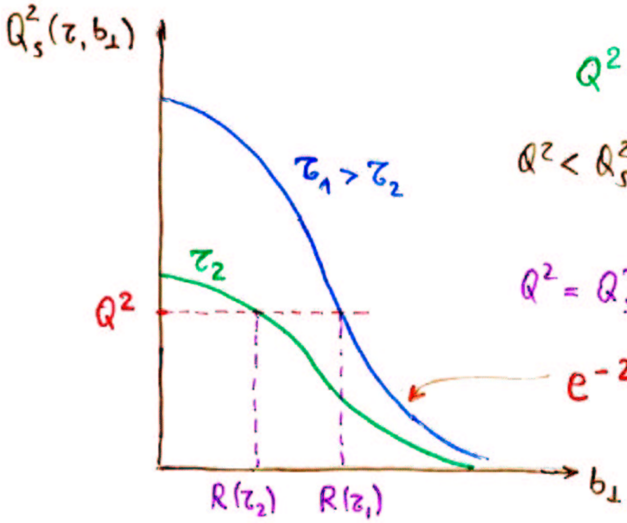
- $Q_s^2(\tau, b_\perp)$ increases with τ $(\sim e^{\omega \alpha_s \tau})$, decreases with $b_\perp$



$\mathcal{N}_\tau \propto r_\perp^2$: "colour transparency"

$b_\perp$ - dependence and the problem of Froissart bound

- The edge of the hadron is not sharp!



$Q^2 \equiv \frac{1}{r_\perp^2}$: dipole resolution

$Q^2 < Q_s^2(\tau, b_\perp) \Rightarrow \mathcal{N}_\tau(Q^2, b_\perp) \simeq 1$
"black"

$Q^2 = Q_s^2(\tau, b)$ for $b = R(\tau, Q^2)$

$e^{-2m_\pi b}$

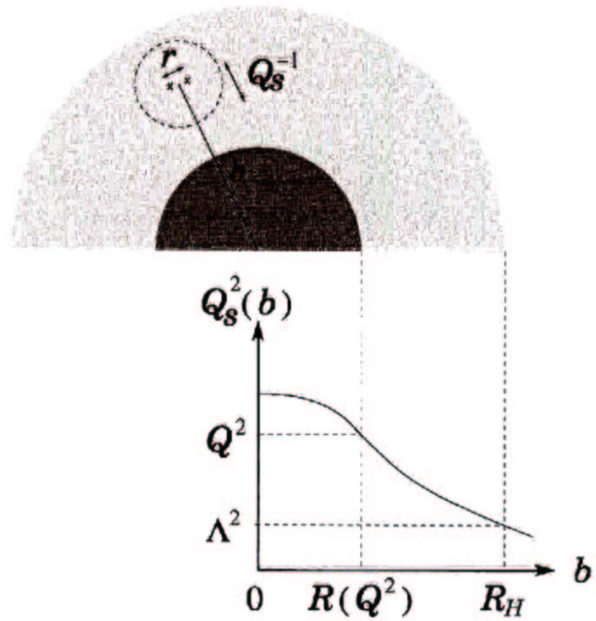
- Assume quantum evolution is local in $b_\perp$

$\Rightarrow \mathcal{N}_\tau(Q^2, b_\perp) \sim e^{\omega \alpha_s \tau} e^{-2m_\pi b} \sim 1$ for $b = R(\tau, Q^2)$

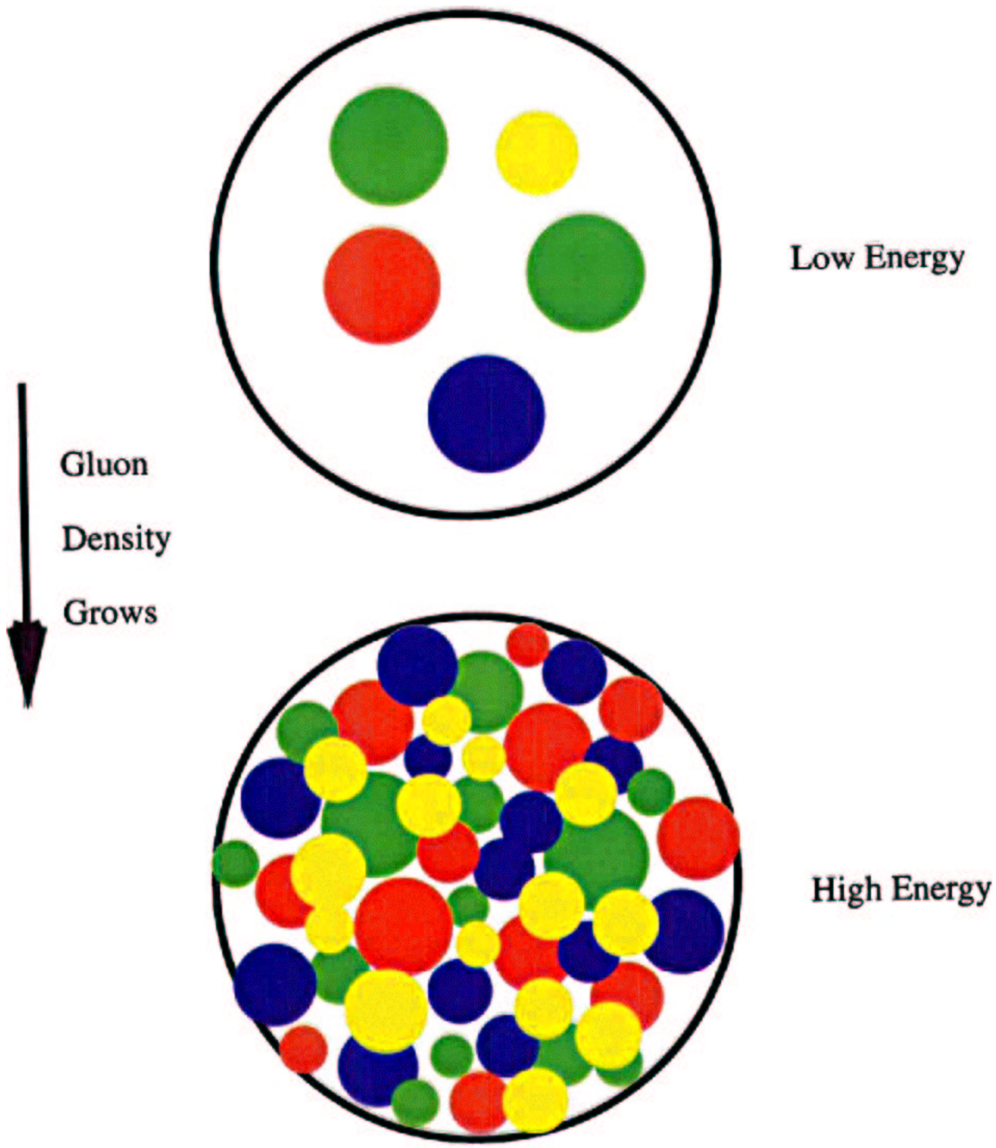
$\Rightarrow R(\tau, Q^2) \simeq \frac{\omega \alpha_s}{2 m_\pi} \tau \rightarrow \ln s$ Heisenberg, 52

$\Rightarrow \sigma_{\text{black disk}} = 2\pi R^2(\tau, Q^2) \sim (\ln s)^2$!

- However: perturbative evolution $\Rightarrow$ massless gluons



12



What happens to the high density gluons ?

- At high density (= small x), gluons overlap in the transverse plane and interact with each other.

Radiation = Recombination

$\Rightarrow$ saturation

(Gribov, Levin, Ryskin, 83)

- Non-linear effects become important when

$$\frac{\alpha_s N_c}{Q^2}\cdot\frac{xG(x,Q^2,b_\perp)}{N_c^2-1}\sim 1$$

i.e. at low transverse momenta: $Q^2 \lesssim Q_s^2(\tau,b_\perp)$

$$Q_s^2(\tau,b_\perp)\simeq \alpha_s\frac{N_c}{N_c^2-1}\,xG(x,Q_s^2,b_\perp)$$

- Unitarization of the (local) scattering amplitude

$\Longleftrightarrow$ Saturation of small-x gluons in the hadron wavefunction

- N.B. In the non-linear regime at saturation, σ_{dipole} is NOT computable in terms of the gluon distribution (a 2-point ftion) alone.

A classical effective theory

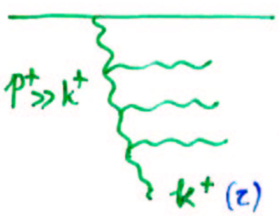
- How to compute in the non-linear regime at small x?

Typical momenta: $k_\perp^2 \sim Q_s^2 \sim e^{\omega \alpha_s \tau} A^{1/3}$

$\Rightarrow \alpha_s(Q_s^2) \ll 1$ for high energy and/or large A

- Large occupation numbers: $x G(x, Q^2; b_\perp) \sim \frac{1}{\alpha_s}$

$\Rightarrow$ Semi-classical regime (McLerran, Venugopalan, 94)



ρ_τ = colour charge density of the "fast" partons: $p^+ \gg k^+$

$[k^+ = x P^+ = e^{-\tau} P^+]$

$\rightarrow$ Lorentz contracted: $\rho_\tau(x^-, x_\perp) \propto \delta_\tau(x^-)$

$[\ln(x^- P^+) \ll \tau]$

$\rightarrow$ Time dilated: ρ_τ is independent of LC-time x^+

$$(D_\nu F^{\nu\mu})_a(x) \simeq \delta^{\mu +} \delta(x^-) \rho_a(x_\perp)$$

$\rightarrow$ Classical random variable, with weight f-tion $W_\tau[\rho]$

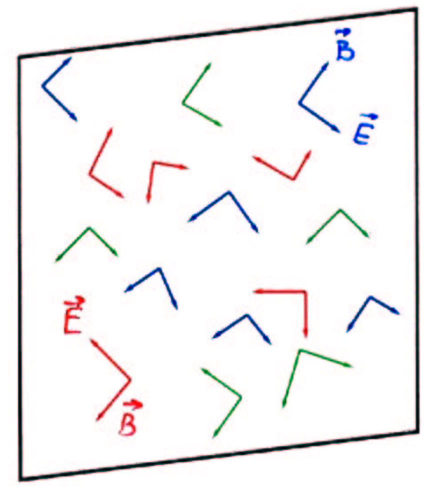
$$\langle F^{+i}(x) F^{+i}(y) \rangle_\tau = \int \mathcal{D}[\rho]\, W_\tau[\rho]\, \underbrace{\mathcal{F}_x^{+i}[\rho]}\, \mathcal{F}_y^{+i}[\rho]$$

$F^{+i} \sim \frac{1}{g}$ at saturation $\Rightarrow$ Exact classical solution

The Classical Solution

A non-Abelian Weiszäcker-Williams field.

$$\begin{cases} E_z = B_z = 0 \\ E_x = B_y; \quad E_y = -B_x \quad (\vec{E}_\perp \cdot \vec{B}_\perp = 0) \end{cases}$$



$$E^i(x^-, x_\perp) = \delta(x^-)\, V(x_\perp)\, \partial^i V^\dagger(x_\perp) \qquad (i = x, y)$$

$$V^\dagger(x_\perp) = P \exp\left(ig \int dx^- \alpha_a(x^-, x_\perp) T^a\right)$$

Wilson line

$-\nabla_\perp^2 \alpha_a = \rho_a$: 2-dimens Coulomb field

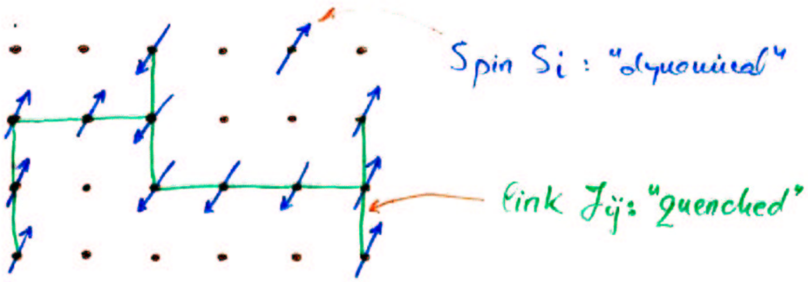
[Light-cone gauge $A^+ = 0$]

Why "Colour Glass" ?

- Spin glass

Random distribution of spins $S_i = \pm 1$

$\Rightarrow H = -\sum_{i,j} J_{ij} S_i S_j$ with random links J_{ij}



$\Rightarrow$ The spins thermalize for a fixed configuration of links

$$F[T] = -T \int [dJ_{ij}]\, W[J]\, \ln Z_T[J]$$

$$Z_T[J] = \sum_{\{S_i\}} e^{-\beta H[J]}$$

- Analogy : $S_i \leftrightarrow F_a^{+i}(x)$ $\quad J_{ij} \leftrightarrow \rho_a(x)$

- Very different from a plasma !

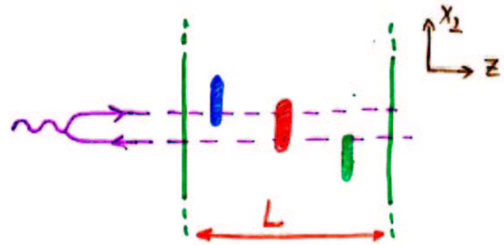
Mobile charges $\Rightarrow$ The current j^μ is determined by the background field A^μ, and not vice-versa!

$$\partial_\nu F^{\nu\mu} = j^\mu[A]$$

Why "condensate" ?

- Coherent state with occupation number $\sim \frac{1}{\alpha_s}$

- MV model : Colour sources $\equiv$ Valence partons



$$\langle \rho_a(x^-, x_\perp)\, \rho_b(y^-, y_\perp) \rangle_A = \delta^{ab}\, \delta(x^- - y^-)\, \delta^2(x_\perp - y_\perp)\, \frac{\mu_A}{L}$$

$$\mu_A \sim \alpha_s A^{1/3}$$

- Gluon density in the transverse phase-space

$$\frac{d^5 N}{d\tau\, d^2k_\perp\, d^2b_\perp} \xleftrightarrow[\text{Wigner tr.}]{} \langle F_a^{+i}(x)\, F_a^{+i}(y) \rangle_A$$

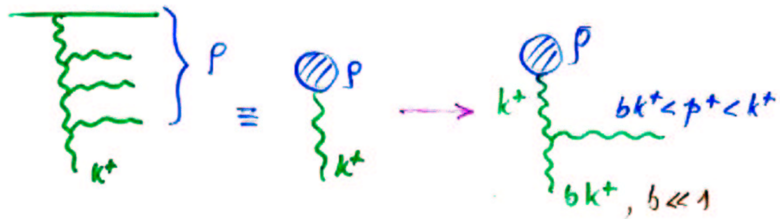
$$\langle F^{+i}(x_\perp)\, F^{+i}(y_\perp) \rangle_A = \frac{1 - \exp\left\{-r_\perp^2\, \alpha_s N_c \mu_A \ln \frac{1}{r_\perp^2 \Lambda^2}\right\}}{\alpha_s\, r_\perp^2}$$

Saturation scale: $Q_s^2(A) = \alpha_s N_c \mu_A \ln \frac{Q_s^2}{\Lambda^2} \sim \alpha_s^2 A^{1/3}$

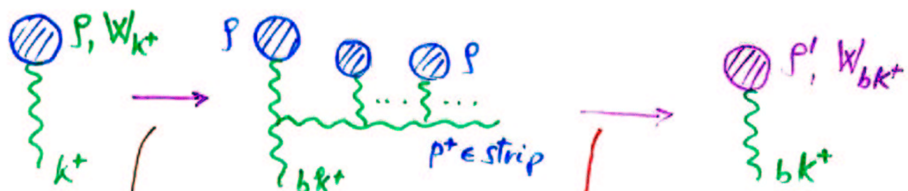
$$\varphi_A(k_\perp) \simeq \begin{cases} \dfrac{\mu_A}{k_\perp^2} \sim A^{1/3} & \text{if } k_\perp \gg Q_s \\[2mm] \dfrac{1}{\alpha_s} \ln \dfrac{Q_s^2}{k_\perp^2} \sim \ln A & \text{if } k_\perp \ll Q_s \end{cases}$$

Non-Linear Quantum Evolution

- How to compute the weight ftion $W_\tau[\rho]$ at high energy?
- Renormalization group in k^+
  Jalilian-Marian, Kovner, Leonidov, Weigert, 97
- Linear evolution $\leftrightarrow$ BFKL

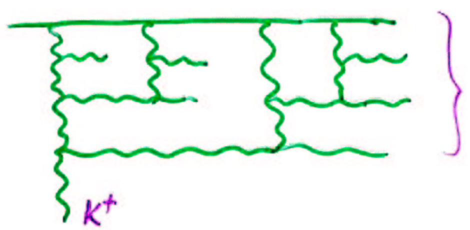


- Non-linear evolution



One-loop background field calculation

Matching to a modified effective theory



ρ : effective source

The Renormalization Group Equation

$$\frac{\partial}{\partial\tau} W_\tau[\alpha] = \frac{1}{2}\int_{x_\perp, y_\perp} \frac{\delta}{\delta\alpha_\tau(x_\perp)}\, \chi_{x_\perp y_\perp}[\alpha]\, \frac{\delta}{\delta\alpha_\tau(y_\perp)}\, W_\tau[\alpha]$$

$$\chi_{x_\perp y_\perp} = \underbrace{\int_{z_\perp} \frac{x^i - z^i}{(x-z)^2}\,\frac{y^i - z^i}{(y-z)^2}}_{\text{BFKL like}}\ \underbrace{\left(1 + V_x^\dagger V_y - V_z^\dagger V_y - V_x^\dagger V_z\right)}_{\text{Wilson lines: } V^\dagger_{x_\perp} = P\, e^{ig\int dx^- \alpha}}$$

- A functional Fokker-Planck eq. (E. I., Leonidov, McLerran 2000)
- Ordinary evolution eqs. for observables

$$\frac{\partial}{\partial\tau}\langle \mathrm{tr}\, V_x^\dagger V_y\rangle = \int \mathcal{D}[\alpha]\ \mathrm{tr}(V_x^\dagger V_y)\ \frac{\partial}{\partial\tau} W_\tau[\alpha]$$

- Equivalent to eqs. established within other formalisms by Balitsky (96), Kovchegov (99), Weigert (2000)
- Path-integral solution (Blaizot, E. I., Weigert, 2002)
- Langevin eq : a random walk on a group manifold

$$V_i^\dagger = e^{ig\alpha_i}\, V_{i-1}^\dagger$$

$$\langle \alpha_i \alpha_i \rangle = \chi[V_{i-1}] \ ; \quad \langle \alpha_i \rangle = \frac{\delta}{\delta\alpha_{i-1}}\chi[V_{i-1}]$$

- Numerical implementation (in progress)
  Rummukainen, Weigert

Approximate Solutions (E.I., McLerran, 2001)

- $Q_s(\tau) \leftrightarrow$ Inverse correlation length for $\langle V_x^\dagger V_y \rangle_\tau$

$$Q_s^2(\tau) \simeq \Lambda^2 e^{c\alpha_s \tau}, \quad c = (4 \div 5) N_c/\pi$$

$$\begin{cases} k_\perp \gg Q_s(\tau): & V_x^\dagger \simeq 1 + ig\alpha(x) \Rightarrow \text{BFKL} \\ k_\perp \ll Q_s(\tau): & g\alpha \sim 1 \Rightarrow V^\dagger \simeq 0 \Rightarrow \text{"free" diffusion} \end{cases}$$

0. Initial condition: the Valence quark model (MV)

$$\langle \rho(k_\perp)\rho(-k_\perp)\rangle_0 \simeq \mu_A = \text{const.} \quad \begin{cases} \text{no evolution} \\ \text{no correlations} \end{cases}$$

I. $k_\perp \gg Q_s(\tau)$: $\langle \rho(k_\perp)\rho(-k_\perp)\rangle_\tau \approx e^{\omega\alpha_s\tau}\, \Phi_\tau(k_\perp)$

- exponential increase with τ
- transverse correlations get built up

II. $k_\perp \ll Q_s(\tau)$: $\langle \rho(k_\perp)\rho(-k_\perp)\rangle_\tau \simeq k_\perp^2 \underbrace{\left(\tau - \bar\tau(k_\perp)\right)}_{\text{diffusion "time"}}$

- colour neutrality: $\langle Q^a Q^a \rangle = 0$ over a scale $\sim 1/Q_s$
- linear increase with $\tau \sim \ln s$: saturated sources

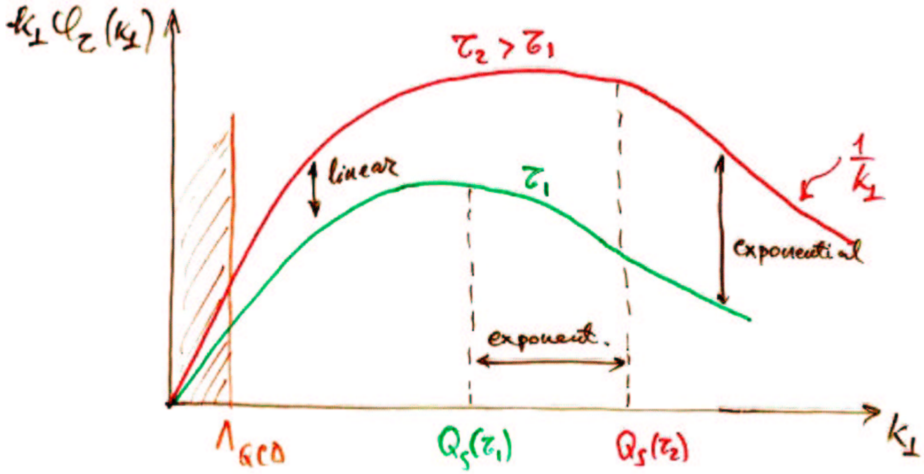
$$\bar\tau(k_\perp) = \frac{1}{c\alpha_s} \ln \frac{k_\perp^2}{\Lambda^2} \quad : \text{critical rapidity where } Q_s = k_\perp$$

Gluon Saturation & Scaling

- Gluon phase-space density:

$$\varphi_\tau(k_\perp) \equiv \frac{d^5 N}{d\tau\, d^2k_\perp\, d^2b_\perp} \simeq \frac{1}{k_\perp^2} \langle \rho_a(k_\perp)\rho_a(-k_\perp)\rangle_\tau$$

$$\boxed{\varphi_\tau(k_\perp) = \tau - \bar\tau(k_\perp) = \frac{1}{c\alpha_s} \ln \frac{Q_s^2(\tau)}{k_\perp^2} \quad \text{for } k_\perp \ll Q_s}$$



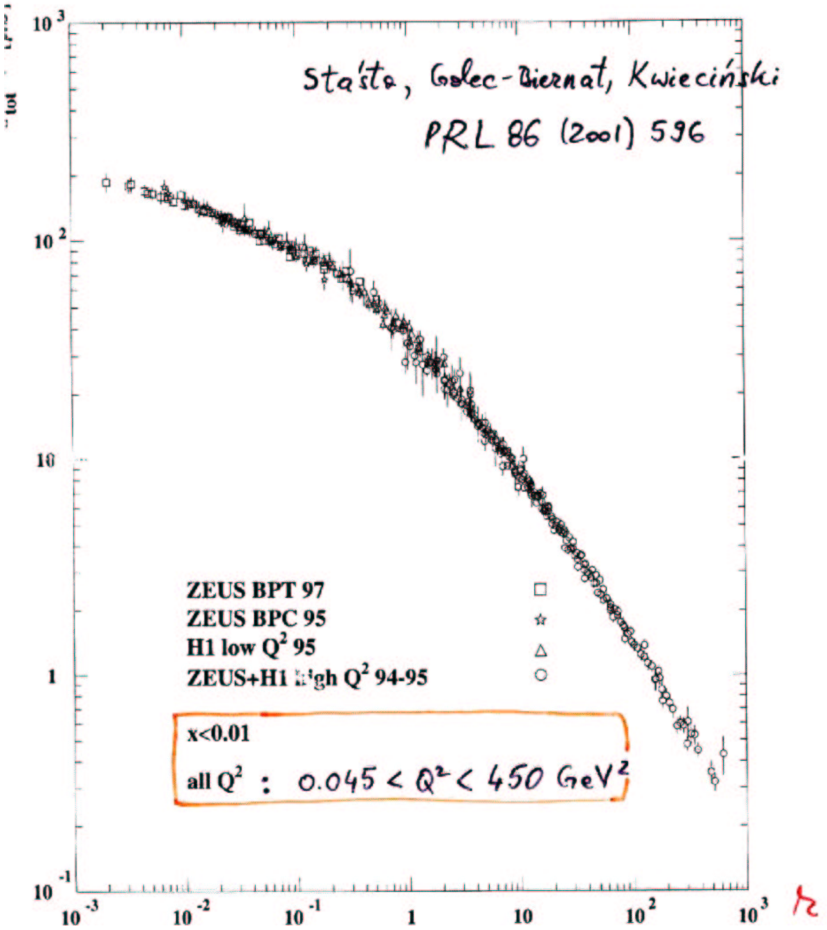
- Geometric scaling: $\varphi_\tau(k_\perp) = f\left(\frac{Q_s^2(\tau)}{k_\perp^2}\right)$

- Holds also above Q_s, up to $k_\perp \sim Q_s^2/\Lambda_{QCD}$.
  (E.I., Itakura, McLerran, 2002)
- Consistent with F_2 data at HERA.

Experimental data on $\sigma_{\gamma^* p}$ plotted versus the scaling variable $\tau \equiv \frac{Q^2}{Q_s^2(x)}$

$$Q_s^2(x) = Q_0^2 \left(\frac{x_0}{x}\right)^\lambda \quad ; \quad \lambda = 0.29, \quad Q_0 = 1\,\mathrm{GeV}, \quad x_0 = 3\cdot 10^{-4}$$

from fit to DIS at $x < 0.01$



The Froissart Bound Revisited

(Ferreiro, E.I., Itakura, McLerran, in preparation)

- Can the perturbative evolution preserve the exponential fall-off of the non-perturbative initial condition?

  - Non-linear evolution eq. for $\mathcal{N}_\tau(r_\perp, b_\perp)$
  - Initial condition: $\mathcal{N}_{\tau_0}(r_\perp, b_\perp) \propto e^{-2m_\pi b}$ at large b

- Potential problem: massless gluons (Kovner, Wiedemann, 2001)

i) At high energy, the cross-section is dominated by the black disk: $\sigma_{dipole}(\tau, r_\perp) \simeq 2\pi R^2(\tau, r_\perp)$

ii) The expansion of the black disk is controlled by scattering within the grey area: $b > R(\tau, r_\perp)$

iii) The scattering in the grey area is dominated by nearby colour sources:

$$|z_\perp - b_\perp| < \frac{1}{Q_s(\tau, b_\perp)} \ll \frac{1}{\Lambda_{QCD}}$$

→ the evolution proceeds locally in $b_\perp$

→ factorization: $\mathcal{N}_\tau(r_\perp, b_\perp) \simeq \underbrace{\mathcal{N}_\tau(r_\perp)}_{\sim \text{BFKL (low density)}} e^{-2m_\pi b}$

Colour neutrality of the saturated gluons is crucial!

$$\mathcal{N}_\tau(r_\perp, b_\perp)\Big|_{\substack{\text{grey}\\\text{area}}} \simeq \underbrace{\text{"short-range"}}_{|z_\perp - b_\perp| < \frac{1}{Q_s}} + \underbrace{\text{"long-range"}}_{z_\perp \in \text{black disk}}$$

$$\simeq \underbrace{\sqrt{r_\perp^2 \Lambda^2}\, e^{\omega\alpha_s \tau - 2m_\pi b}}_{\text{BFKL solution}} + \underbrace{\frac{r_\perp^2}{b_\perp^4}\int_0^\tau d\tau' R^2(\tau')}_{\text{dipole-dipole inter.}}$$

$\mathcal{N}_\tau(r_\perp, b_\perp) \sim 1$ for $b_\perp = R(\tau, r_\perp)$

$\Rightarrow$ "short-range" $= \mathcal{O}(1)$ and "long-range" $\ll 1$

$$\Rightarrow \boxed{R(\tau) \simeq \frac{\omega\alpha_s}{2m_\pi}\tau\,, \qquad \omega = 4(\ln 2)\frac{N_c}{\pi}}$$

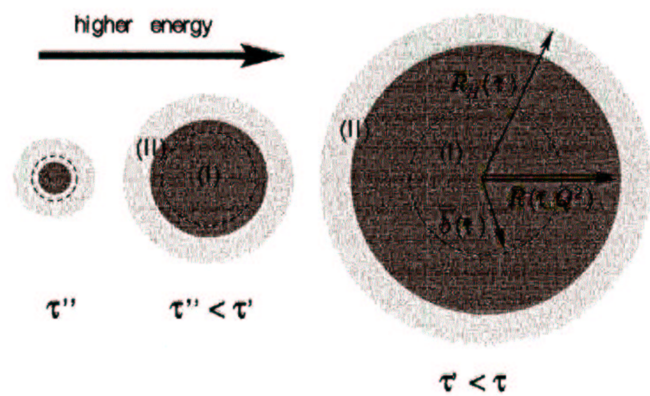
N.B. $\sigma_{BFKL} \sim s^{\omega\alpha_s} \leftrightarrow \sigma_{\text{Froissart}} \sim \left(\frac{\omega\alpha_s}{m_\pi}\ln s\right)^2$

- Without transverse correlations: $\langle \rho(k_\perp)\rho(-k_\perp)\rangle = \mu$

$\Rightarrow$ "long-range" $= \frac{\mu r_\perp^2}{b_\perp^2}\int_0^\tau d\tau' R^2(\tau')$

$\Rightarrow R^2(\tau) \sim \exp(\alpha_s \mu r_\perp^2 \tau) \sim s^{\alpha_s \mu r_\perp^2}$

$\Rightarrow$ no Froissart bound!



30

Perspectives and open problems

- Detailed numerical studies of the non-linear evolution eqs. (including $b_\perp$-dependence)
  (E.g., to numerically confirm Froissart bound, and to compare with F_2 data at HERA)
- Next-to-leading order formalism (non-linear generalization of NLO-BFKL)
  - necessary in order to compare with data
- Extension to hadron-hadron collisions (e.g. heavy ions): "factorization formulae"

  Any hadron may be viewed as a collection of "dipoles" (for large N_c and high energy).