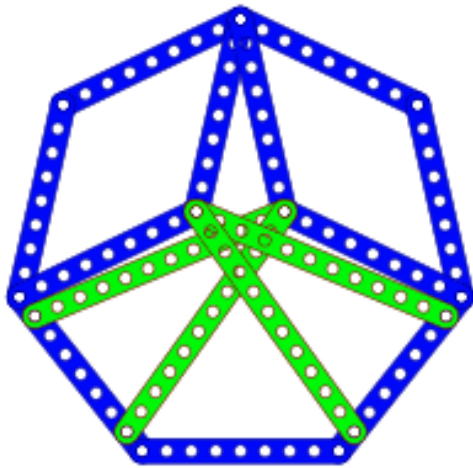


# Lifting Heptagon Symbols to Functions Through 4 Loop NMHV



**Lance Dixon (SLAC)**

LD, Y.-T. (Andy) Liu, 2007.12966

KITP Reunion Conference August 4, 2020

# Scattering Amplitudes in Planar N=4 SYM

- Maximally supersymmetric N=4 super-Yang-Mills theory (SYM), gauge group  $SU(N_c)$ , in the large  $N_c$  (planar) limit
- Structure of  $n$ -point amplitudes is very rigid:

$$A_n = \sum_i \text{rational}_i \times \text{transcendental}_i$$

- For planar N=4 SYM, understand rational structure quite well, focus on the transcendental functions.
- Space of functions is so restrictive, and physical constraints are so powerful, can write  $L$  loop answer as linear combination of known weight  $2L$  polylogarithms.
- Unknown coefficients found simply by solving linear constraints (sometimes a large number)

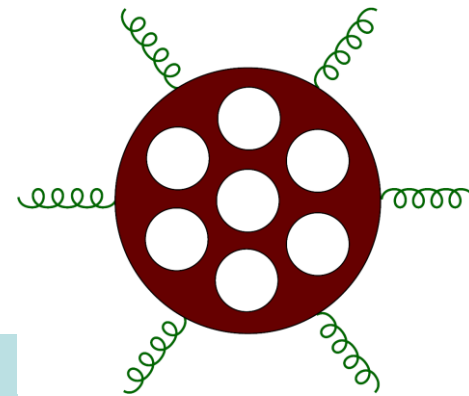
# Hexagon function bootstrap

- 3 LD, Drummond, Henn, 1108.4461, 1111.1704;  
4,5 Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington,  
1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;  
6,7 Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou,  
1903.10890 and 1906.07116; LD, Dulat, 2008.nnnnn (NMHV 7 loops)

Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, **without ever peeking inside the loops**

First step toward doing this **nonperturbatively** (no loops to peek inside) for general kinematics

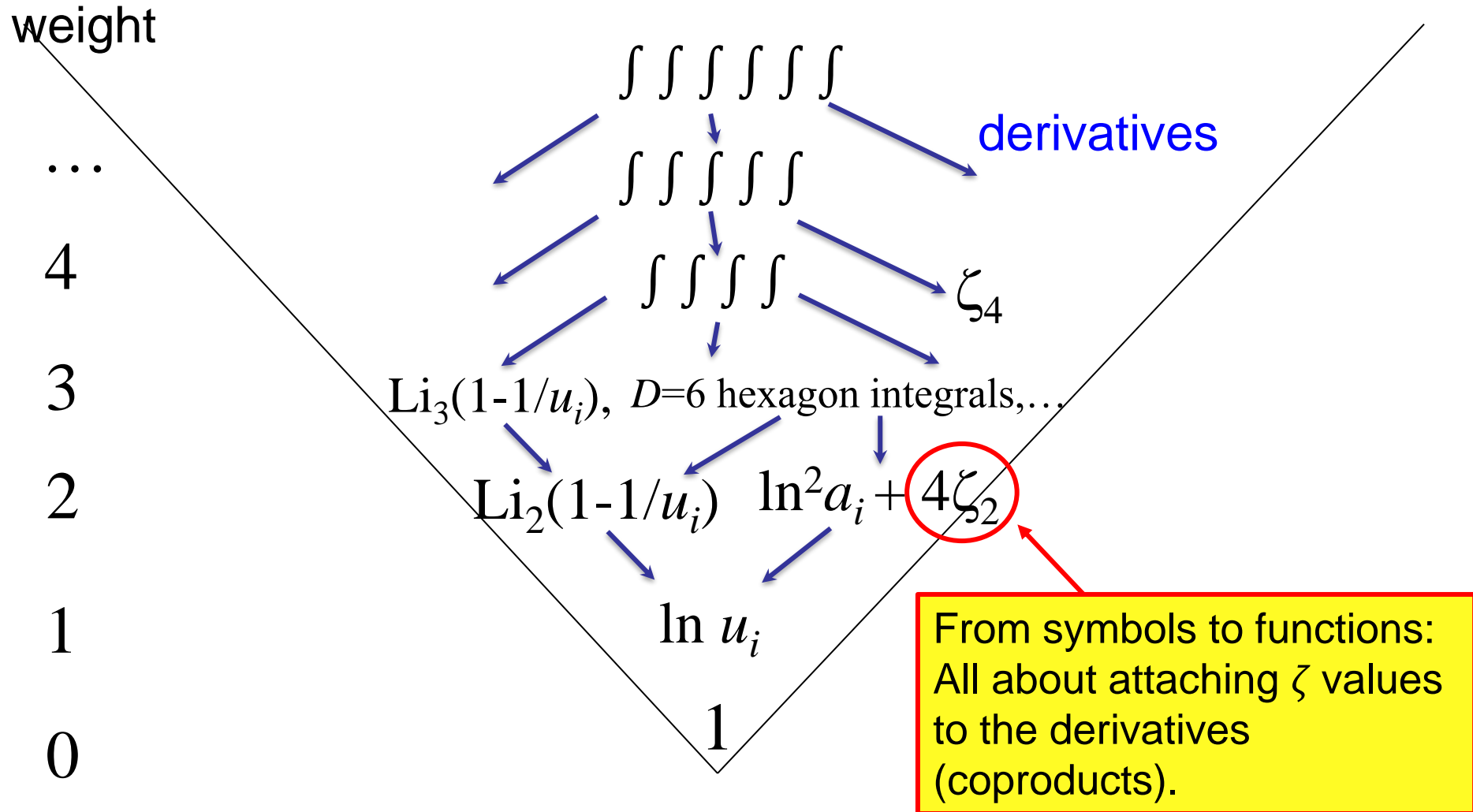
**But at function level it was mostly limited to 6 point amplitudes!**



# High loop orders beyond 6 points

- **Cluster algebras** provide strong clues to right polylogarithmic function space  
Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Foster, Gurdogan, 1710.10953
- **Symbol** of 3-loop MHV 7-point amplitude bootstrapped first.  
**More rigid than 6 points; fewer constraints needed!**  
Drummond, Papathanasiou, Spradlin 1412.3763
- With **Steinmann relations**, could go to 4-loop MHV and 3-loop non-MHV **symbol** LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976
- **Extended Steinmann**  $\rightarrow$  4-loop NMHV **symbol**  
Drummond, Gurdogan, Papathanasiou, 1812.04640
- **Still need to go from symbols  $\rightarrow$  actual functions!**

# Heuristic view of hexagon or heptagon function space



# Why functions?

## Motivation 1: "LHC doesn't measure symbols"

Lots

$n = 7$

SIXTY Ψ SYMBOLS

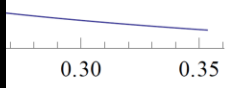
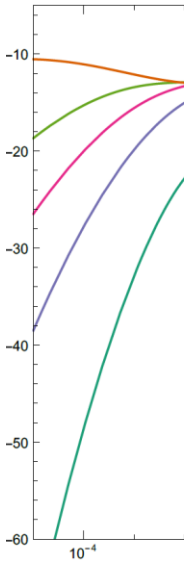
Videos about the symbols of physics and astronomy

The University of Nottingham

\* = recently updated

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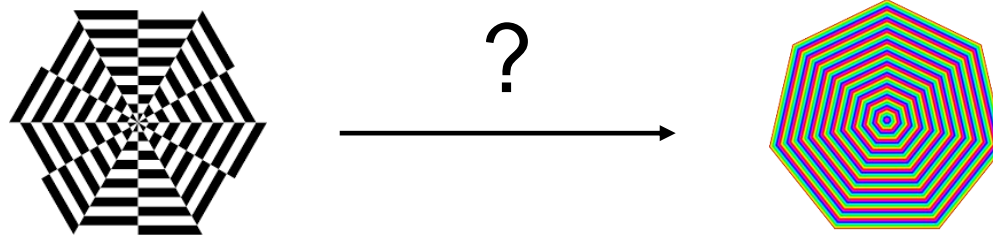
	Pa	$\chi$	$\nu_e$	$\text{♀}$	<u>LHC</u>	$E=mc^2$ *	$\infty$	$\bar{\eta}$	DM
$\text{♂}$	$R_H$ *	$\blacklozenge$	K	$\Psi$	q	$\text{♁}$	$\text{♁}_P$		[ ]
B		$\eta$		$\rightarrow$	Z		Re	P	?*
$\text{♁}$	$\delta$	Gyr		$\pi$		$\Sigma$	W	$\text{♀}$	
		$R_p$			$\Delta$			m	
$D_f$	$\vec{L}$	$\Psi$	pc	G2V	$\eta$	$\sigma$		$\Gamma$	



$n =$   
LD, Du  
Caron-  
McLeo

0.12181;  
2.05690

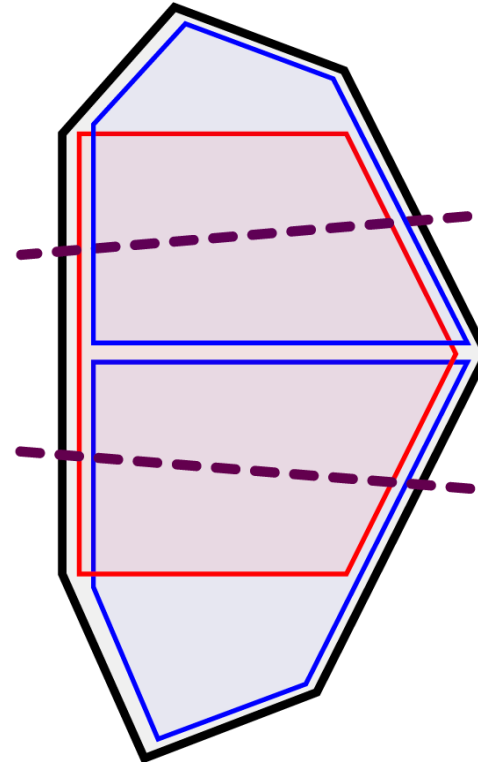
## Motivation 2: Every Superhero/amplitude needs an Origin Story



- For  $n = 6$ , when all 3 cross ratios  $u_i \rightarrow 0$ , logarithm of MHV amplitude is remarkably simple,  $\propto c(\zeta) \ln^2 u_i$  through  $L = 7$   
Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1906.07116
- Zeta-valued coefficients determined by “tilted” BES kernel  
Basso, LD, Papathanasiou, 2001.05460
- **How about  $n = 7$ ?**
- Analogous origin (due to Gram det): first 6  $u_i \rightarrow 0$ ,  $u_7 \rightarrow 1$
- MHV symbol, known to 4 loops, is consistent with this, but provides no  $\zeta$ -valued information  
LD, Drummond, Harrington, McLeod, Papathanasiou, Spradlin, 1612.08976



# Motivation 3: Near collinear (OPE) limit



Test full Pentagon OPE framework  
 Basso, Sever, Vieira, 1303.1396,...  
 to higher loop orders at function level

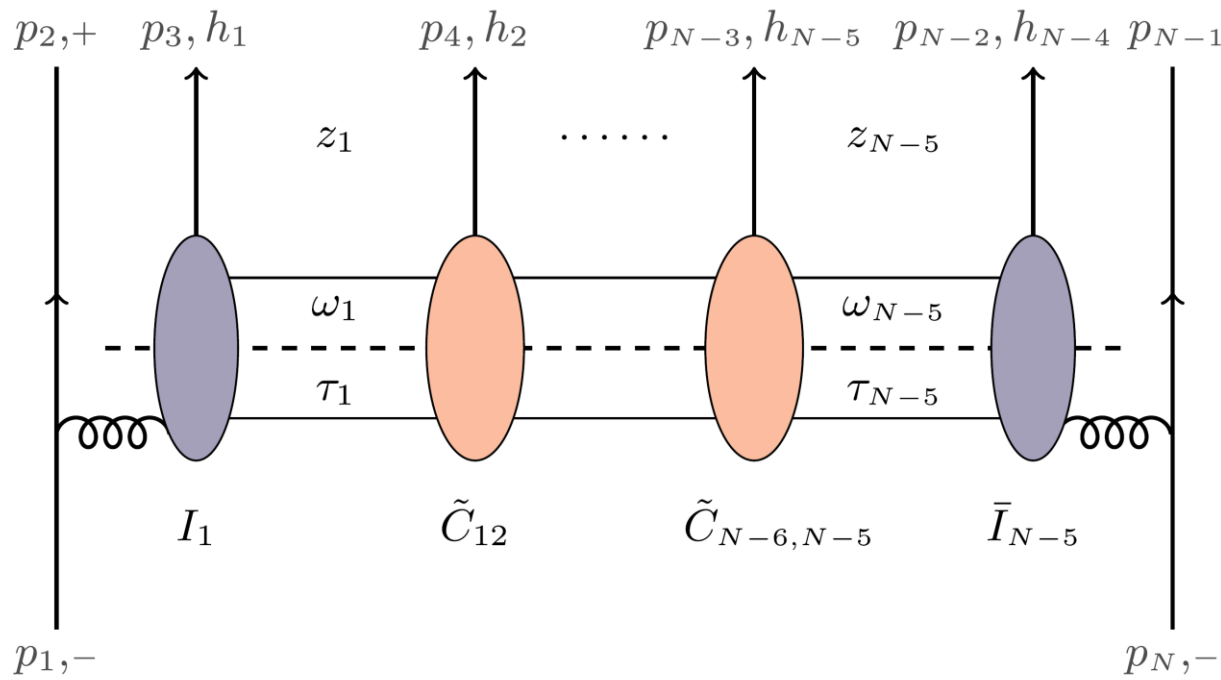
$$\mathcal{W}_{\text{hep}} = \sum_{\mathbf{a}, \mathbf{b}} \int d\mathbf{u} d\mathbf{v} P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{ab}}(\bar{\mathbf{u}}|\mathbf{v}) P_{\mathbf{b}}(\bar{\mathbf{v}}|0) e^{-E(\mathbf{u})\tau_1 + ip(\mathbf{u})\sigma_1 + im_1\phi_1 - E(\mathbf{v})\tau_2 + ip(\mathbf{v})\sigma_2 + im_2\phi_2}$$



# Motivation 4: Multi-Regge limit

- Test recent **all orders conjecture** for **central emission block**, which first appears at  $n = 7$

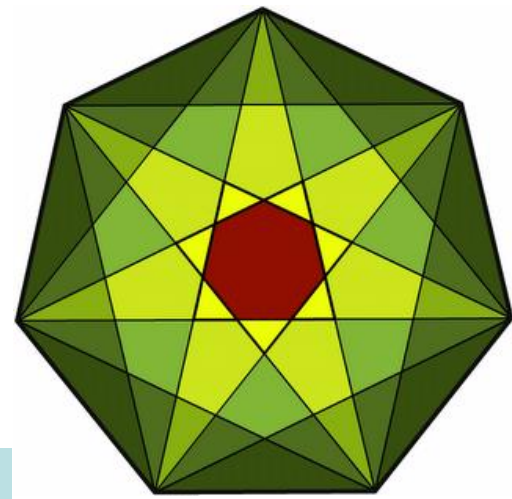
Del Duca, Druc, Drummond, Duhr, Dulat, Marzucca, Papathanasiou, Verbeek, 1912.00188



# Motivation 5: Coaction Principle

- **Restricted set** of zeta values & functions appear for  $n = 6$
- For example, **no independent  $\zeta_2$  or  $\zeta_3$**   
Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1906.07116
- In line with previous observations of a **coaction principle** for  $\phi^4$  theory, electron  $g^{-2}$   
Schnetz, 1302.6445; Brown, 1512.06409; Panzer, Schnetz, 1603.04289;  
Schnetz, 1711.05118
- **Lower loop/weight “missing” values constrain/predict missing values at higher loops.**
- **What happens for  $n = 7$ ?**

# Outline of rest of talk



1. Multiple polylog/symbol refresher
2. “CO” surface interpolating between origin and soft/collinear limits
3. Fixing  $\zeta$  values using branch cut conditions
4. Fixing amplitudes
5. Plots & comments
6. Conclusions

# Multiple polylogarithms

Chen; Goncharov; Brown

- Special class of  $n$ -fold iterated integrals, or weight  $n$  transcendental functions  $f$ .

- Define by derivatives: 
$$d f = \sum_{s_k \in \mathcal{S}} f^{s_k} d \ln s_k$$

$\mathcal{S}$  = finite set of rational expressions, “symbol letters”, and

$f^{s_k} \equiv \{n - 1, 1\}$  component of a “coproduct”  $\Delta$

$f^{s_k}$  are also polylogs, weight  $n-1$ . [still contain  $\zeta$  values]

- **Iterate:**  $d f^{s_k} \Rightarrow f^{s_j, s_k} \equiv \{n - 2, 1, 1\}$  component
  - **Symbol** =  $\{1, 1, \dots, 1\}$  component (maximally iterated) [no  $\zeta$ 's]
- Goncharov, Spradlin, Vergu, Volovich, 1006.5703

# Symbol letters = polylog arguments

Projectively invariant ratios of 4-brackets of momentum twistors  $Z_i$

Hodges, 0905.1473

$$\langle ijkl \rangle \equiv \det(Z_i Z_j Z_k Z_l)$$

- Hexagon, 3 variables ( $u, v, w$ ), 9 letters:

$$\mathcal{S} = \{u, v, w, 1 - u, 1 - v, 1 - w, y_u, y_v, y_w\}$$

$$u = \frac{\langle 1236 \rangle \langle 3456 \rangle}{\langle 1346 \rangle \langle 2356 \rangle} \quad 1 - u = \frac{\langle 1356 \rangle \langle 2346 \rangle}{\langle 1346 \rangle \langle 2356 \rangle} \quad y_u = \frac{\langle 1236 \rangle \langle 2456 \rangle \langle 1345 \rangle}{\langle 1235 \rangle \langle 1246 \rangle \langle 3456 \rangle}$$

+ 6 more letters from cyclic permutations

Goncharov, Spradlin, Vergu, Volovich, 1006.5703;

LD, Drummond, Henn, 1108.4461

# Heptagon: 6 variables, $u_i$ , 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$$a_{11} = \frac{\langle 1234 \rangle \langle 1567 \rangle \langle 2367 \rangle}{\langle 1237 \rangle \langle 1267 \rangle \langle 3456 \rangle},$$

$$a_{41} = \frac{\langle 2457 \rangle \langle 3456 \rangle}{\langle 2345 \rangle \langle 4567 \rangle},$$

$$a_{21} = \frac{\langle 1234 \rangle \langle 2567 \rangle}{\langle 1267 \rangle \langle 2345 \rangle},$$

$$a_{51} = \frac{\langle 1(23)(45)(67) \rangle}{\langle 1234 \rangle \langle 1567 \rangle}$$

$$a_{31} = \frac{\langle 1567 \rangle \langle 2347 \rangle}{\langle 1237 \rangle \langle 4567 \rangle},$$

$$a_{61} = \frac{\langle 1(34)(56)(72) \rangle}{\langle 1234 \rangle \langle 1567 \rangle}$$

$$\langle a(bc)(de)(fg) \rangle \equiv \langle abde \rangle \langle acfg \rangle - \langle abfg \rangle \langle acde \rangle$$

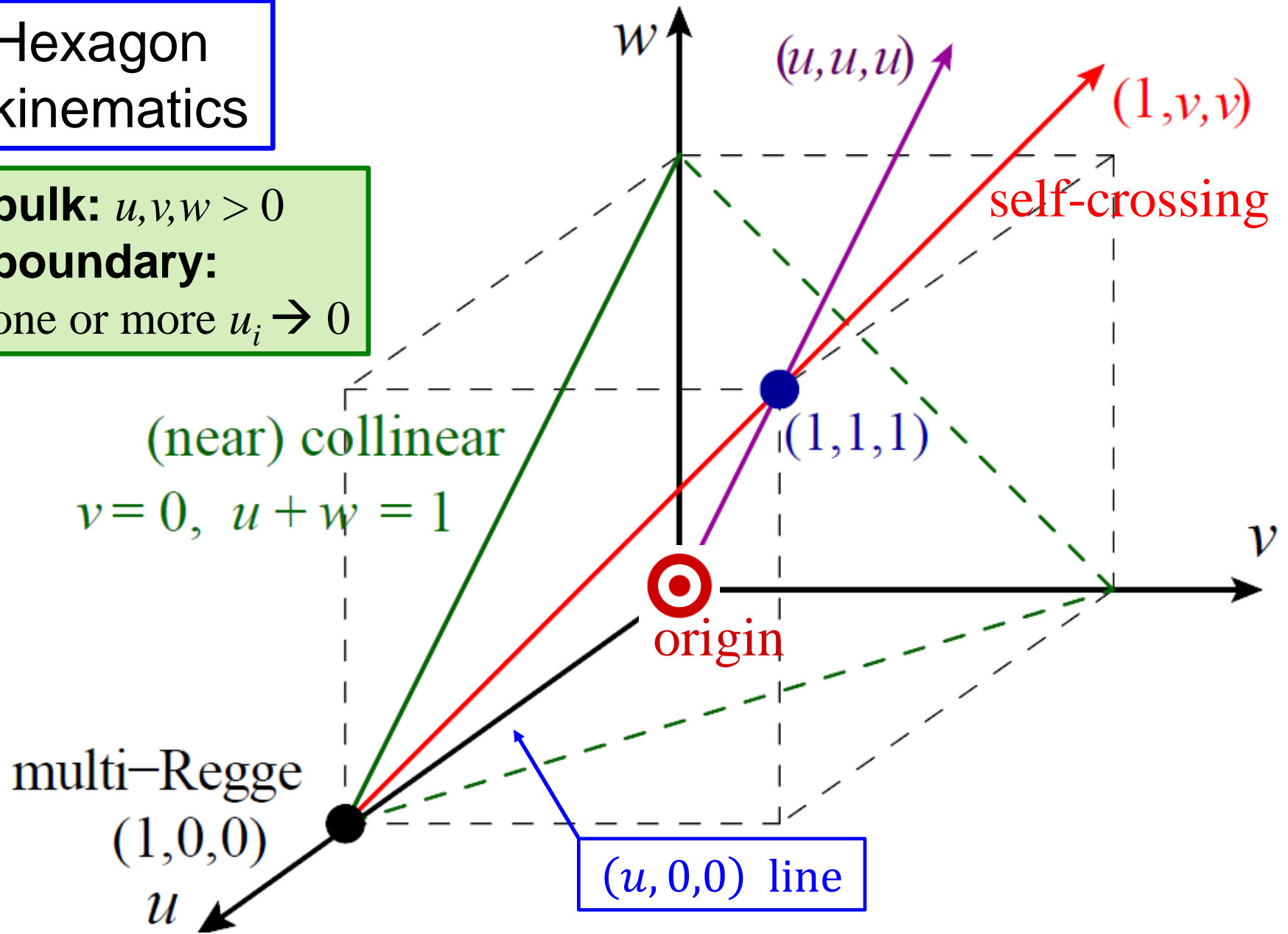
$$u_1 = \frac{a_{17}}{a_{13} a_{14}}$$

plus cyclic,  $i \rightarrow i+1 \pmod{7}$ ,  $a_{ji} \rightarrow a_{j,i+1}$  (6 x 7 = 42 letters)

Gram determinant 4<sup>th</sup> order constraint  $G(u_i) = 0$

Hexagon kinematics

bulk:  $u, v, w > 0$   
boundary: one or more  $u_i \rightarrow 0$

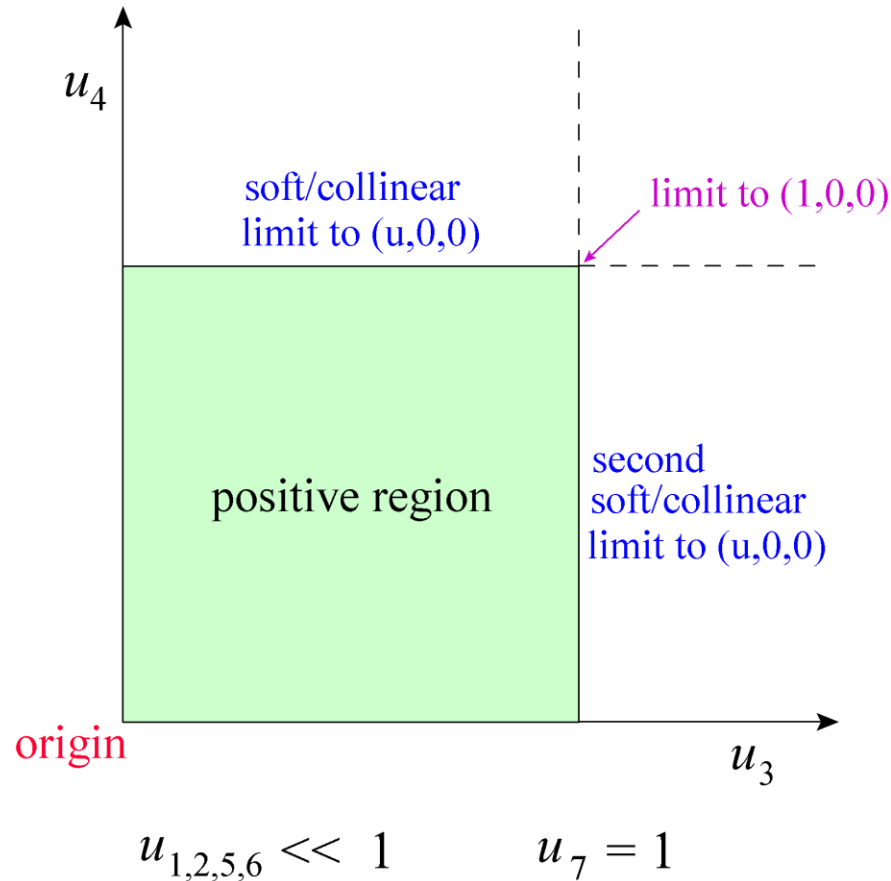




# Very useful 2d slice of heptagon kinematics:

“CO” surface on boundary

interpolates between Soft/Collinear limits and Origin



# Symbol letters on CO surface

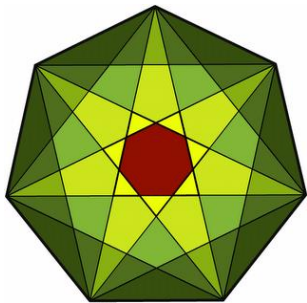
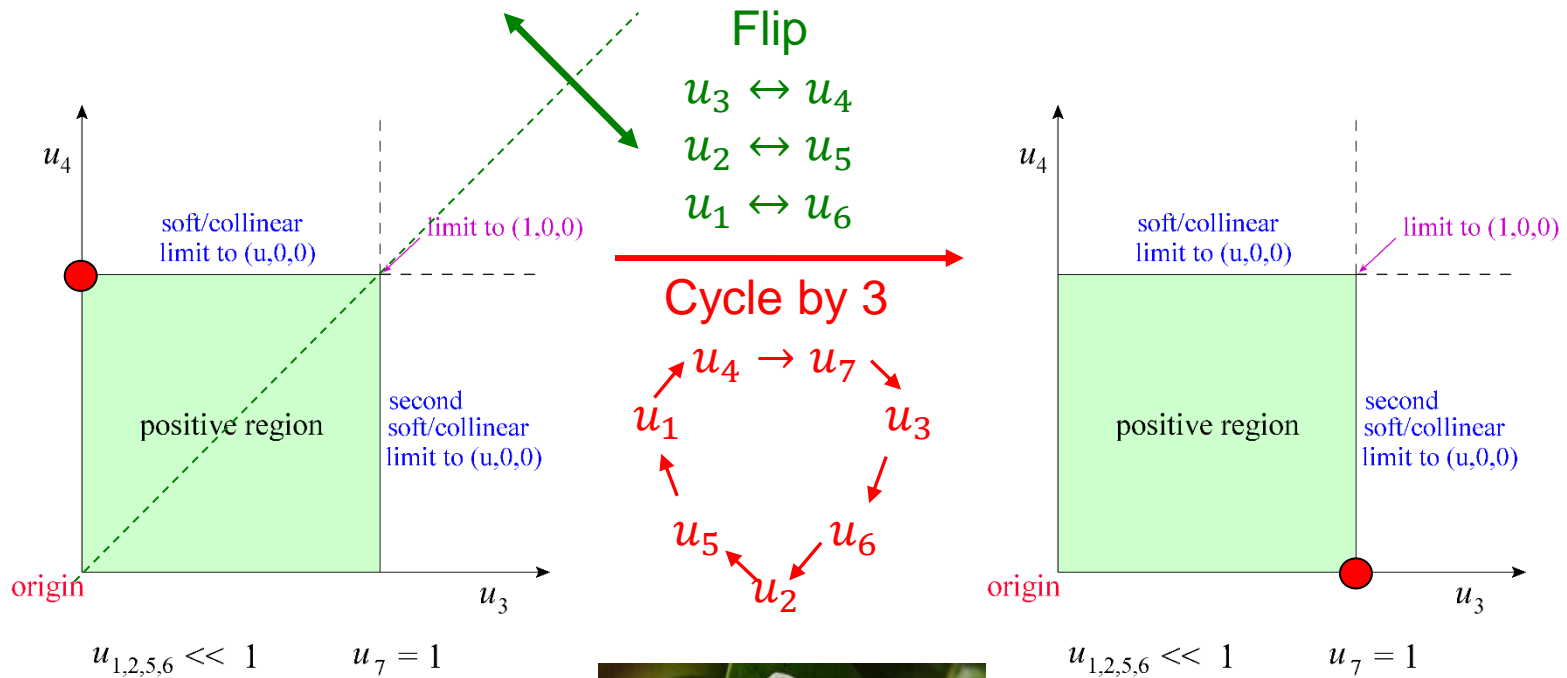
$$\{u_1, u_2, u_5, u_6, u_3, 1 - u_3, u_4, 1 - u_4, 1 - u_7\}$$

- Remarkably simple! In fact, coefficients of  $1 - u_7$  coproduct always vanish, as a branch-cut constraint.
- Function space is just single-argument harmonic polylogs:

$$\{\ln(u_i), i = 1, 2, 5, 6\} \otimes \{H_{\bar{w}}(u_3)\} \otimes \{H_{\bar{w}}(u_4)\}, \quad w_k \in \{0, 1\}$$

- “CO” surface  $\sim$  two copies of 6-point  $(u, 0, 0)$  line

# Dihedral $D_7$ symmetry and CO surface



touching at **points** is enough to carry constants of integration from one “petal” to another

# CO surface as triple scaling limit

- BSV parametrization:  $T_i, S_i, F_i, i = 1, 2$

Basso, Sever, Vieira  
1306.2058

$$(Z_1 \dots Z_7) = \begin{pmatrix} \frac{S_1}{\sqrt{F_1}} & 1 & -1 & -S_2\sqrt{F_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{T_2\sqrt{F_2}} & \frac{S_2+T_2F_2}{T_2S_2\sqrt{F_2}} & 1 & \frac{1}{S_1\sqrt{F_1}} \\ \frac{\sqrt{F_1}}{T_1} & 0 & 0 & -\frac{1}{T_2\sqrt{F_2}} & -\frac{1}{T_2\sqrt{F_2}} & 0 & \frac{\sqrt{F_1}}{T_1} \\ T_1\sqrt{F_1} & 0 & 1 & \frac{1+T_2S_2F_2+T_2^2}{T_2\sqrt{F_2}} & \frac{1}{T_2\sqrt{F_2}} & 0 & 0 \end{pmatrix}$$

- OPE limit:  $T_i \rightarrow \epsilon T_i, \epsilon \rightarrow 0$  BSV, 1407.1736
- Double scaling limit:  $T_i \rightarrow \epsilon T_i, F_i \rightarrow \frac{F_i}{\epsilon}, \epsilon \rightarrow 0$
- CO limit:  $T_i \rightarrow \epsilon T_i, S_i \rightarrow \frac{S_i}{\epsilon}, F_i \rightarrow \frac{F_i}{\epsilon^2}, \epsilon \rightarrow 0$
- Similar limits for  $n > 7$  Basso, LD, Liu, Papathanasiou, in progress

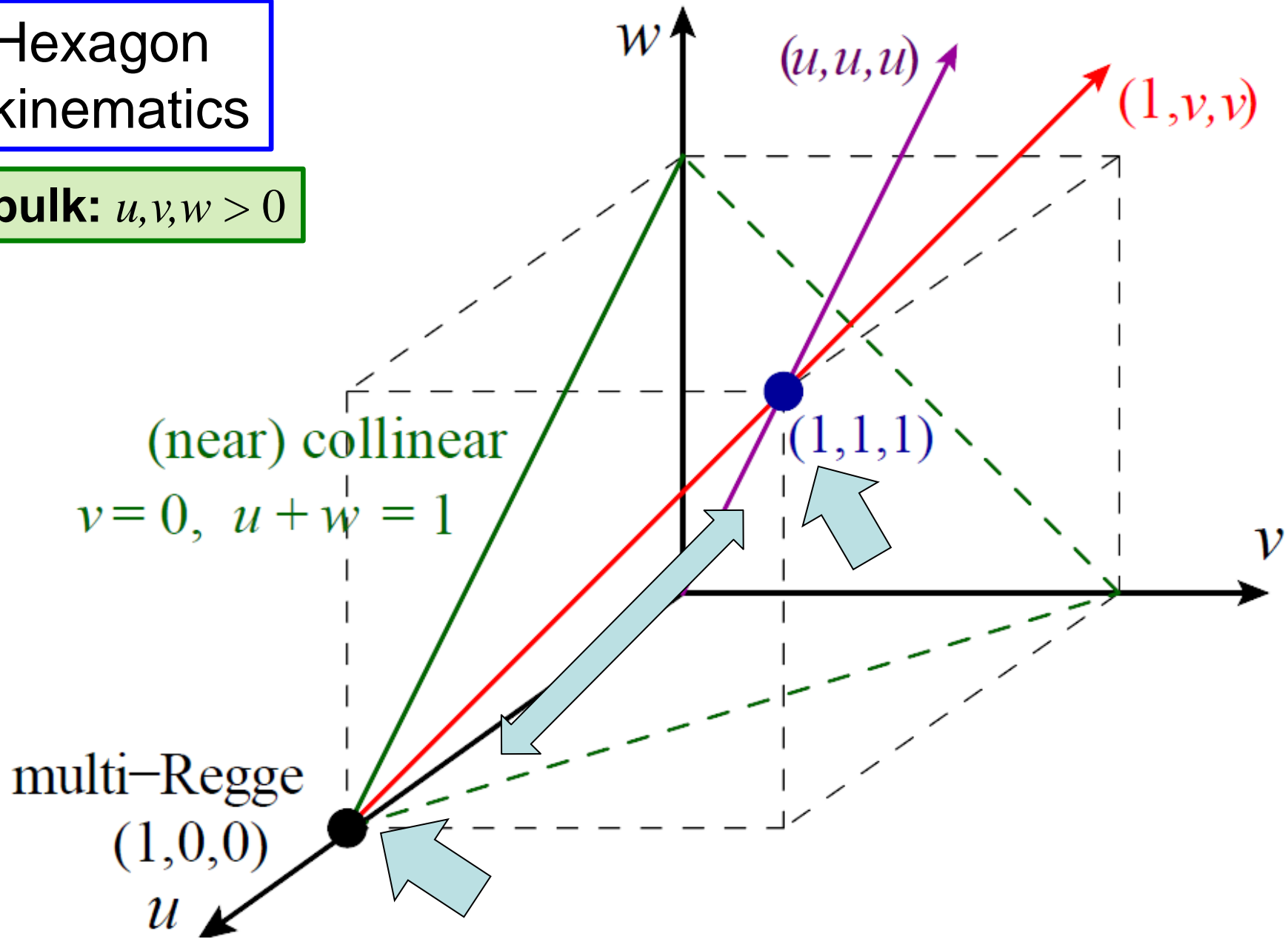
# Branch cut conditions

- **The way to fix all  $\zeta$  values, build consistent functions**  
LD, Drummond, von Hippel, Pennington, 1308.2276
- Functions  $\zeta_m \ln s_k$  for  $s_k \neq u_k$  have **unphysical branch cuts**.  
Not independent functions, but get “attached” to other functions to make their branch cuts all physical.
- **Where to fix?** Near  $s_k = 0$ , initially.
- **Hexagon natural base point:**  $(u, v, w) = (1, 1, 1)$
- Finite, dihedrally invariant, all hexagon functions evaluate to multiple zeta values (MZV's) there.
- **“Bulky” choice of branch cut conditions:** no  $\ln(1 - u_i)$  at  $(1, 1, 1)$   
no  $\ln(y_u)$  or  $\ln(y_v)$  at  $(u, u, 1)|_{u \rightarrow 0}$
- Transport results back up  $(u, u, 1)$  line to  $(1, 1, 1)$ ; remarkably only a **restricted set of MZV's** appear there:  $1, \zeta_2, [no \zeta_3], \zeta_4, 5\zeta_5 - 2\zeta_2\zeta_3, \dots$

Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1906.07116

Hexagon kinematics

bulk:  $u, v, w > 0$



# 7-point branch cut conditions

- **No simple bulk analog of (1,1,1) for heptagons, so work on boundary (CO surface)**
- Vanishing of  $\ln(1 - u_7)$  on entire CO surface
- No  $\ln(1 - u_3)$  at  $u_3 = 1$
- No  $\ln(1 - u_4)$  at  $u_4 = 1$
- Also one branch cut condition on nearby “soft” surface (or can apply additional CO conditions)
- Parity-odd functions vanish at  $u_3 = u_4 = 1$  because this point touches 6-point  $\Delta = 0$  surface
- **These conditions, along with integrability and extended Steinmann relations, suffice to fix all  $\zeta$  values and fully define heptagon functions**
- Big computational assist from previous symbol level information, complete through weight 6 LD, Drummond, McLeod, Harrington, Papathanasiou, Spradlin, 1612.08976; Drummond, Gurdogan, Papathanasiou, 1812.04640



# Many interesting surfaces intersect CO

- For example, cancellation of spurious poles in the NMHV amplitude implies

LD, Drummond, Harrington, McLeod, Papathanasiou, Spradlin, 1612.08976

$$\text{Spurious I: } E_{47} |_{\langle 1356 \rangle = 0} = 0$$

$$\text{Spurious II: } (E_{23} - E_{25}) |_{\langle 1467 \rangle = 0} = 0$$

- $\langle 1356 \rangle = 0$  intersects CO at  $u_4 = 1$
- $\langle 1467 \rangle = 0$  intersects CO at  $u_3 = 1$
- Easy to impose these constraints at function level, and uniquely determine MHV and NMHV amplitudes, using also their soft/collinear limits. (A little help needed from OPE for 4 loop NMHV to fix 2 problematic parameters.)

# $R_7^{(2)}$

Golden, Spradlin, 1406.2055

- For generic kinematics, Golden-Spradlin formula has 49,677 terms, 112 featuring the non-classical polylog

$$L_{2,2}(x, y) \equiv \frac{1}{2} \int_0^1 \frac{dt}{t} \left[ \text{Li}_2(-tx) \text{Li}_1(-ty) - \text{Li}_2(-ty) \text{Li}_1(-tx) \right]$$

- When we drop onto CO surface, the 112  $L_{2,2}$  functions collapse to 3 pairs, which can all be removed using the identity

$$\begin{aligned} L_{2,2}(x, y) + L_{2,2}(1/y, 1/x) = & 3 \text{Li}_4(x/y) + \text{Li}_4(-x) - \text{Li}_4(-y) \\ & - \ln(x/y) \text{Li}_3(x/y) - \ln y \text{Li}_3(-x) + \ln x \text{Li}_3(-y) \\ & + \frac{1}{4} (\ln^2 y + 2\zeta_2) \text{Li}_2(-x) - \frac{1}{4} (\ln^2 x + 2\zeta_2) \text{Li}_2(-y) \\ & + \frac{1}{24} \ln^2 y \ln(x/y) (\ln y - 3 \ln x) - \frac{\zeta_2}{4} \ln(x/y) (\ln x - 3 \ln y) - 3 \zeta_4 \end{aligned}$$

# $R_7^{(2)}$ on CO surface

- Using also standard classical polylog identities, Golden-Spradlin expression collapses to:

$$\begin{aligned}
 R_7^{(2)}(u_{1,2,5,6} \ll 1, u_7 = 1) = & 6 \operatorname{Li}_4\left(\frac{-u_3}{1-u_3}\right) - \frac{1}{2} \left[ \operatorname{Li}_2\left(\frac{-u_3}{1-u_3}\right) \right]^2 - 2 \ln\left(\frac{u_3}{u_1 u_2 u_4}\right) \operatorname{Li}_3\left(\frac{-u_3}{1-u_3}\right) \\
 & + \left[ \ln\left(\frac{1-u_3}{u_1 u_2 u_4}\right) \ln u_3 + \ln(u_1 u_4) \ln u_2 - \frac{1}{2} \ln^2(1-u_3) - 3 \zeta_2 \right] \operatorname{Li}_2\left(\frac{-u_3}{1-u_3}\right) \\
 & + \frac{1}{8} \ln^4(1-u_3) - \left[ \frac{1}{6} \ln^2(1-u_3) + \zeta_2 \right] \ln(1-u_3) \left[ \ln u_3 + 2 \ln(u_1 u_2 u_4) \right] \\
 & + \frac{1}{2} \ln^2(1-u_3) \left[ \ln u_1 \ln u_2 + \ln(u_1 u_2) \ln u_3 + \ln(u_2 u_3) \ln u_4 + 3 \zeta_2 \right] \\
 & - \ln(1-u_3) \ln u_3 \ln u_2 \ln(u_1 u_4) \\
 & + \frac{\zeta_2}{2} \left[ \left( \ln u_4 + 2 \ln(u_1 u_2 u_5) \right) \ln u_3 + 2 \ln u_1 \ln u_2 \right] + \frac{17}{4} \zeta_4 \\
 & + \{ u_1 \leftrightarrow u_6, u_2 \leftrightarrow u_5, u_3 \leftrightarrow u_4 \}
 \end{aligned}$$

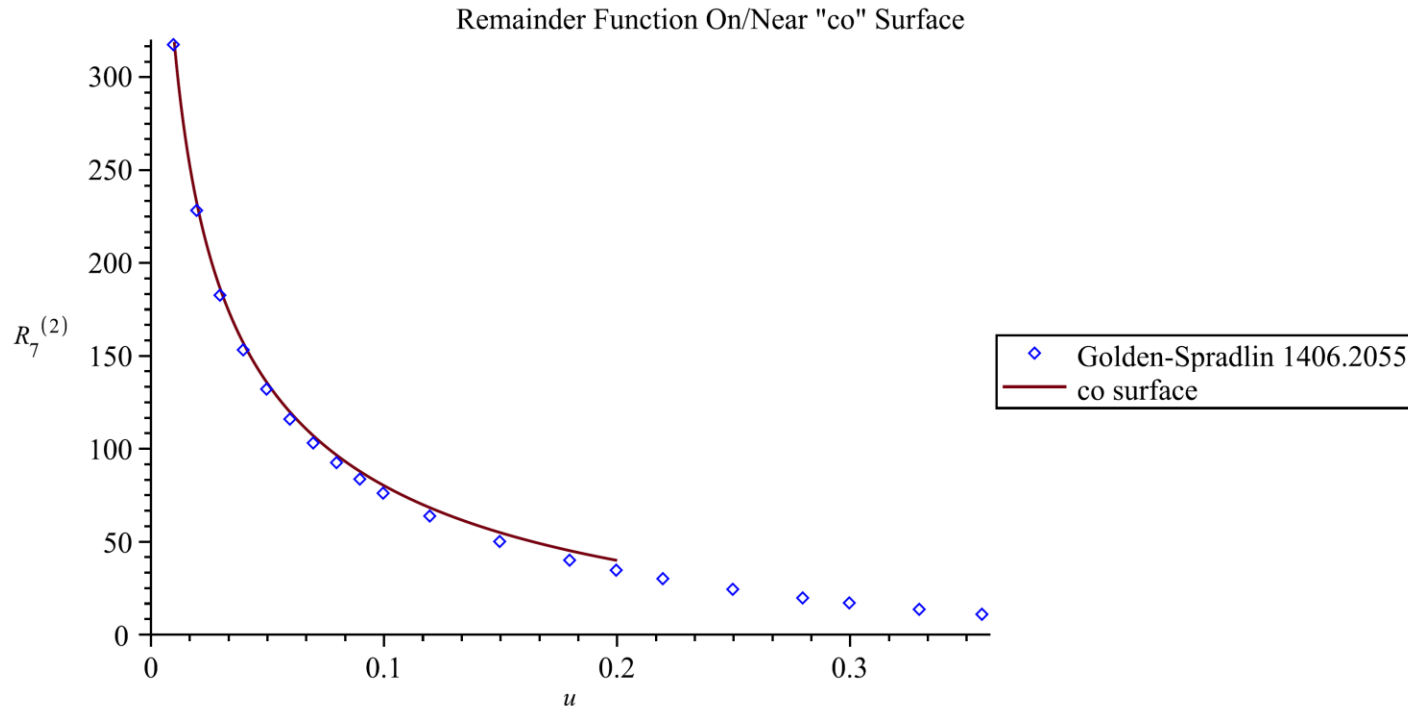
only terms surviving at origin,  $u_3, u_4 \rightarrow 0$

# “Symmetric line” near CO surface

$$\begin{aligned}
 R_7^{(2)}(u, u, u, u, u, u, 1) = & 12 \operatorname{Li}_4\left(\frac{-u}{1-u}\right) - \left[\operatorname{Li}_2\left(\frac{-u}{1-u}\right)\right]^2 + 8 \ln u \operatorname{Li}_3\left(\frac{-u}{1-u}\right) \\
 & - \left(2 \ln^2 u - 2 \ln u \ln(1-u) + \ln^2(1-u) + 6\zeta_2\right) \operatorname{Li}_2\left(\frac{-u}{1-u}\right) \\
 & - \frac{1}{12} \ln(1-u) \left(48 \ln^3 u - 60 \ln^2 u \ln(1-u) + 28 \ln u \ln^2(1-u) - 3 \ln^3(1-u)\right) \\
 & - \zeta_2 \ln(1-u) \left(14 \ln u - 3 \ln(1-u)\right) + 9 \zeta_2 \ln^2 u + \frac{17}{2} \zeta_4
 \end{aligned}$$

- Doesn't really make sense unless  $u \ll 1$ , since Gram determinant  $G(u) \propto u$  doesn't vanish. But we can still plot it for smallish  $u$

$R_7^{(2)}$  on  $(u, u, u, u, u, u, \frac{(1-u-u^2)^2}{1-2u^2})$  vs. on **co surface**  $(u, u, u, u, u, u, 1)$

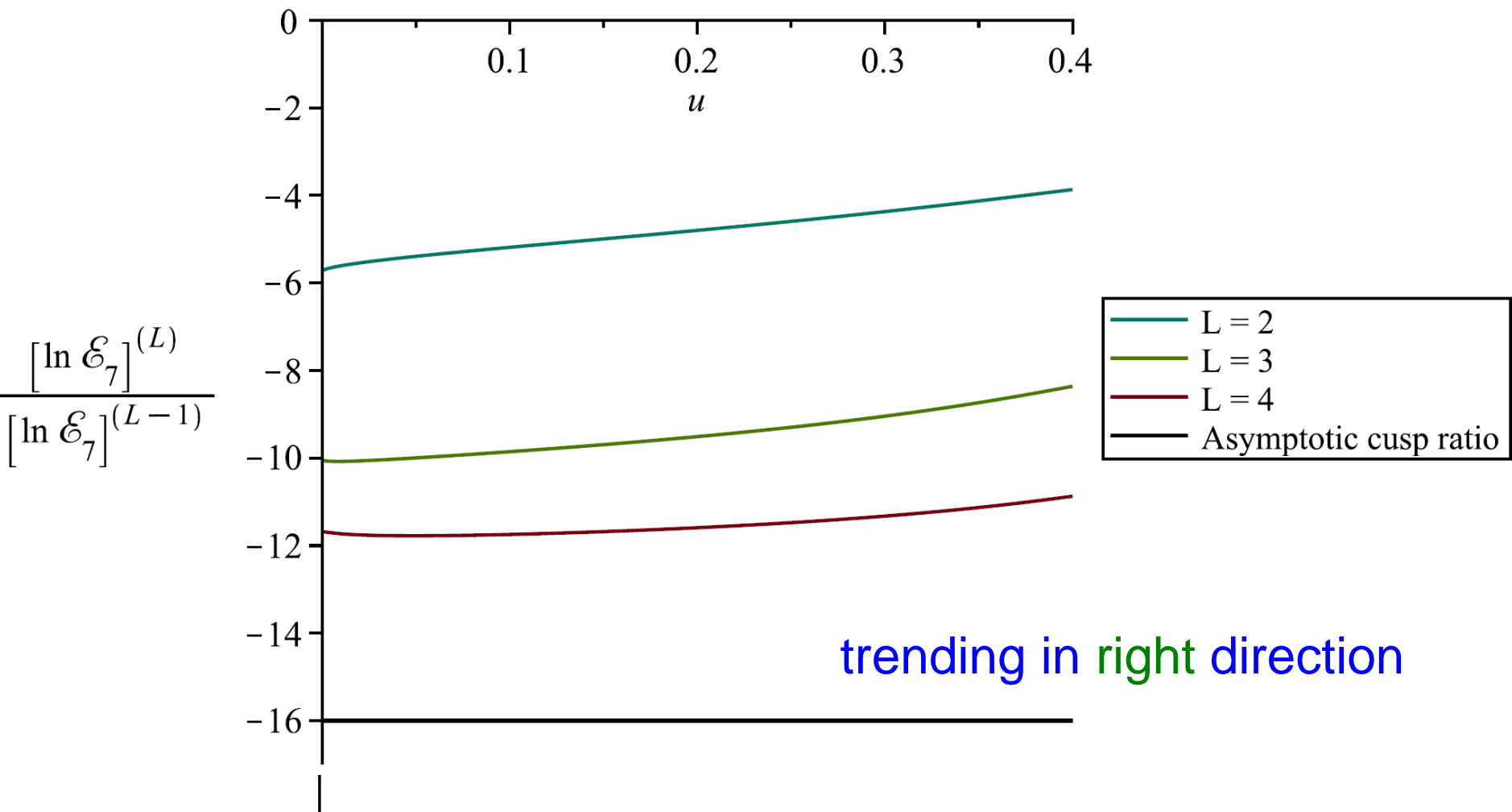


- In fact, we can compute **any amplitude** on actual  $(u, u, u, u, u, u, \frac{(1-u-u^2)^2}{1-2u^2})$ , as a **power series expansion** around  $u = 0$ .
- Integration constants fixed at **origin**,  $(0,0,0,0,0,0,1)$ .
- Forty terms suffice to converge past end of positive region at  $u = 0.35689\dots$

# “High” Order Perturbative Behavior

- Planar N=4 SYM has **finite radius of convergence** of perturbative expansion (unlike QCD, QED, whose perturbative series are **asymptotic**)
- For BES solution to cusp anomalous dimension, using coupling  $g^2 = \frac{\lambda}{16\pi^2}$ , this radius is  $\frac{1}{16}$
- Ratio of successive terms  $\frac{\Gamma_{\text{cusp}}^{(L)}}{\Gamma_{\text{cusp}}^{(L-1)}} \rightarrow -16$
- Based on 6-point experience, expect ratios of successive loop orders for **amplitudes** to also approach  $-16$ , at least in the bulk

Logarithm of MHV Amplitude Ratios on Symmetric Line  $\left( u, u, u, u, u, u, \frac{(1-u-u^2)^2}{1-2u^2} \right)$



trending in right direction



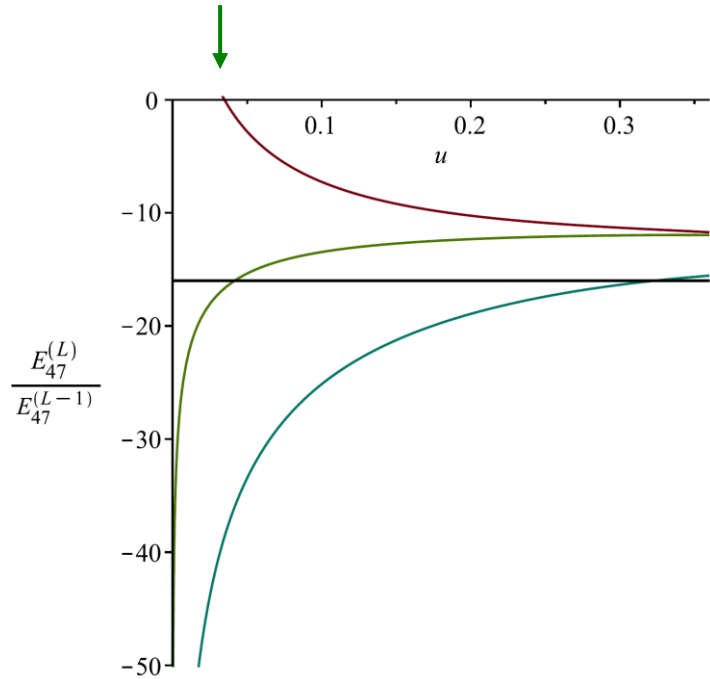
# NMHV successive loop ratios, through 4 loops

3 of 15 components shown

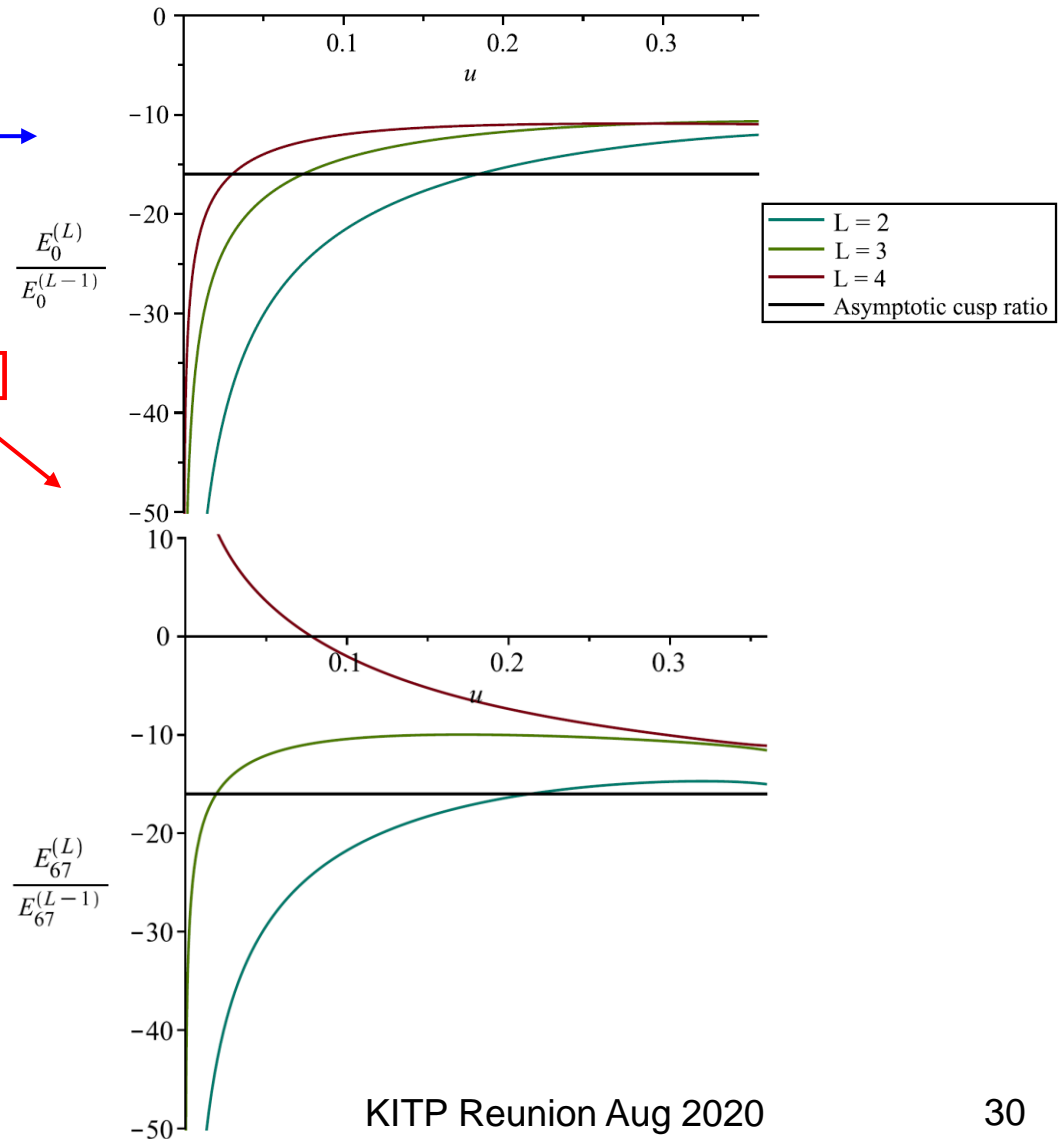
BDS-like Normalized Amplitude Ratios on Symmetric Line  $\left(u, u, u, u, u, u, \frac{(1-u-u^2)^2}{1-2u^2}\right)$

tree  $\rightarrow$

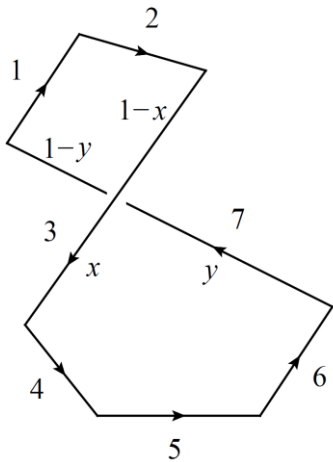
(47) = [12356]      (67) = [12345]



L. Dixon      Heptagons to 4 loop NMHV



KITP Reunion Aug 2020



# “Self-crossing” surface

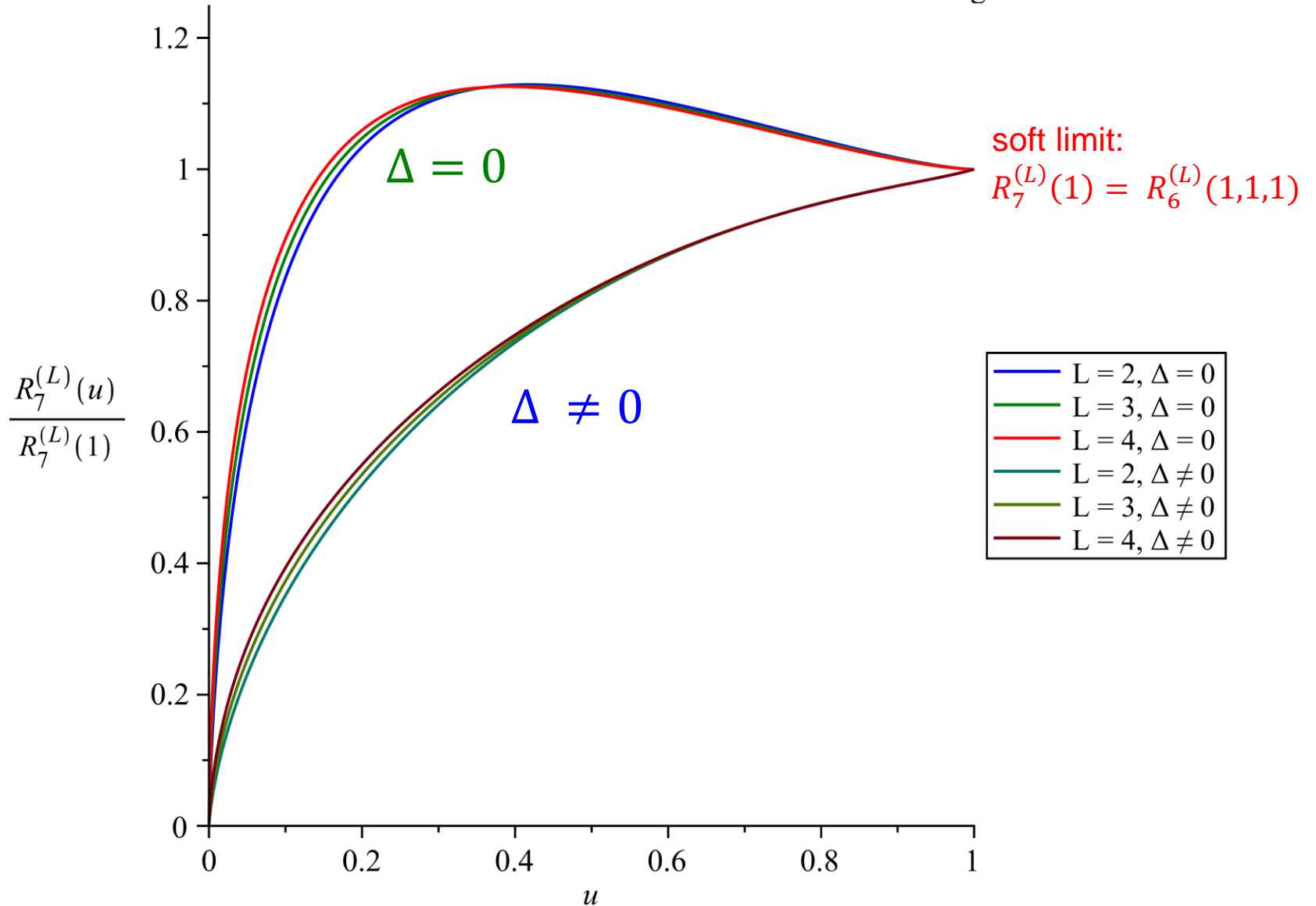
LD, Esterlis, 1602.02107

→ 4 parameter surface

$$u_7 = 1, \quad u_3 = 1 - \frac{u_1 u_5}{u_6}, \quad u_4 = 1 - \frac{u_2 u_6}{u_1}$$

- Work on Euclidean sheet, as another example of **bulk kinematics**
- Intersects co surface at  $u_3 = u_4 = 1$
- For simplicity, consider **two different 1D “lines”** in this surface:
  - $(u, u, 1 - u, 1 - u, u, u, 1)$  and  $(\frac{4u}{(1+u)^2}, u, 1 - u, 1 - u, u, \frac{4u}{(1+u)^2}, 1)$
  - $\Delta \neq 0$   $\Delta = 0$
- Parity-odd functions **nonvanishing** vs. **vanishing**
- Both lines intersect co surface at  $u = 0$ , and leave Euclidean region at same **soft limit** point,  $u = 1 \Rightarrow (1, 1, 0, 0, 1, 1, 1)$ .
- Symbol alphabet in  $\Delta \neq 0$  case still has square roots, but can integrate up as power series around both  $u = 0$  and  $u = 1$ .

# Normalized Remainder Function on Two Lines in the Self Crossing Surface



# Coaction Principle?

- By taking coproducts of 4 loop MHV and 3 loop NMHV amplitudes, we find that  $\zeta_3$  is an **independent constant function** in the heptagon function space
- **Big surprise**, because  $\zeta_3$  is **not independent at 6 points**; it does not even appear as a value of any hexagon function at  $(u, v, w) = (1, 1, 1)$ ! CDDvHMP, 1906.07116
- $\zeta_2$  is not independent, just as at 6 points  
→ **coaction principle can be nontrivial (at function level)**
- $\zeta_4$  is independent, just as at 6 points
- Can't yet determine independence of  $\zeta_5$  or  $\zeta_2\zeta_3$ ; need at least 5 loop MHV amplitude
- **But so far it seems like 7 points and 6 points will behave quite differently**

# “Parity odd $\zeta$ values”

Number of independent  $\{n, 1, \dots, 1\}$  coproducts at L loops  
 parity even sector                      parity odd sector

weight $n$	0	1	2	3	4	5	6	7	8
$L = 1$	1	7	15						
$L = 2$	1	7	28	63	15				
$L = 3$	1	7	28	92	239	126	16		
$L = 4$	1	7	28	92	288	753	638	154	16

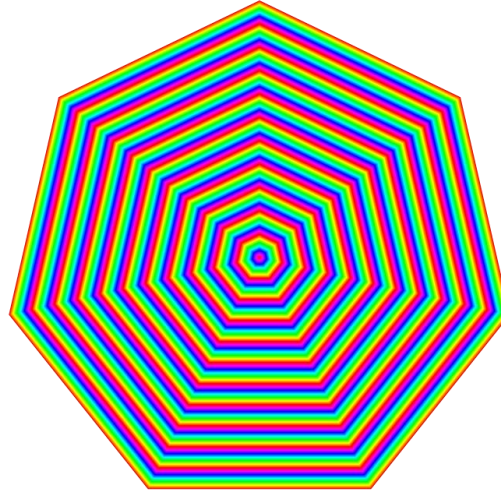
weight $n$	0	1	2	3	4	5	6	7	8
$L = 1$	0	0	0						
$L = 2$	0	0	0	6	14				
$L = 3$	0	0	0	6	28	118	15		
$L = 4$	0	0	0	6	28	120	406	154	15

green  $\rightarrow$  saturation of bottom-up number

- P-odd saturates earlier, but **406** is symbol number, not the expected  $412 = 406 + 6$  [ $\zeta_3 \times$  weight 3 odd functions]
- **The weight 3 odd functions may be “locked” to the symbol-level weight 6 functions.**
- **Also: So far, all P-odd, weight  $2k+1$  functions have no  $\zeta_{2k+1}$  at origin.**  
*( $k=3,5$ , small part of 7)*

# Lightning dive to the origin

B. Basso, LD, Y.-T. Liu and G. Papathanasiou, in progress



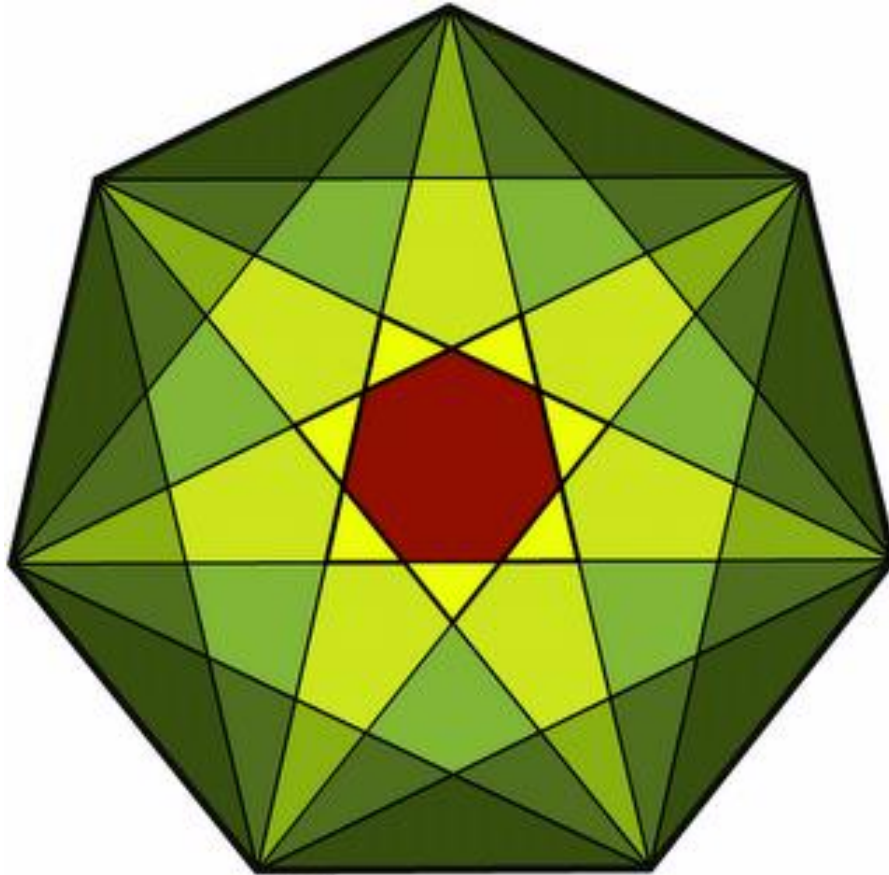
**6-points:**  $\ln \varepsilon^{MHV}$  is quadratic in logarithms through at least 9 loops, and almost certainly to all orders. Tilted BES kernel controls  $\zeta$  values. [Basso, LD, Papathanasiou, 2001.05460](#)

**7-points:**  $\ln \varepsilon^{MHV}$  is quadratic in logarithms through at least 4 loops

# Conclusions

- Heptagon function bootstrap now well underway:
- Every known 7-point amplitude symbol  $\rightarrow$  function.
- Helped greatly to find a **simple CO surface** on the boundary where enough **branch cut conditions** can be imposed to **fix all zeta values**
- Can integrate up along different lines into the bulk
- CO surface touches many interesting regions (**origin**, **MRK**, **OPE**, ...) and with dihedral symmetry will allow zeta values to be fixed there too
- Similar **multi-scaling surfaces** will be very useful for  $n > 7$  too!

# Extra Slides





# Dual conformal invariance

- Wilson  $n$ -gon invariant under inversion:  $x_i^\mu \rightarrow \frac{x_i^\mu}{x_i^2}$ ,  $x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}$   
 $x_{ij}^2 = (k_i + k_{i+1} + \dots + k_{j-1})^2 \equiv s_{i,i+1,\dots,j-1}$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

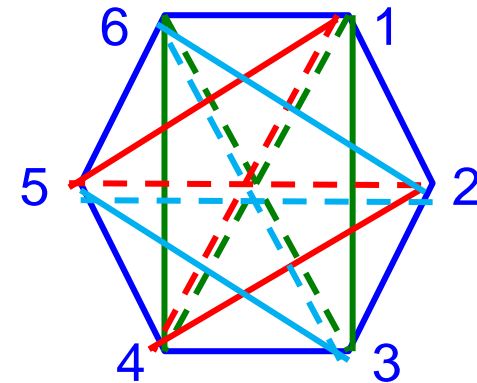
- $x_{i,i+1}^2 = k_i^2 = 0 \rightarrow$  no such variables for  $n = 4, 5$

$n = 6 \rightarrow$  precisely 3 ratios:

$$u_1 = u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

$$u_2 = v = \frac{s_{23} s_{56}}{s_{234} s_{123}}$$

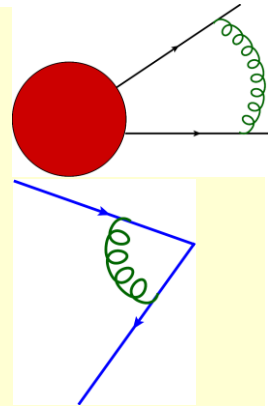
$$u_3 = w = \frac{s_{34} s_{61}}{s_{345} s_{234}}$$



$n = 7 \rightarrow$  6 independent ratios

# Removing Divergences

- On-shell amplitudes **IR divergent** due to long-range gluons
- Polygonal Wilson loops **UV divergent** at cusps, anomalous dimension  $\Gamma_{\text{cusp}}$   
 – known to all orders in planar N=4 SYM:  
 Beisert, Eden, Staudacher, hep-th/0610251
- Both removed by dividing by **BDS-like ansatz**  
 Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant.
- **BDS-like** also maintains important relation due to causality (Steinmann).



$$\mathcal{E}_n(u_i) = \lim_{\epsilon \rightarrow 0} \frac{\mathcal{A}_n(s_{i,i+1}, \epsilon)}{\mathcal{A}_n^{\text{BDS-like}}(s_{i,i+1}, \epsilon)} = \exp\left[\mathcal{R}_n + \frac{\Gamma_{\text{cusp}}}{4} \mathcal{E}_n^{(1)}\right]$$

# BDS-like ansatz ( $n = 6$ )

$$\frac{\mathcal{A}_6^{\text{BDS-like}}}{\mathcal{A}_6^{\text{MHV}(0)}} = \exp \left[ \sum_{L=1}^{\infty} a^L \left( f^{(L)}(\epsilon) \frac{1}{2} \hat{M}_6(L\epsilon) + C^{(L)} \right) \right]$$

where  $f^{(L)}(\epsilon) = \frac{1}{4} \gamma_K^{(L)} + \epsilon \frac{L}{2} \mathcal{G}_0^{(L)} + \epsilon^2 f_2^{(L)}$  are constants, and

$$\begin{aligned} \hat{M}_6(\epsilon) &= M_6^{1\text{-loop}}(\epsilon) + Y(u, v, w) \\ &= \sum_{i=1}^6 \left[ -\frac{1}{\epsilon^2} (1 - \epsilon \ln s_{i,i+1}) - \ln s_{i,i+1} \ln s_{i+1,i+2} + \frac{1}{2} \ln s_{i,i+1} \ln s_{i+3,i+4} \right] + 6\zeta_2 \end{aligned}$$

- $Y$  is dual conformally invariant part of one-loop amplitude  $M_6^{1\text{-loop}}$  containing all 3-particle invariants:

$$Y(u, v, w) = -\mathcal{E}^{(1)} = -\text{Li}_2 \left( 1 - \frac{1}{u} \right) - \text{Li}_2 \left( 1 - \frac{1}{v} \right) - \text{Li}_2 \left( 1 - \frac{1}{w} \right)$$

- More minimal BDS-like ansatz contains all IR poles, but **no 3-particle invariants**.

# Branch cut condition

- All massless particles  $\rightarrow$  all branch cuts start at origin in

$$s_{i,i+1}, s_{i,i+1,i+2}$$

- **6-point:** Branch cuts all start from 0 or  $\infty$  in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad \text{or } v \text{ or } w$$

$\rightarrow$  Only 3 weight 1 functions, not 9:  $\{ \ln u, \ln v, \ln w \}$

- **7-point:** Only 7 weight 1 functions, not 42  $\{ \ln u_i \}$

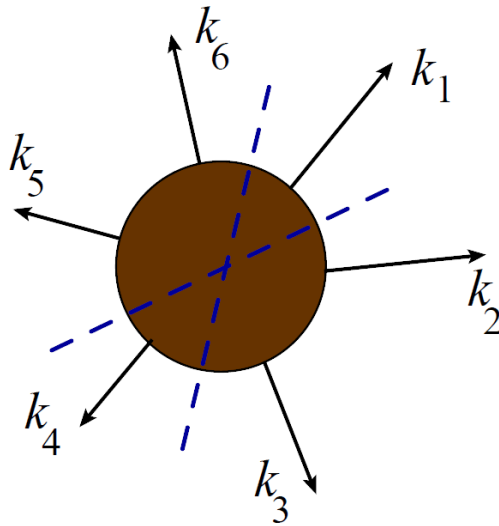
- Discontinuities commute with branch cuts
- Require derivatives of higher weight functions to obey branch-cut condition too  $\rightarrow$  very powerful constraint.

# Steinmann relations

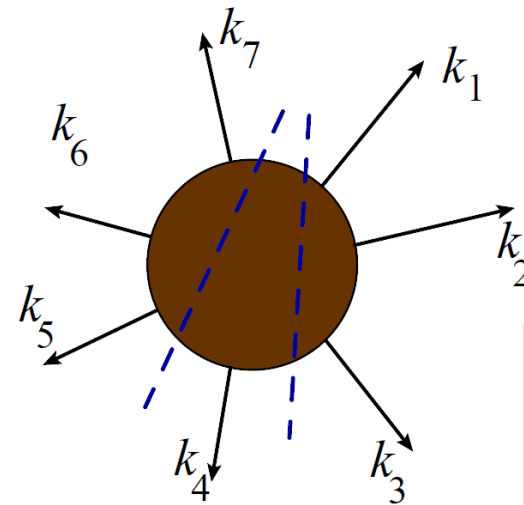
Steinmann, *Helv. Phys. Acta* (1960)    Bartels, Lipatov, Sabio Vera, 0802.2065

Bourjaily, Hannesdottir, McLeod, Schwartz, Vergu, 2007.13747

- Amplitudes should not have **overlapping** branch cuts:



Not Allowed



Allowed

can't apply to  
2 particle cuts in  
**massless** case  
because they are  
**not independent**

$$\text{Disc}_{s_{234}} \left[ \text{Disc}_{s_{123}} \mathcal{E}(u, v, w) \right] = 0$$

Violated by BDS  
normalized amplitudes!

# Steinmann constraints ( $n = 6$ )

S. Caron-Huot, LD, M. von Hippel, A. McLeod, 1609.00669

$$\text{Disc}_{s_{234}} \left[ \text{Disc}_{s_{123}} \mathcal{E}(u, v, w) \right] = 0 \quad + \text{cyclic conditions}$$

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \quad v = \frac{s_{23}s_{56}}{s_{234}s_{123}} \quad w = \frac{s_{61}s_{34}}{s_{345}s_{234}}$$

$\ln^2 u$        $\ln^2 \frac{uv}{w}$   
**NO**          **OK**

$$\frac{uv}{w} = \frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}$$

Analogous  
 constraints for  $n=7$

LD, J. Drummond,  
 T. Harrington, A. McLeod,  
 G. Papathanasiou,  
 M. Spradlin, 1612.08976

Weight 2 functions restricted to 6 out of 9:

$$\text{Li}_2(1 - 1/u) \quad \text{Li}_2(1 - 1/v) \quad \text{Li}_2(1 - 1/w)$$

$$\ln^2 \frac{uv}{w} \quad \ln^2 \frac{vw}{u} \quad \ln^2 \frac{wu}{v}$$