Yang-Mills amplitudes with manifest color-kinematics

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Work together with:

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(1608.00006, 1605.06501, 1509.02169, 1508.03627,
1507.00997, 1506.06137, 1403.4553)

New prescription for perturbative amplitudes

CHY formula (Cachazo, He and Yuan) (See also Ricardo's and Yvonne's talks)

$$\mathcal{A}_n = \int \frac{d^n\sigma}{\mathrm{volSL(2,c)}} \prod_a{}' \delta\left(\sum_{a\neq b} \frac{k_a\cdot k_b}{z_a - z_b}\right) \left(\frac{\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}\cdots T^{a_n})}{(z_1-z_2)(z_2-z_3)\cdots(z_n-z_1)} + \cdots\right)^{2-s} (\mathrm{Pf}'\Psi)^s$$

Color trace factor, very alike to cyclic trace in MHV amplitudes

Algebraic solutions

Pfaffian (depends on polarizations and momenta)

The scattering equations

$$S_i = \sum_{j \neq i} \frac{k_i \cdot k_j}{z_i - z_j} = 0 \qquad z_1 = 0, z_{N-1} = 1 \text{ and } z_N = \infty$$

The scattering equations: reminiscent of early work on dual models / high-energy string scattering (Fairlie and Roberts; Gross and Mende)

Some results:

- Proof of CHY formalism (Dolan and Goddard)
- In Ambi-twistor-space (Mason and Skinner; Adamo, Casali and Skinner)
- In pure spinor formalism (Berkovits; Gomez, Yuan)
- Some work on explicit solutions (e.g. Dolan and Goddard; Søgaard and Zhang; Cardona and Gomez; Zlotnikov; Gomez)

Simplest case:
The N-point scalar amplitude

For the N-point scalar amplitude (s = 0) one has

$$A_N = \int \prod_i{}' \delta(S_i) \frac{(z_1 - z_{N-1})(z_1 - z_N)(z_{N-1} - z_N)}{\prod_{i=1}^{N}(z_i - z_{i+1})^2} \prod_{i=2}^{N-1} dz_i$$

Here

$$S_i \equiv \sum_{j \neq i} \frac{s_{ij}}{(z_i - z_j)} = 0$$

are the scattering equations where

$$z_1 = 0, z_{N-1} = 1 \text{ and } z_N = \infty$$

Sum over solutions

Generally complicated solutions at higher points. N-roots of Polynomial equations.
(can be complex)

Much like standard Kobe-Nielsen gauge fixing

Illustrating the 4-point scalar amplitude

Following the prescription we have :

$$A_4 = \int dx \frac{\delta(S_i)}{(z_{12})^2 (z_{23})^2}$$

We have the following total (not-independent) scattering equations

$$\left.\begin{aligned} \frac{s_{12}}{z_{12}} + \frac{s_{13}}{z_{13}} &= 0 \\ -\frac{s_{12}}{z_{12}} + \frac{s_{23}}{z_{23}} &= 0 \\ \frac{s_{13}}{z_{13}} + \frac{s_{23}}{z_{23}} &= 0 \end{aligned}\right\}$$

$$s_{12} = s, \quad s_{13} = u, \quad s_{23} = t$$

$$z_4 = \infty, \quad z_1 = 0, \quad z_2 = x, \quad z_3 = 1$$

$$z_{12} = -x, \quad z_{23} = x - 1$$

Solution: $x = \dfrac{s}{s+t}$

Illustrating the 4-point scalar amplitude

So that:

$$A_4 = \int dx \frac{\delta(S_i)}{(z_{12})^2(z_{23})^2} = \int dx \frac{\delta\left(\frac{s}{x} - \frac{t}{(1-x)}\right)}{x^2(1-x)^2}$$

$$= \frac{st}{(s+t)^3}\frac{(s+t)^2}{s^2}\frac{(s+t)^2}{t^2} = \frac{(s+t)}{st} = \frac{1}{s} + \frac{1}{t}$$

The correct result!

4-point scalar ‘stringy’ amplitude

We have:

$$\mathcal{A}(1,2,3,4) = \int_0^1 dx (x)^{\alpha' s_{12}-1}(1-x)^{\alpha' s_{14}-1}$$

so by integration we have

$$= -(\alpha')\frac{\Gamma(-\alpha' s_{12})\Gamma(-\alpha' s_{14})}{\Gamma(-\alpha'(s_{12}+s_{14}))} = \left(\frac{1}{s}+\frac{1}{t}\right) \boxed{+\, O(\alpha')}$$

Same leading order result.

Different logic!

$$\alpha' \to 0$$

The N-point gluon amplitude

For gluons (s = 1) we have

$$A_N = \int \mathrm{Pf}'\Psi_N(z_i) \prod_i{}' \delta(S_i) \prod_{i=1}^{N} \frac{1}{(z_i - z_{i+1})} \prod_{i=2}^{N-2} dz_i \,,$$

Sum over solutions

Cyclic trace

$$\mathrm{Pf}'\Psi \equiv \frac{(-1)^{i+j}}{(z_i - z_j)} \mathrm{Pf}\left(\Psi^{ij}_{ij}\right)$$

$$\Psi \equiv \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

Polarizations and momenta

Gluon amplitudes from string theory

$$\mathcal{A}_n = \lim_{\alpha'\to 0} \alpha'^{(n-4)/2} \int \prod_{i=3}^{n-1} dz_i \, \frac{(z_1-z_2)(z_2-z_n)(z_n-z_1)}{\prod_{i=1}^{n}(z_i-z_{i+1})} \int d^n\theta \, d^n\varphi \prod_{i<j} (z_i - z_j - \theta_i\theta_j)^{\alpha' s_{ij}}$$

$$\times \prod_{i<j} \exp\left[\frac{\sqrt{2\alpha'}(\theta_i-\theta_j)\Big(\varphi_i(\epsilon_i\cdot k_j)+\varphi_j(\epsilon_i\cdot k_j)\Big)}{(z_i-z_j)} - \frac{\varphi_i\varphi_j\epsilon_{ij}}{(z_i-z_j)} - \frac{\theta_i\theta_j\varphi_i\varphi_j\epsilon_{ij}}{(z_i-z_j)^2} \right]$$

Subtraction of fermionic and bosonic degrees of freedom

Koba-Nielsen factor:
important signs introduced from orderings of integrations

Auxiliary Grassmann integrations introduce multi-linearity in polarizations just as Pfaffian does in the CHY formalism

(Other integrands possible to consider as well⋯)

String Theory and CHY

- Interesting feature: Integration by parts identities in string theory are in this viewpoint related to the scattering equations.

$$\exp[-\alpha' s \log(x) - \alpha' t \log(1-x)]$$

- E.g. :

$$\partial_x \exp[-\alpha' s \log(x) - \alpha' t \log(1-x)]$$

$$= \alpha' \left(-\frac{s}{x} + \frac{t}{(1-x)}\right) \exp[-\alpha' s \log(x) - \alpha' t \log(1-x)]$$

(E.g. Polchinski; Broedel, Schlotterer, Stieberger)

Open question: CHY Scattering equations: is the Kobe-Nielsen factor missing??!

Analogy between prescriptions

String theory

Integration in an ordered manner along the real line.

Poles comes from pinching regions.

$$z_i - z_j \to 0$$

Scattering eq. prescription

Integral saturated by delta-function and amplitude becomes localized.

Solutions not necessarily on real line.

$$\lim_{\alpha'\to 0} \int \prod dz_i \Big[\prod_{i=1}^{n-1} \Theta[z_{\sigma(i+1)\sigma(i)}] \leftrightarrow \frac{\prod_{n-3} \delta(S_i)}{z_{\sigma(1)\sigma(2)}\, z_{\sigma(2)\sigma(3)} \cdots z_{\sigma(n)\sigma(1)}} \Big] \times \prod_{i<z} |z_{ij}|^{\alpha' s_{ij}} \times H(z)$$

(NEJB, Damgaard, Tourkine, Vanhove)

Point of view :

- CHY formalism can be viewed as truncation of low-energy string scattering. (NEJB, Damgaard, Tourkine, Vanhove)
  - Useful: no need for integrations
  - Advantages: Certain string considerations/symmetries can carry over…
    - E.g. both CHY formalism and string theory share invariance under Mobius transformations
    - Amplitudes are built up in similar ways.

Using the scattering eq. formalism

- Basically currently three options for evaluation:
  - Direct numerical solutions
    - Numerically very hard beyond 7pt .. Normally (real) numerical results from 6pt up.
  - Using rules for evaluation of residues: scattering eq. rules for scalars, see *e.g.* (Cachazo, He and Yuan; Baadgaard Jepsen, NEJB, Bourjaily, Damgaard, Feng; Gomez) and recent extension to gluons (NEJB, Bourjaily, Damgaard, Feng; Cardona, Feng, Gomez and Huang)
  - Finally some techniques for direct integration, see *e.g.* (Dolan and Goddard; Cardona and Gomez; Zlotnikov; Søgaard and Zhang; Gomez)

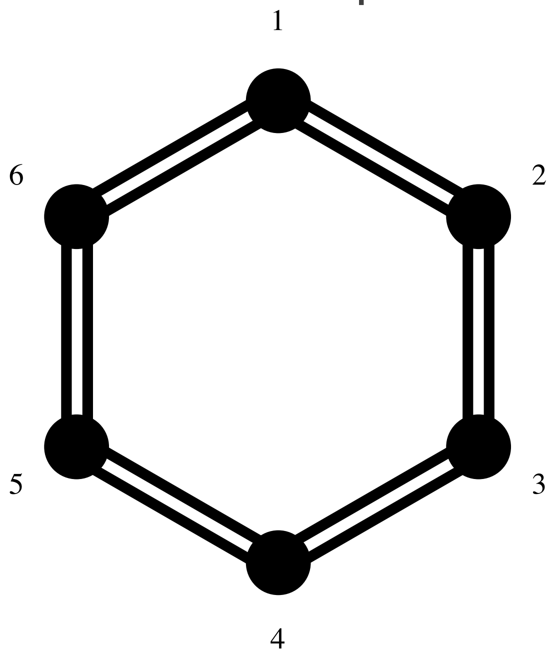
Integration rules for scattering eq.

Scalar theories

Think about the scalar amplitude integrand

$$\mathcal{A}_n^{\varphi^3} = \int d\Omega_{\mathrm{CHY}} \left(\frac{1}{(z_1-z_2)^2(z_2-z_3)^2 \cdots (z_n-z_1)^2} \right)$$

This can be thought of diagrammatically as a double line between points 1 to n.



It will as its result after summing over the (n-3)! solutions give the result for the n point φ^3 tree.

Question: Can we identify the individual diagrams in this tree amplitude?

Integration rules for scattering eq.

- It turns out we can!
- Again such rules comes very natural from a comparison of the scattering equations and string theory.

$$\lim_{\alpha'\to 0}\int \prod dz_i \Big[\prod_{i=1}^{n-1} \Theta[z_{\sigma(i+1)\sigma(i)}] \leftrightarrow \frac{\prod_{n-3}\delta(S_i)}{z_{\sigma(1)\sigma(2)}\, z_{\sigma(2)\sigma(3)} \cdots z_{\sigma(n)\sigma(1)}}\Big] \times \prod_{i<z} |z_{ij}|^{\alpha' s_{ij}} \times H(z)$$

Poles comes where $z_i - z_j \to 0$
so we can do a counting of how many
points pinching we need to have given a pole.

From this 'count' we
get scattering eq.
Integration rules

Integration rules for scattering eq.

The rules

- Have integrand H(z) with weight 2 in all variables (Mobius invariance).
- The integration rule is

(Baadsgaard Jepsen, NEJB, Bourjaily, Damgaard)

$$\prod_{a=1}^{n-3} \left(1/s_{q_a}\right)$$

If:

Integrand has $2|q_a| - 2$ factors $z_i - z_j$ where $\{i, j\} \subset q_a$

All pairs of set have to satisfy that either they are nested (or their compliment are).

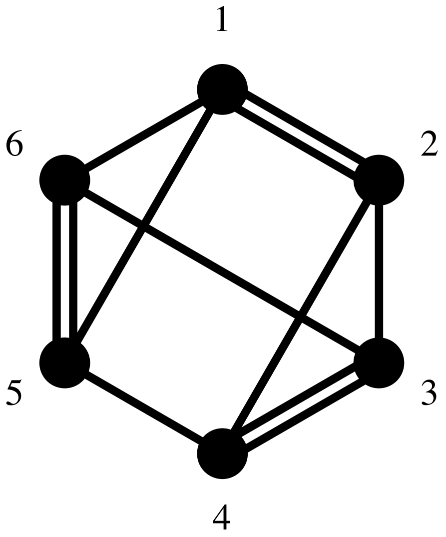
Set of points (or compliment set of points)

Starting point: find nested sets in diagram.

Integration rules for scattering eq.

Example

$$H(z) = \frac{1}{(z_1 - z_2)(z_1 - z_5)(z_2 - z_4)(z_3 - z_4)(z_3 - z_6)(z_5 - z_6)}$$



$\{3,4\}$: two variables, one factor connecting them

$\{5,6\}$: two variables, one factor connecting them

$\{2,3,4\}$: three variables, two factors connecting them

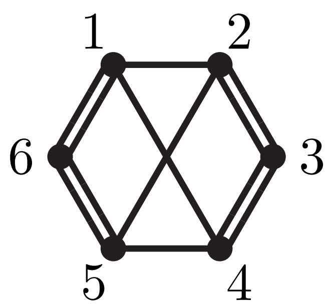
$\{3,4,5,6\}$: four variables, three factors connecting them

$$\tau_1 \equiv \{\{3,4\},\{5,6\},\{2,3,4\}\}$$
$$\tau_2 \equiv \{\{3,4\},\{5,6\},\{3,4,5,6\}\}$$

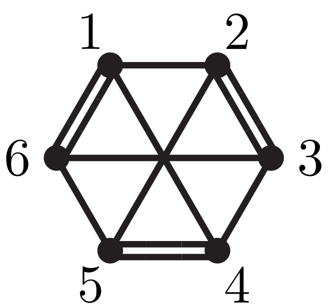
$$\frac{1}{s_{34}s_{56}}\left(\frac{1}{s_{234}} + \frac{1}{s_{3456}}\right)$$

Diagrammatic interpretation

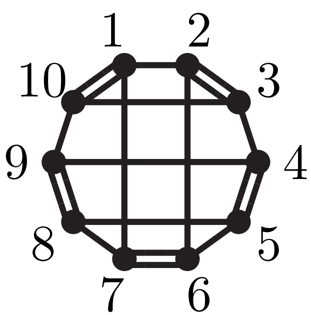
Examples



$$\Longrightarrow \left(\frac{1}{s_{23}s_{56}s_{561}} + \frac{1}{s_{23}s_{61}s_{561}} + \frac{1}{s_{34}s_{56}s_{561}} + \frac{1}{s_{34}s_{61}s_{561}} \right)$$



$$\Longrightarrow \frac{1}{s_{23}s_{45}s_{61}}$$

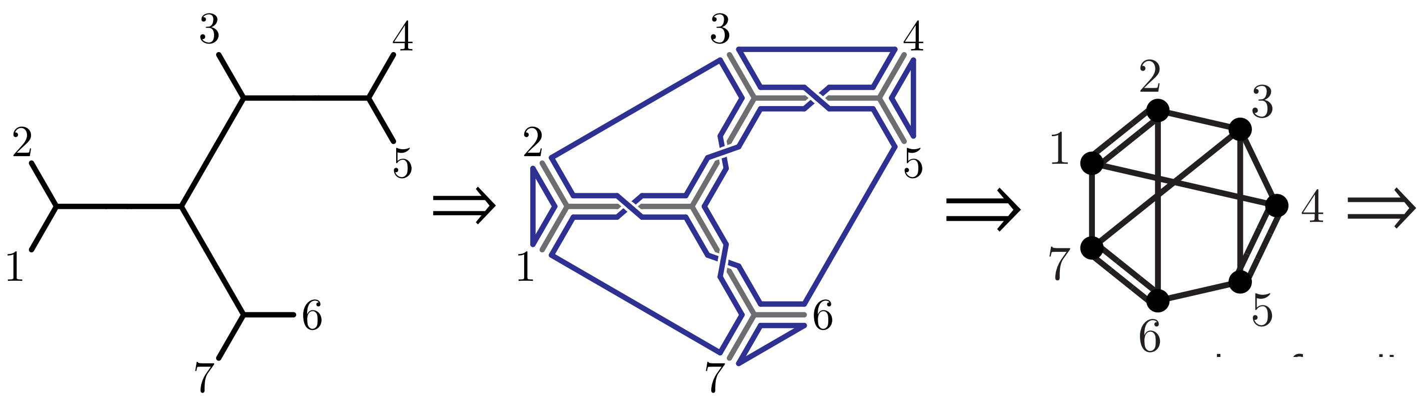


$$\Longrightarrow \frac{1}{s_{23}s_{45}s_{67}s_{89}s_{101}s_{10123}s_{4589}}$$

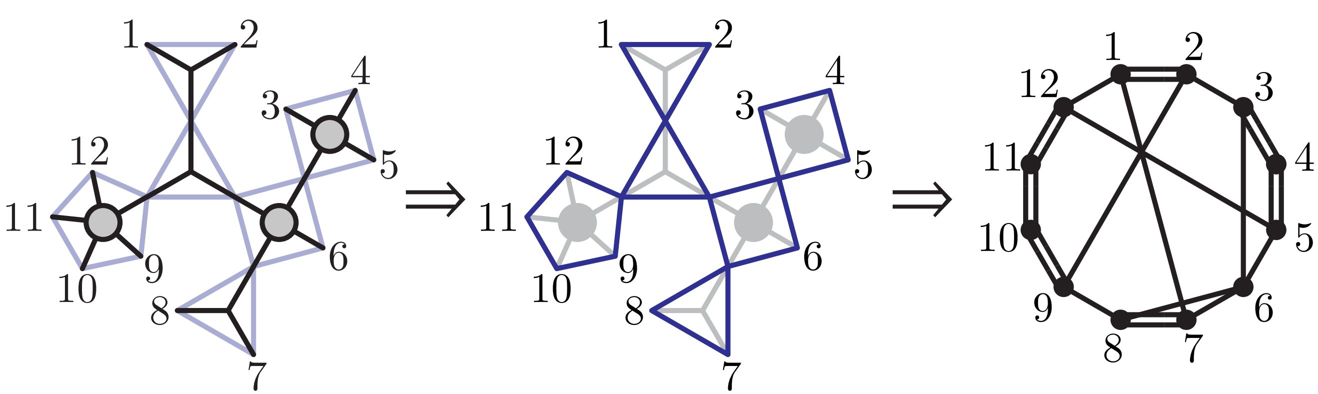
Diagrammatic interpretation

Weaving diagrams

Given the integration rules it is now possible to make a intuitive connection back to scalar Feynman diagrams:

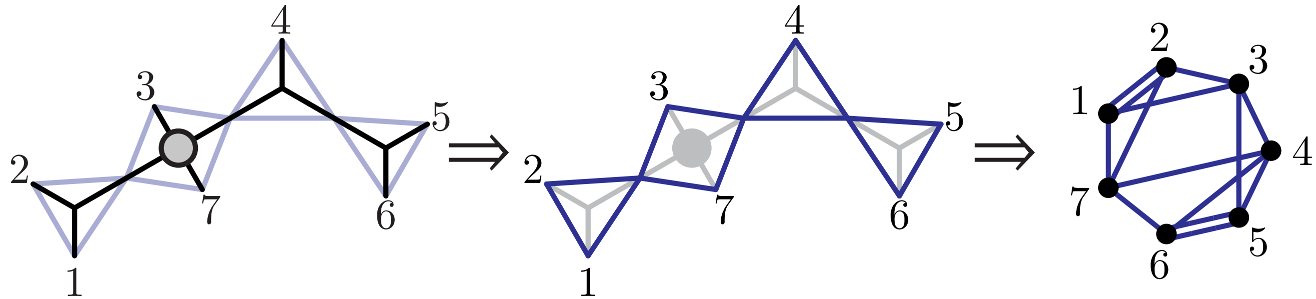


$$\Rightarrow \frac{1}{s_{12}s_{45}s_{67}s_{345}}$$

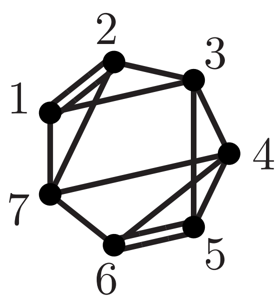


Diagrammatic interpretation

We can also decompose even further using partial fractioning identities:



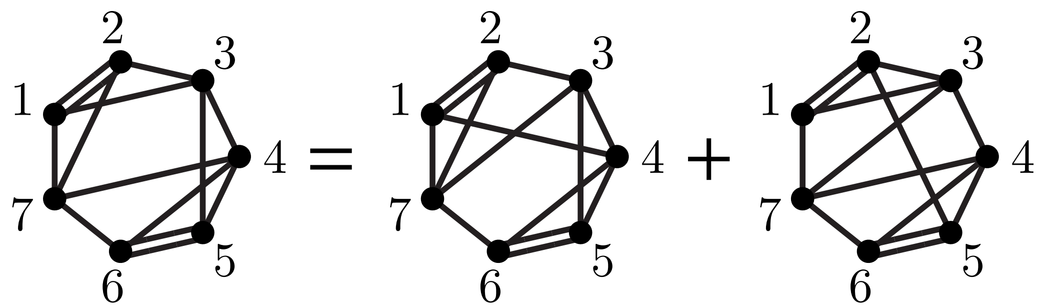
We have:



$$\Longleftrightarrow \left\{ \begin{array}{l} \dfrac{1}{(z_1-z_2)(z_2-z_3)(z_3-z_4)(z_4-z_5)(z_5-z_6)(z_6-z_7)(z_7-z_1)} \\ \times \dfrac{1}{(z_1-z_2)(z_2-z_7)(z_7-z_4)(z_4-z_6)(z_6-z_5)(z_5-z_3)(z_3-z_1)} \end{array} \right.$$

Diagrammatic interpretation

Now:



$$\Leftrightarrow \begin{cases} \dfrac{1}{(z_1-z_2)^2(z_2-z_3)(z_3-z_4)(z_4-z_5)(z_5-z_6)^2(z_6-z_7)(z_7-z_1)} \\ \times \dfrac{1}{(z_3-z_7)(z_4-z_6)}\left[\dfrac{1}{(z_1-z_4)(z_2-z_7)(z_3-z_5)}+\dfrac{1}{(z_2-z_5)(z_7-z_4)(z_1-z_3)}\right] \end{cases}$$

We see that all contributions are now decomposed into φ^3 diagrams.

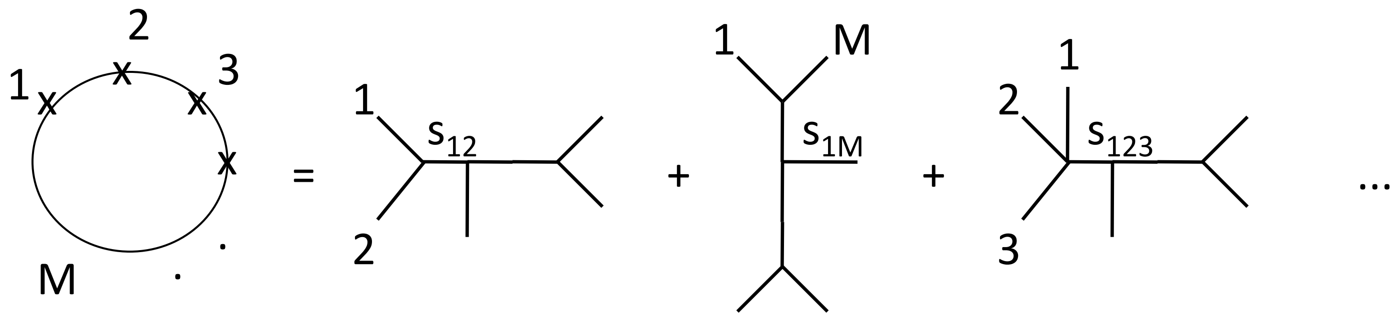
String theory

Useful laboratory

String theory adds channels up..

<->

Feynman diagrams sums separate kinematic poles



String theory

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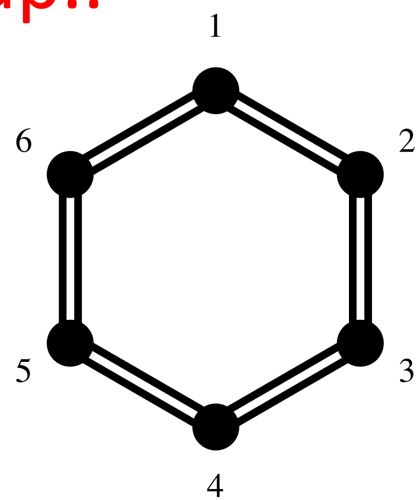

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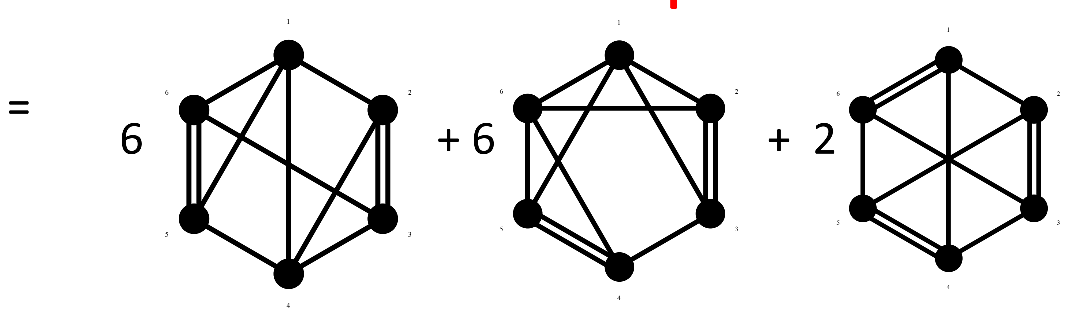


Gluon amplitudes

Providing analytic trees for Yang-Mills from traditional methods difficult in arbitrary dimension.

- The scattering equation formalism appear to be the perfect place to start.
- Formalism naturally combines the beautiful aspects of string theory in a concrete formalism that avoid integrations.
- As we will see 'integration rules for gluons' not straightforward…..but still possible...
  (NEJB, Bourjaily, Damgaard, Feng)

Starting point for gluon amplitudes

Starting point is the integrand:

$$\mathcal{A}_n \equiv (-1)^{\lfloor n/2 \rfloor} \int \Omega_{\text{CHY}} \, \frac{\text{Pf}\,'\Psi(z_i)}{(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_1)}$$

Where we have:

$$\Omega_{\text{CHY}} \equiv \frac{d^n z}{\text{vol}(SL(2))} \prod_i{}' \delta(S_i) \equiv (z_r - z_s)^2 (z_s - z_t)^2 (z_t - z_r)^2 \prod_{i \in \mathbb{Z}_n \setminus \{r,s,t\}} dz_i \, \delta(S_i)$$

$$\text{Pf}\,'\Psi \equiv \frac{(-1)^{i+j}}{(z_i - z_j)} \text{Pf}(\Psi^{ij}_{ij}), \quad \text{where} \quad \Psi \equiv \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

Definition of Pfaffian

$$s_{ij} \equiv 2k_i \cdot k_j$$
$$\epsilon_{ij} \equiv 2\epsilon_i \cdot \epsilon_j$$
$$(\epsilon_i \cdot k_j) \equiv 2\epsilon_i \cdot k_j$$

One has $\Psi \equiv \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$

$$A_{i\neq j} \equiv \frac{s_{ij}}{(z_i - z_j)}, \qquad B_{i\neq j} \equiv \frac{\epsilon_{ij}}{(z_i - z_j)}, \qquad C_{i\neq j} \equiv \frac{(\epsilon_i \cdot k_j)}{(z_i - z_j)},$$

$$A_{i=j} \equiv 0, \qquad B_{i=j} \equiv 0, \qquad C_{i=j} \equiv -\sum_{l\neq i} \frac{(\epsilon_i \cdot k_l)}{(z_i - z_l)}$$

Generic integrand multi-linear in polarizations:

4pt : 8 x 8 matrix reduction → Pfaffian of 6 x 6 matrix
5pt : 10 x 10 matrix reduction → Pfaffian of 8 x 8 matrix
N-pt : 2N x 2N matrix reduction → Pfaffian of 2(N-1)x2(N-1) matrix

Gluon integrands from Pfaffian

For example at 4pt we have *e.g.* two types of terms:

$$\frac{\epsilon_{13}\epsilon_{24}s_{34}}{(-z_1+z_3)(-z_2+z_4)(-z_3+z_4)}$$

$$\frac{\epsilon_{12}(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_1)}{(-z_1+z_2)(-z_1+z_3)(-z_1+z_4)}$$

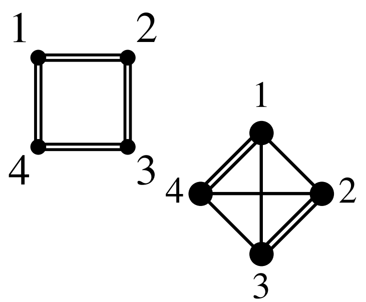
Two observations:

1) multi-linearity always automatically satisfied
2) integrations follow contractions

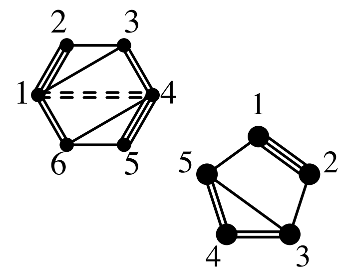
Obstacles

New terms to deal with:

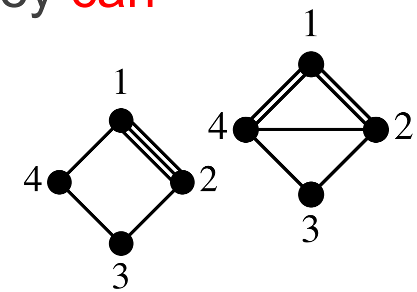
- Integrands: scalar type (only double or single lines everywhere): can be immediately integrated using the rules



- Integrand: ‘tuple type’ (they have for example a triple line or a cluster of double lines in a corner). Such integrands cannot be immediately integrated using the scattering eq. rules.



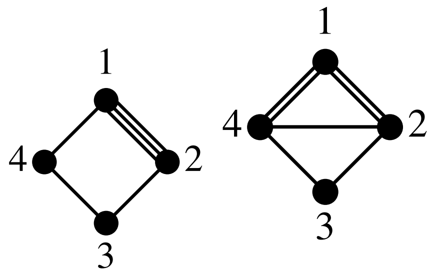
- Not manifestly Mobius invariant integrands. They need rewriting (using momentum conservation) before they can be integrated. Some of such diagrams are tuple diagrams.)



Dealing with diagonal terms in C

- We start with contributions that are of the mobius violating type:

$$C_{i=j} \equiv -\sum_{l\neq i} \frac{(\epsilon_i\cdot k_l)}{(z_i - z_l)}$$



- Now we can use partial fractioning identities to write

$$-\frac{(\epsilon_i\cdot k_l)}{(z_i - z_l)} = \frac{(\epsilon_i\cdot k_l)}{(z_a - z_i)} + \frac{(\epsilon_i\cdot k_l)(z_l - z_a)}{(z_a - z_i)(z_i - z_l)} \quad \text{for} \quad i \neq a$$

by momentum conservation

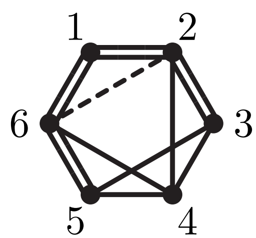
$$\Rightarrow \sum_{l\neq i,a} \frac{(\epsilon_i\cdot k_l)(z_l - z_a)}{(z_a - z_i)(z_i - z_l)}$$

New feature: numerators!

Dealing with diagonal terms in C

- Now such diagrams will have numerator contributions but are still possible to compute using the basic scalar rules. Basically a numerator factor is like a denominator factor but counts as -1.

- We can consider



$$H(z) = \frac{(z_2 - z_6)}{(z_1-z_2)^2(z_1-z_6)^2(z_2-z_3)^2(z_2-z_4)(z_3-z_4)(z_3-z_5)(z_4-z_5)(z_4-z_6)(z_5-z_6)^2}$$

$\{1,2\}$, $\{1,6\}$, $\{2,3\}$, $\{5,6\}$ } two points, two lines

$\{1,2,3\}$, $\{2,3,4\}$ } three vertices, four lines

$\{1,2,6\}$ Not in because of dotted

Dealing with diagonal terms in C

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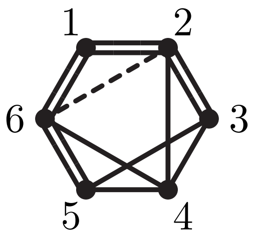


$$H(z) = \frac{(z_2 - z_6)}{(z_1 - z_2)^2 (z_1 - z_6)^2 (z_2 - z_3)^2 (z_2 - z_4)(z_3 - z_4)(z_3 - z_5)(z_4 - z_5)(z_4 - z_6)(z_5 - z_6)^2}$$

$\{1,2\}, \{5,6\}, \{1,2,3\}$

$\{1,6\}, \{2,3\}, \{2,3,4\}$

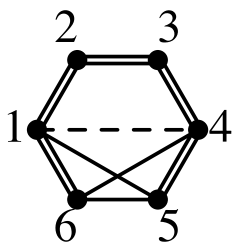
$\{2,3\}, \{5,6\}, \{1,2,3\}$

$\{2,3\}, \{5,6\}, \{2,3,4\}$

$$\left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right)\frac{1}{s_{56}s_{123}} + \left(\frac{1}{s_{16}} + \frac{1}{s_{56}}\right)\frac{1}{s_{23}s_{234}}$$

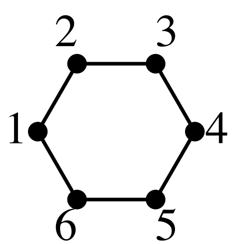
Dealing with tuple diagrams

- Here the link to string theory will be important. We will consider integrals in the scattering equation formalism such as



$$\Leftrightarrow \frac{(z_1-z_4)}{(z_1-z_2)^2(z_2-z_3)^2(z_3-z_4)^2(z_4-z_5)^2(z_5-z_6)(z_1-z_6)^2(z_1-z_5)(z_4-z_6)}$$

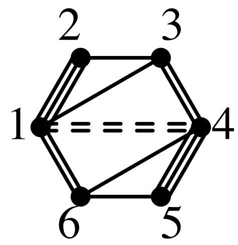
- We will sometimes for convenience focus on the outer rim which we will denote



$$\Leftrightarrow \frac{1}{(z_1-z_2)(z_2-z_3)(z_3-z_4)(z_4-z_5)(z_5-z_6)(z_1-z_6)} = -PT(1,2,3,4,5,6)$$

Dealing with tuple diagrams

- The integrations we will consider how to deal with will be of the form:



$$\Leftrightarrow \frac{(z_1-z_4)^2}{(z_1-z_2)^3(z_2-z_3)(z_3-z_4)^2(z_4-z_5)^3(z_5-z_6)(z_1-z_6)^2(z_1-z_3)(z_4-z_6)}$$

Here we have problems in lines: $\{1,2\}, \{4,5\}, \{1,2,3\},$ and $\{1,2,6\}$

Now we in the following in a systematic way see how to deal with such integrals. Here the 'link' to string theory is useful, *i.e.*:

$$\mathcal{I}_n = \lim_{\alpha'\to 0} \alpha'^{\,n-3} \int \prod_{i=3}^{n-1} dz_i \, (z_1-z_2)(z_2-z_n)(z_n-z_1) \prod_{1\le i<j\le n} |z_i - z_j|^{\alpha' s_{ij}} H(z)$$

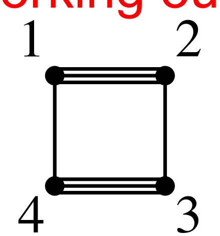
4 point gluon amplitudes

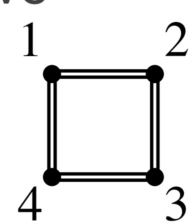
- We will start with the four point gluon amplitude to illustrate the procedure. What we do here will extend to higher points.

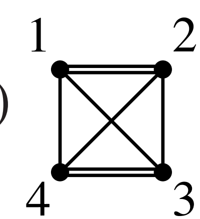
- For the four amplitude we have the following decomposition:

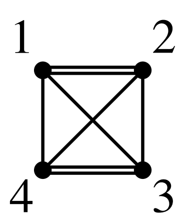
$$\mathcal{A}_4 = \alpha_1\, \epsilon_{12}\epsilon_{34} + \alpha_2\, \epsilon_{13}\epsilon_{24} + \beta_1\, \epsilon_{12} + \beta_2\, \epsilon_{13} + \text{cyclic}$$

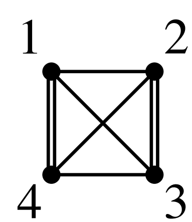
- From working out the Pfaffian we have

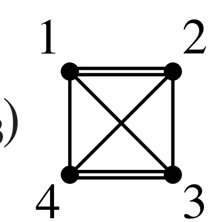
$\alpha_1 \equiv s_{12}$ 

$\beta_1 \equiv \quad (\epsilon_3 \cdot k_2)(\epsilon_4 \cdot k_1)$ 

$- (\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_2)$ 

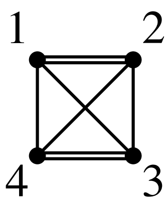
$\alpha_2 \equiv -s_{12}$ 

$\beta_2 \equiv -(\epsilon_2 \cdot k_3)(\epsilon_4 \cdot k_1)$ 

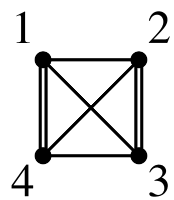
$- (\epsilon_2 \cdot k_1)(\epsilon_4 \cdot k_3)$ 

4 point gluon amplitudes

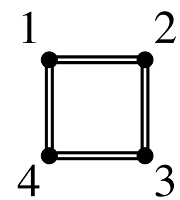
- Now we can use the scalar integration rules to write:



$$= -\frac{1}{s_{12}},$$



$$= -\frac{1}{s_{23}},$$



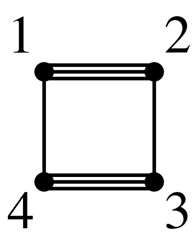
$$= -\left(\frac{1}{s_{12}} + \frac{1}{s_{23}}\right)$$

- So that

$$\beta_1 = \frac{(\epsilon_3 \cdot k_1)(\epsilon_4 \cdot k_2) s_{23} + (\epsilon_3 \cdot k_2)(\epsilon_4 \cdot k_1) s_{13}}{s_{12} s_{23}}, \qquad \beta_2 = \frac{(\epsilon_2 \cdot k_1)(\epsilon_4 \cdot k_3) s_{23} + (\epsilon_2 \cdot k_3)(\epsilon_4 \cdot k_1) s_{12}}{s_{12} s_{23}}$$

- However this diagram is a problem:

$$\alpha_1 \equiv s_{12}$$



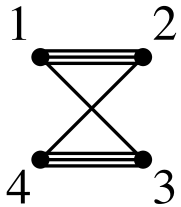
4 point gluon amplitudes

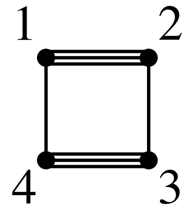
- Now we will use that we have a dual description in terms of string theory type integrations. At four points we can write:

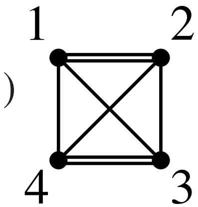
$$0 = \int_{-\infty}^{0} dz H(z)(-z)^{\alpha' s_{12}}(1-z)^{\alpha' s_{23}}$$

$$+ e^{i\alpha' s_{12}} \int_{0}^{1} dz\, H(z)(z)^{\alpha' s_{12}}(1-z)^{\alpha' s_{23}} + e^{i\alpha'(s_{12}+s_{23})} \int_{1}^{\infty} dz\, H(z)(z)^{\alpha' s_{12}}(z-1)^{\alpha' s_{23}}$$

- This gives for the type of integrand we are considering:

$0 =$ 

$+ \; e^{i\alpha' s_{12}}$ 

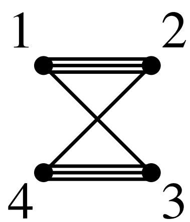
$- \; e^{i\alpha'(s_{12}+s_{23})}$ 

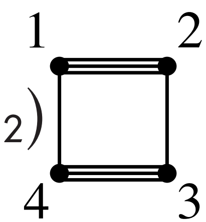
Feature: Kobe-Nielsen factor important!

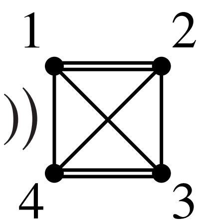
4 point gluon amplitudes

- This identity natural splits in two ways: (like string theory monodromy) (NEJB, Damgaard, Vanhove; Stieberger)

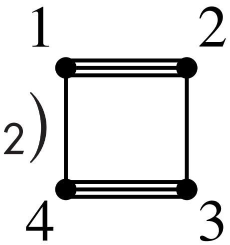
- Real part:

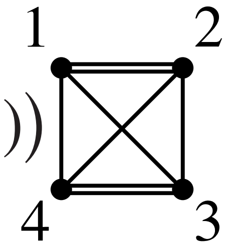
$0 =$ 

$+ \cos(\alpha' s_{12})$ 

$- \cos(\alpha'(s_{12}+s_{23}))$ 

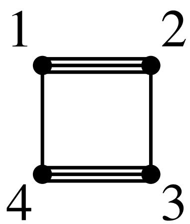
- Imaginary part:

$0 = \sin(\alpha' s_{12})$ 

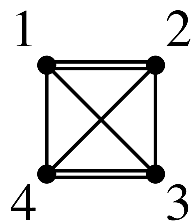
$- \sin(\alpha'(s_{12}+s_{23}))$ 

4 point gluon amplitudes

- Now in the field theory limit we have:



$$= \frac{s_{12}+s_{23}}{s_{12}}$$



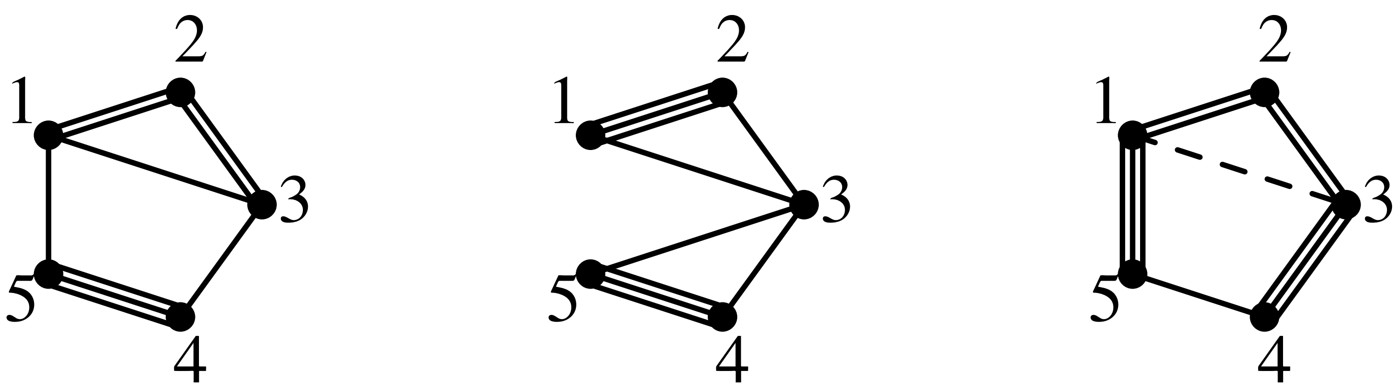
$$= -\frac{s_{12}+s_{23}}{s_{12}^2} = \frac{s_{13}}{s_{12}^2}$$

- Thus we have the following simple expression for the four gluon amplitude:

$$\mathcal{A}_4 = \Big[\epsilon_{13}\epsilon_{24} + \frac{1}{s_{12}}\Big(\epsilon_{12}\epsilon_{34}s_{13} + \epsilon_{12}\big((\epsilon_3\cdot k_1)(\epsilon_4\cdot k_2) + (\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1)\big) + \epsilon_{13}(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_3)\Big)$$
$$+ \frac{1}{s_{23}}\Big(\epsilon_{12}(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1) + \epsilon_{13}(\epsilon_2\cdot k_3)(\epsilon_4\cdot k_1)\Big)\Big] + \text{cyclic.}$$

Higher point gluon amplitudes

- At higher point we of course get more problematic tuples as well. For example at 5 point we have:



- Using the notation:

$$PT(1,2,\ldots,n) \equiv \frac{1}{(z_1 - z_2)(z_2 - z_3)(z_3 - z_4)\cdots(z_n - z_1)}$$

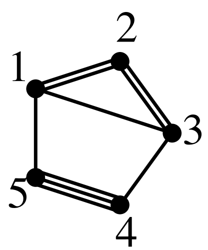
Higher point gluon amplitudes

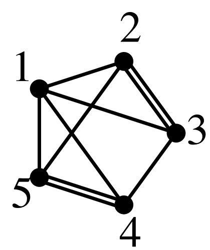
- It is now clear that using the same type of trick as for four points at higher points (2 – tuple identity)

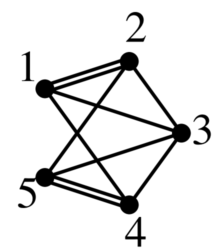
$$0 = s_{12} PT(1, 2, \dots, n) + \sum_{k=3}^{n-1} (s_{12} + s_{2(3\cdots k)}) PT(1, \dots, k, 2, k+1, \dots, n)$$

- Or $\quad \mathrm{Id}_{\{1,2\}} \equiv -\sum_{k=3}^{n-1} \left(\frac{s_{12} + s_{2(3\cdots k)}}{s_{12}}\right) \frac{PT(1, \dots, k, 2, k+1, \dots, n)}{PT(1, 2, \dots, n)} = 1$

- So that we *e.g.* have

$\mathrm{Id}_{\{4,5\}}$ 

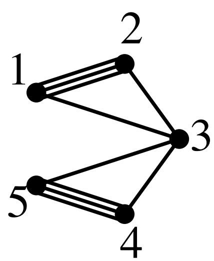
$= \frac{s_{45} + s_{15}}{s_{45}}$ 

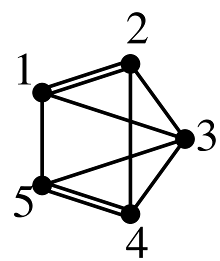
$+ \frac{s_{45} + s_{(12)5}}{s_{45}}$ 

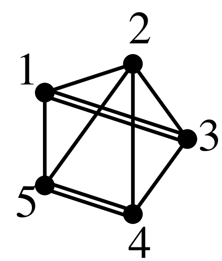
$$= \frac{1}{s_{45}^2} \left(\frac{s_{45} + s_{15}}{s_{23}} - \frac{s_{35}}{s_{12}} \right).$$

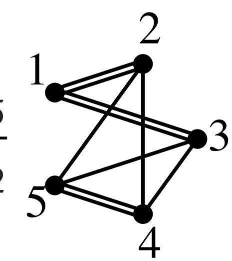
Higher point gluon amplitudes

- Similarly we can consider

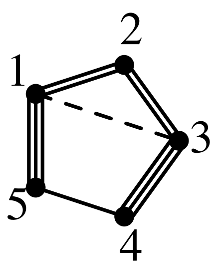
$\mathrm{Id}_{\{4,5\}}\mathrm{Id}_{\{1,2\}}$ 

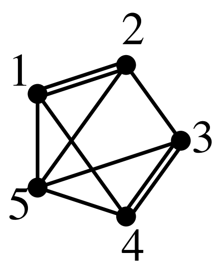
$= -\frac{s_{(12)3}s_{(34)5}}{s_{12}s_{45}}$ 

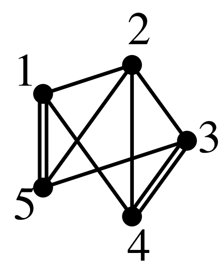
$-\frac{s_{(12)3}s_{25}}{s_{12}s_{45}}$ 

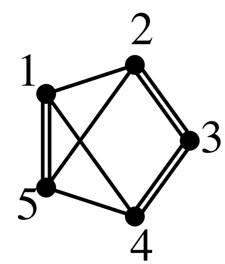
$-\frac{s_{25}}{s_{12}}$ 

$$= \frac{s_{(12)3}}{s_{12}s_{13}s_{45}}\left(\frac{s_{13}s_{(34)5}}{s_{12}s_{45}} - \frac{s_{25}}{s_{45}} + \frac{s_{25}}{s_{12}}\right)$$

$\mathrm{Id}_{\{5,1\}}\mathrm{Id}_{\{3,4\}}$ 

$= \frac{s_{1(25)}s_{4(35)}}{s_{15}s_{34}}$ 

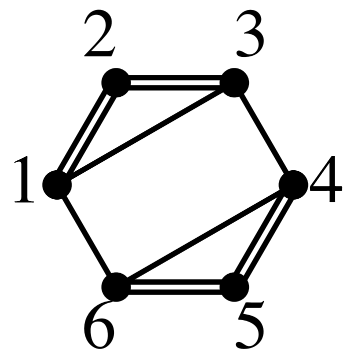
$-\frac{s_{1(25)}s_{24}}{s_{15}s_{34}}$ 

$-\frac{s_{14}}{s_{15}}$ 

$$= \frac{1}{s_{15}s_{34}}\left(\frac{s_{1(25)}s_{4(35)}}{s_{12}s_{34}} - \frac{s_{1(25)}s_{24}}{s_{15}s_{34}} - \frac{s_{3(24)}s_{14}}{s_{15}s_{23}}\right)$$

Generalizations and higher point gluon amplitudes

- For 2-tuples the identities before are fine but for diagrams like



we need yet another generalization. Here again ‘monodromy’ guides the way. Here we have identities like

$$
\begin{aligned}
0 = \;& PT(1,2,3,4,5,6)s_{123} + PT(1,2,4,3,5,6)(s_{123}+s_{34}) \\
& + PT(1,2,4,5,3,6)(s_{123}+s_{3(45)}) + PT(1,4,2,3,5,6)(s_{123}+s_{(23)4}) \\
& + PT(1,4,2,5,3,6)(s_{123}+s_{(23)4}+s_{35}) + PT(1,4,5,2,3,6)(s_{123}+s_{(23)(45)})
\end{aligned}
$$

Generalizations and higher point gluon amplitudes

- This can be written as

$$0 = \sum_{\sigma \in (\{2,\dots,k\} \sqcup \{k+1,\dots,n\text{-}1\})} PT(1,\sigma_1,\dots,\sigma_{n\text{-}2},n)\Big(s_{1\dots k} + \sum_{\{i,j\}|\sigma_i>\sigma_j} s_{\sigma_i\,\sigma_j}\Big)$$

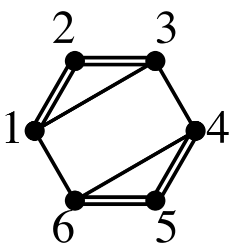
- Giving the following tuple identity

$$\mathrm{Id}_{\{1,\dots,k\}} \equiv \frac{-1}{PT(1,\dots,n)s_{1\dots k}} \sum_{\sigma \in (\{2,\dots,k\} \,\widetilde{\sqcup}\, \{k+1,\dots,n\text{-}1\})} PT(1,\sigma_1,\dots,\sigma_{n\text{-}2},n)\Big(s_{1\dots k} + \sum_{\{i,j\}|\sigma_i>\sigma_j} s_{\sigma_i\,\sigma_j}\Big) = 1$$

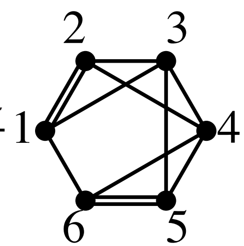
- Now such identities provide the remaining problematic diagrams.

Generalizations and higher point gluon amplitudes

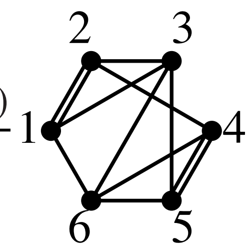
- For example



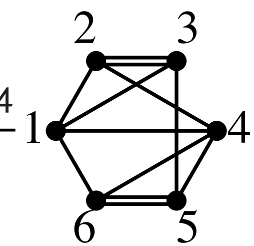
$= \frac{s_{123}+s_{34}}{s_{123}}$



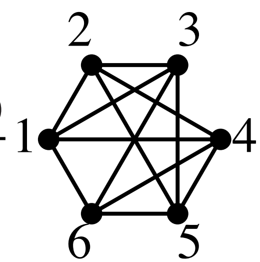
$+ \frac{s_{123}+s_{3(45)}}{s_{123}}$



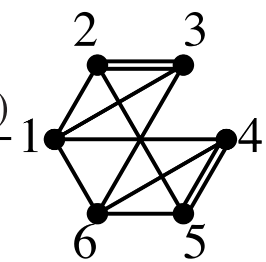
$+ \frac{s_{123}+s_{(23)4}}{s_{123}}$



$- \frac{s_{123}+s_{24}+s_{3(45)}}{s_{123}}$



$+ \frac{s_{123}+s_{(23)(45)}}{s_{123}}$



$$= -\frac{1}{s_{123}^2}\left(\frac{s_{123}+s_{34}}{s_{12}s_{56}} + \frac{s_{123}+s_{3(45)}}{s_{12}s_{45}} + \frac{s_{123}+s_{(23)4}}{s_{23}s_{56}} + \frac{s_{123}+s_{(23)(45)}}{s_{23}s_{45}}\right)$$

Five and six point amplitudes

- Through the new techniques we can now expand the Pfaffian terms and just integrate the various contributions.
- Procedure works as follows: First one computes all basic scalar integrations, next all C diagonal terms are converted into Mobius invariant terms with possible tuples.
- Next all tuple diagrams are rewritten to basic scalar integrands via the monodromy type relations.
- This immediately provides results for five and six gluon amplitudes.
  - Beyond six point, same procedure works – manipulations do become more complicated.

What is learned

- We have seen that analytic expressions for gluon amplitudes can be directly written down using the integration rules as well as the monodromy prescription.
- This gives yet another method for computation of amplitudes in D-dimensions.
- We will now see how the result can be refined so that we also can directly generate analytic results for BCJ numerators.
- Results can be compared to previous results in the literature from either analytic integration (Medina et al), or pure spinor results (Mafra, Schlotterer, Stieberger; Mafra, Schlotterer).

Color-Kinematics Duality

It follows from CHY that if we can expand

$$\mathrm{Pf}'\Psi = \sum_{\sigma} n_{1,\sigma,n} \times \mathrm{PT}(1, \sigma(2), \ldots, \sigma(n-1), n)$$

Then the coefficients $n_{1,\sigma,n}$ are KK Jacobi BCJ numerators.

This is required from demanding consistency of KLT squaring in the CHY formalism.

Color-Kinematics Duality

The starting point is the directly computed integrand that arises from the Pfaffian.

We have seen how to reduce the various contributions to integrands that can be readily integrated using the integration rules.

New goal: to bring Pfaffian directly to the form:

$$\mathrm{Pf}'\Psi = \sum_{\sigma} n_{1,\sigma,n} \times \mathrm{PT}(1, \sigma(2), \ldots, \sigma(n-1), n)$$

The reduction procedure will also be a very useful tool for many other integrands : i.e. reduction to single closed Hamiltonian cycles.

Example 4 points

- Starting point is:

$$\mathrm{Pf}'\Psi = \frac{n_1}{\langle 1234\rangle} + \frac{n_2}{\langle 1324\rangle} + \frac{n_3}{\langle 14\rangle\langle 23\rangle} + \frac{n_4}{\langle 124\rangle\langle 3\rangle} + \frac{n_5}{\langle 134\rangle\langle 2\rangle} + \frac{n_6}{\langle 14\rangle\langle 2\rangle\langle 3\rangle}$$

we get: (reducing $\langle 14\rangle$ with $\{a,b\} \equiv \{1,3\}$)

$$\frac{1}{\langle 1243\rangle} = -\left(\frac{1}{\langle 1234\rangle} + \frac{1}{\langle 1324\rangle}\right)$$

$$\frac{1}{\langle 14\rangle\langle 23\rangle} = -\frac{s_{24}}{s_{14}}\frac{1}{\langle 1324\rangle}$$

$$\frac{1}{\langle 124\rangle\langle 3\rangle} = \frac{\epsilon k_{31}}{\langle 1243\rangle} - \frac{\epsilon k_{32}}{\langle 1234\rangle} = -\left(\frac{\epsilon k_{31} + \epsilon k_{32}}{\langle 1234\rangle} + \frac{\epsilon k_{31}}{\langle 1324\rangle}\right)$$

$$\frac{1}{\langle 134\rangle\langle 2\rangle} = \frac{\epsilon k_{21}}{\langle 1243\rangle} - \frac{\epsilon k_{23}}{\langle 1324\rangle} = -\left(\frac{\epsilon k_{21}}{\langle 1234\rangle} + \frac{\epsilon k_{21} + \epsilon k_{23}}{\langle 1324\rangle}\right)$$

$$\frac{1}{\langle 14\rangle\langle 2\rangle\langle 3\rangle} = -\frac{\epsilon k_{31}}{\langle 134\rangle\langle 2\rangle} - \frac{\epsilon k_{32}}{\langle 14\rangle\langle 2\rangle}\frac{(2,4)}{(4,3)(3,2)}$$

Example 4 points

Thus we can finally reduce:

$$\frac{1}{\langle 14\rangle\langle 2\rangle}\frac{(2,4)}{(4,3)(3,2)} = -\frac{\epsilon k_{21}}{\langle 1234\rangle} - \frac{\epsilon k_{23}}{\langle 14\rangle\langle 23\rangle}$$

So that we arrive

$$\frac{1}{\langle 14\rangle\langle 2\rangle\langle 3\rangle} = \frac{\epsilon k_{21}(\epsilon k_{31}+\epsilon k_{32})}{\langle 1234\rangle} + \left(\epsilon k_{31}(\epsilon k_{21}+\epsilon k_{23}) - \frac{s_{24}\,\epsilon k_{23}\,\epsilon k_{32}}{s_{14}}\right)\frac{1}{\langle 1324\rangle}$$

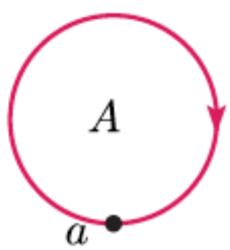
And now

$$\mathrm{Pf}'\Psi \equiv \frac{n_{1,\{2,3\},4}}{\langle 1234\rangle} + \frac{n_{1,\{3,2\},4}}{\langle 1324\rangle}$$

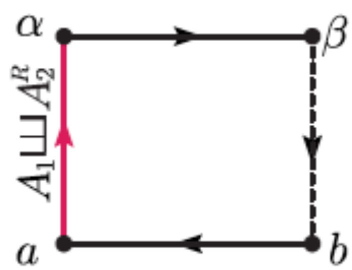
The freedom in picking reductions can be used to derive different numerator decompositions

A systematic algorithm for integrands

Given the amplitudes considered the previous slides we can generate the following generic relation:



$$= \sum_{\substack{\alpha\in A \\ \beta\in A^c \\ \sigma\in(A_1\sqcup A_2^R)}} (-1)^{|A_2|}\, n_{\alpha,\beta}$$



$$, \quad n_{\alpha,\beta} \equiv \begin{cases} s_{\alpha\beta}/s_A & |A|>1 \\ \epsilon k_{\alpha\beta} & |A|=1 \end{cases}$$

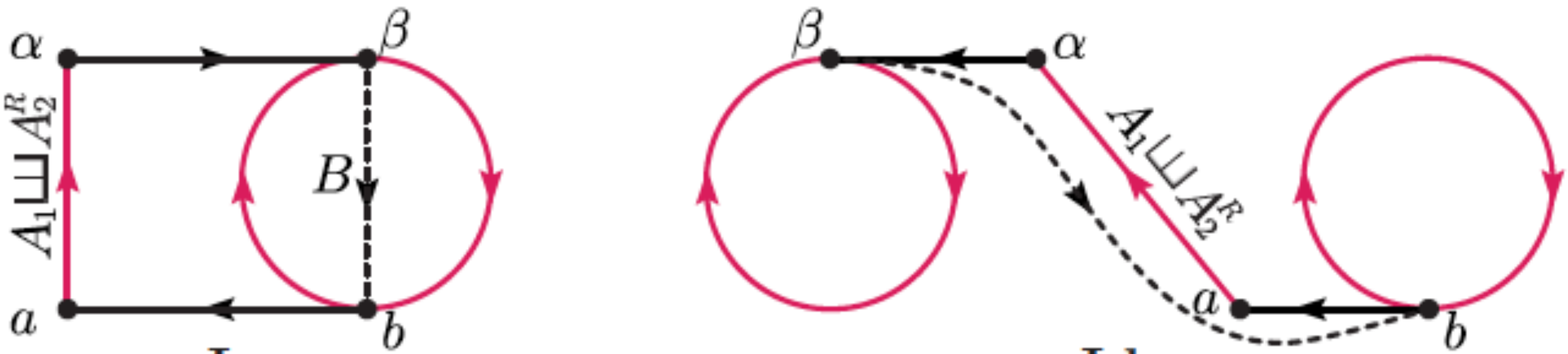
That is any cycle A can (given two points a in A and b not in A) be written in the alternative form ‘left’ here the the KK type relations has been used:

$$\mathrm{PT}(a, A_1, \alpha, A_2) = (-1)^{|A_2|} \sum_{\sigma\in A_1\sqcup A_2} \mathrm{PT}(a, \sigma, \alpha)$$

(Feature: monodromy type rewriting of terms important)

A systematic algorithm for integrands

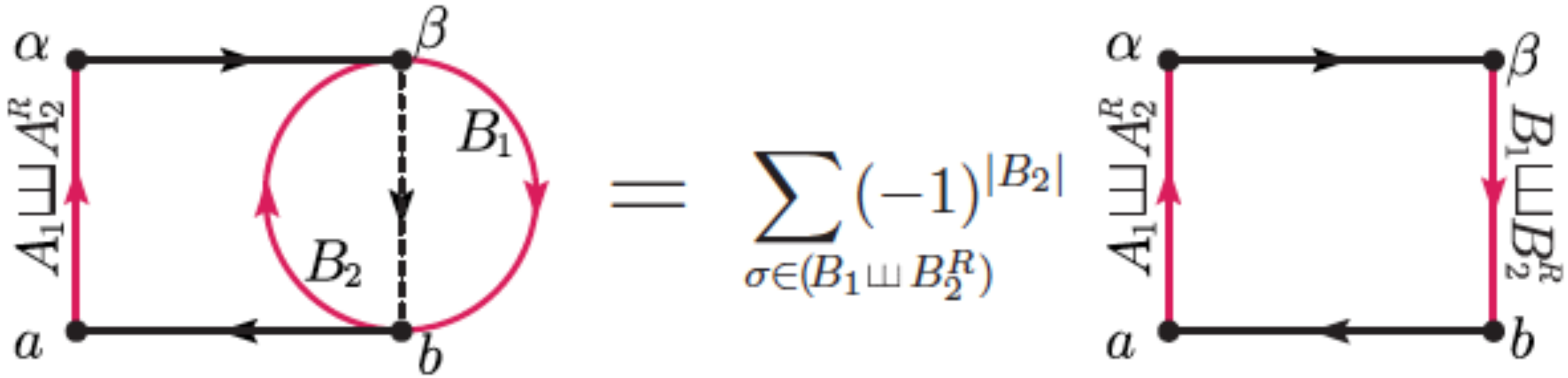
- Starting point is a generic diagram with a number of disjoint cycles:
- Reducing a cycle A relative to points a and b gives two different cases after a reduction, depending on if β lies in the cycle same as b or in a different one:



I.a I.b

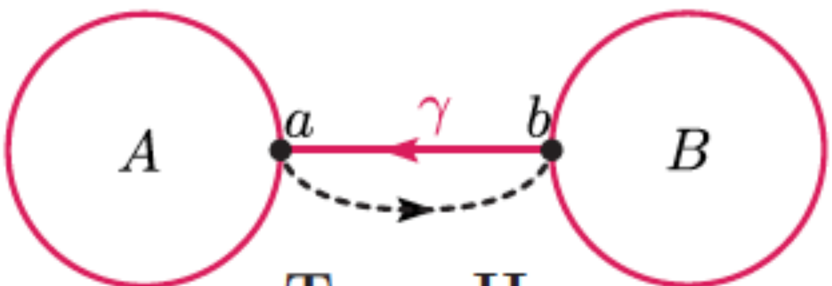
A systematic algorithm for integrands

- As we can easily verify:



so only need to worry about type I.b.

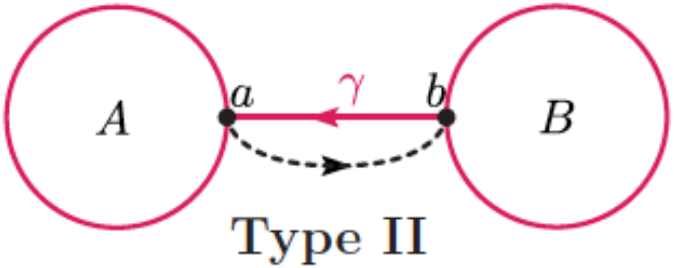
Terms type I.b will be of a type where two cycles are connected by a link, we will now consider reduction of such contributions.



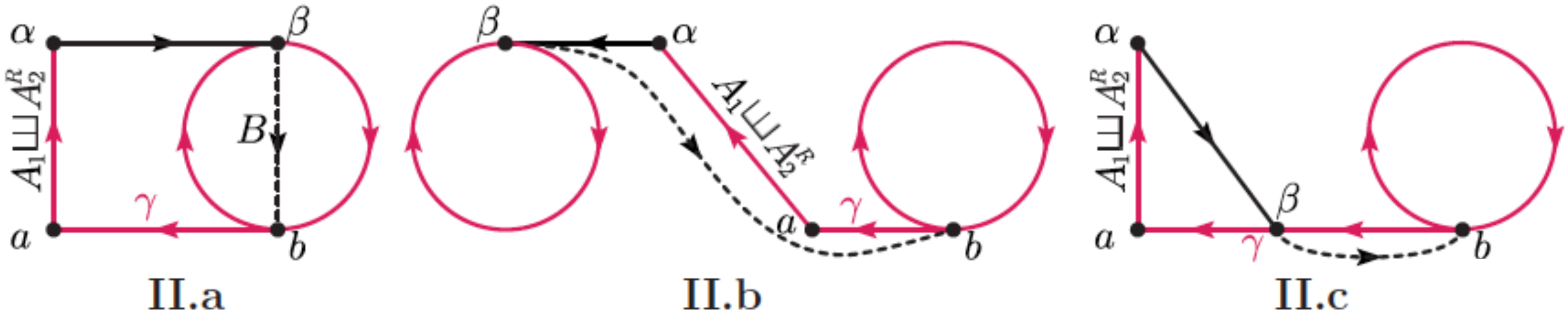
Type II

A systematic algorithm for integrands

- The starting point for reduction of diagrams of type II



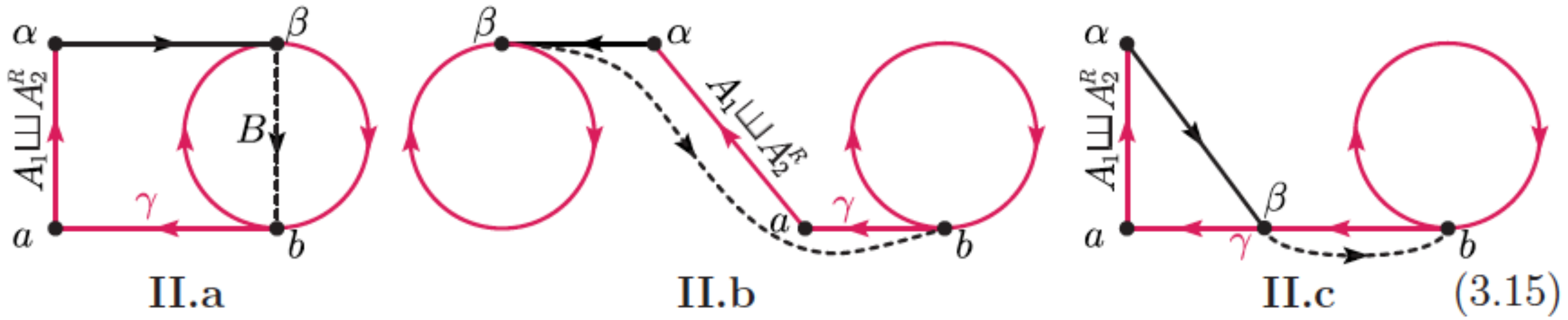
is a reduction of cycle A with respect to points a and b in the link:



Again we characterize according to if β lies in same cycle as b, in another cycle, or in the link γ.

A systematic algorithm for integrands

- Now II.a and II.b are of same type as I.a and I.b



Thus we will iterate to get fewer cycles. Only issue is II.c but it now contains a shorter link than the starting point type II, so it will always be possible reduce until no link exist.

A systematic application of these reductions thus guaranties (after a finite number of steps) that we end up an integrand consisting of only single Hamiltonian cycles.

Conclusion

- Integration methods gives a clear path forward.
- We can provide analytic and covariant expressions in many cases.
- Useful tool for rewriting.
- Another point: Use 'string theory' for inspiration to write down CHY integrands.
- Many new applications for various CHY integrands.

  (See e.g. Fu, Du, Huang and Feng)

Conclusions

Open questions:

- Needed: better fundamental 'mathematical' understanding of the scattering eq. formalism?
- Question: Can the map between string theory and the scattering eq. formalism and become more precise?

(Mathematical identity in a limit linking very different integrands…)

What is the precise mathematical connection???

$$\lim_{\alpha'\to 0}\int\prod dz_i\Big[\prod_{i=1}^{n-1}\Theta[z_{\sigma(i+1)\sigma(i)}]\leftrightarrow\frac{\prod_{n-3}\delta(S_i)}{z_{\sigma(1)\sigma(2)}\,z_{\sigma(2)\sigma(3)}\cdots z_{\sigma(n)\sigma(1)}}\Big]\times\prod_{i<z}|z_{ij}|^{\alpha' s_{ij}}\times H(z)$$

Conclusion

- Method: A clear way forward for many different theories. We can provide analytic and covariant expressions.
- Observation: solutions to the scattering equations not very important.
- Goal: extend analytic methods to many other types of theories
  - General relativity/Gravity: need to consider integrands multiplied with Pfaffian squared.
  - Loops: forward limits / Q-cuts
- Many new interesting aspects to consider in this regard!