The correlahedron

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based on: arXiv:1701.00453 with Eden, Mason
as well as a number of papers with:
Bourjaily, Chicherin, Doobary, Eden, Korchemsky, Mason, Sokatchev, Tran.

Idea

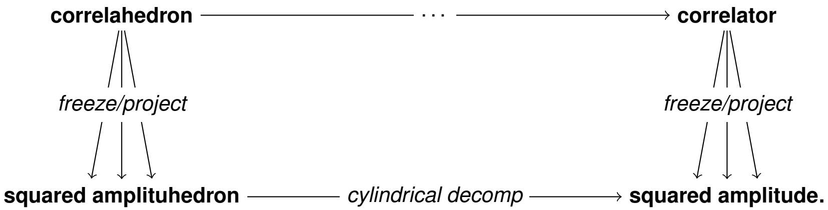
- Amplituhedron $\Leftrightarrow$ Amplitudes
- Correlators $\rightarrow$ Amplitudes (squared) (multiple lightlike limits)
- Strongly suggests the existence of a bigger geometric object:

The correlahedron

- Correlators $\Leftrightarrow$ correlahedron
- Correlahedron $\rightarrow$ (squared) amplituhedron

Plan of talk

- super-amplitudes, super-correlators and the amplitude/correlator duality via lightlike limit in superspace
- bosonised superspace and the lightlike limit as "freeze and project"
- geometry: amplituhedron, squared amplituhedron and correlahedron
- "freeze and project" correlahedron $\to$ amplituhedron
- algebra from geometry: cylindrical decomposition



Superamplitudes

Superamplitude integrands in planar $\mathcal{N} = 4$ SYM

- Divide by MHV tree
- Momentum supertwistor space [Hodges, Mason Skinner]

$$\mathbb{C}^{4|4} \ni \mathcal{Z}_i^{\mathcal{A}} = (Z_i^A | \chi_i^I)$$

(Z_i related to momentum of particle, χ_i particle type)

- Structure of *n*-point superamplitude

$$\mathcal{A}_n = \sum_{\ell,k} a^\ell \mathcal{A}_{n;k}^{(\ell)}$$

where $\mathcal{A}_{n;0}^{(0)} = 1$ and $\mathcal{A}_{n;k}^{(\ell)} = O(\chi^{4k})$ is an N^{k}MHV superamplitude.

1st non-trivial example: $\mathcal{A}_{5;1}^{(0)} = \dfrac{\delta^4(\chi_1\langle 2345\rangle + ... + \chi_5\langle 1234\rangle)}{\langle 1234\rangle ... \langle 5123\rangle}$ where $\langle ijkl\rangle = det(Z_i Z_j Z_k Z_l)$

Correlators in $\mathcal{N} = 4$

AdS/CFT

Supergravity/String theory on $AdS_5 \times S^5 \quad = \quad \mathcal{N}$=4 super Yang-Mills

- Correlation functions of gauge invariant operators in SYM $\leftrightarrow$ string scattering in $AdS \times S$
- Stress-tensor multiplet $\rightarrow$ gravity in AdS
- Contain data about anomalous dimensions of operators and 3 point functions via OPE $\rightarrow$integrability / bootstrap
- Big Bonus more recently: Correlators give scattering amplitudes

Super-correlators

Correlation function integrands of chiral stress-tensor multiplets in planar $\mathcal{N} = 4$ SYM

- Chiral superspace = space of 2-planes in supertwistor space

$$Gr(2,(4|4)) \ni X_{i\alpha}^{\mathcal{A}} = \left(1_2, x_{i\alpha\dot{\alpha}} \,\middle|\, \theta_{i\alpha}^{I}\right)$$

- Structure of n-point supercorrelators

$$\langle \mathcal{O}(X_1)\mathcal{O}(X_2)\ldots\mathcal{O}(X_n)\rangle = \sum_{\ell,k} a^{\ell} G_{n;k}^{(\ell)}$$

- $\mathcal{O}$ is the stress-tensor multiplet

$$\mathcal{O}(X_i) = \ldots + \theta_{IJ,KL}^{4} tr(\Phi_{IJ}(x_i)\Phi_{KL}(x_i)) + \ldots + \theta^{8} L(x_i)$$

- chiral superspace instead of usual analytic superspace
- insight from twistor Feynman diagram approach [Chicherin Doobary Korchemsky Mason Sokatchev PH]

So far looks very similar to the superamplitude, but:

- Complication: Much higher power of superspace variables,

$$G^{(\ell)}_{n;k} = O(\theta^{4(n+k)})$$

(in chiral superspace - also possible to use analytic superspace which requires the introduction of further bosonic variables)

- Simplification:

$$G^{(\ell)}_{n;k} = \int d^8\theta_{n+1} \ldots d^8\theta_{n+\ell} G^{(0)}_{n+\ell;k+\ell}$$

therefore only need consider tree-level correlators $G^{(0)}_{n;k}$ get loops for free!

Correlators → Amplitudes

[Eden Korchemsky Sokatchev, Alday Eden Korchemsky Maldacena Sokatchev, Eden Korchemsky Sokatchev PH, Adamo Bullimore Mason Skinner]

The amplitude/correlator duality states

$$\lim \frac{G_n}{G^{(0)}_{n;0}} = (\mathcal{A}_n)^2$$

- Here "lim" is the lightlike limit: the 2-planes X_i consecutatively intersect in twistor space.
- Eg Choose a basis of the plane such that:

$$X_{i1} \to Z_{i-1} \qquad X_{i2} \to Z_i$$

Each side is an expansion (in both superspace variables and coupling). Expanding:

$$\lim \frac{G^{(\ell)}_{n;k}}{G^{(0)}_{n;0}} = (\mathcal{A}^2)^{(\ell)}_{n;k} = \sum_{k_1+k_2=k,\ell_1+\ell_2=\ell} \mathcal{A}^{(\ell_1)}_{n;k_1} \mathcal{A}^{(\ell_2)}_{n;k_2} \quad = \quad 2\mathcal{A}^{(\ell)}_{n;k} + \dots$$

Putting this with the relation between loop and tree correlators: a single tree-level correlator contains many $(k+1)$ different loop-level amplitudes! (also in any ordering of points)

$$
G^{(0)}_{n;k} \to \begin{cases}
(A^2)^{(0)}_{n;k} & n\text{-point lightlike limit} \\
(A^2)^{(1)}_{n-1;k-1} & n-1\text{-point lightlike limit} \int d^8\theta_n \\
(A^2)^{(2)}_{n-2;k-2} & n-2\text{-point lightlike limit,} \int d^8\theta_{n-1} d^8\theta_n \\
\vdots & \\
(A^2)^{(k)}_{n-k;0} & n-k\text{-point lightlike limit,} \int d^8\theta_{n-k+1} \ldots d^8\theta_n
\end{cases}
$$

Bosonised superspace for the amplitude

[Hodges, Arkani-Hamed Trnka]

- Key feature: bosonised supertwistors $\mathbb{C}^{4|4} \to \mathbb{C}^{4+k}$
- Introduce $4k$ global fermionic variables ϕ_I^a

$$(Z_i^A, \chi_i^I) \in \mathbb{C}^{4|4} \quad \to \quad \mathcal{Z}_i^{\mathcal{A}} = (Z_i^A, \chi_i^I \phi_I^a) \in \mathbb{C}^{4+k}$$

Eg $\mathcal{A}_{5;1}^{(0)}$ becomes

$$\mathcal{A}_{5;1}^{(0)} = \frac{\delta^4(\chi_1\langle 2345\rangle + ... + \chi_5\langle 1234\rangle)}{\langle 1234\rangle ... \langle 5123\rangle} \to \frac{\langle 12345\rangle^4}{\langle 1234\, Y_0\rangle ... \langle 5123\, Y_0\rangle}$$

- angle brackets $\to$ 5×5 determinants
- $Y_0 = (0,0,0,0,1)$ projecting onto the original twistors
- Get back to superspace simply by integrating out all the ϕ's

Bosonised superspace for the correlator

- Clear generalisation to correlators. But now $4(n+k)$ ϕ's

$$
\begin{array}{ll}
(X^A_{i\alpha}, \theta^I_{i\alpha}) & \to \mathcal{X}^{\mathcal{A}}_{i\alpha} = (X^A_{i\alpha}, \theta^I_{i\alpha}\phi^a_I) \\
\in & \in \\
Gr(2,4|4) & Gr(2,4+n+k)
\end{array}
$$

- correlator becomes a function of n 2-planes in $4+n+k$ dimensions

Eg for $G^{(0)}_{5;1}$ the points become $\mathcal{X}^{\mathcal{A}}_{i\alpha} \in Gr(2,10)$

$$G^{(0)}_{5;1} = \frac{\langle X_1 X_2 X_3 X_4 X_5 \rangle^4}{\prod_{i<j} \langle X_i X_j Y_0 \rangle}$$

where the brackets are 10×10 determinants and $Y_0 = \begin{pmatrix} 0_{4\times 6} \\ 1_{6\times 6} \end{pmatrix}$

- Note: traditional analytic superspace approach numerator = huge polynomial structure hard to see immediately [Eden Schubert Sokatchev]
- Here it takes the conceptually very simple form $\langle X_1 X_2 X_3 X_4 X_5 \rangle^4$

Known correlators

summary of what is known

- We know $G^{(0)}_{n;n-4}$ explicitly for all $n \leq 14$ (equivalently the 4-point correlator to 10 loops)
  [Eden Schubert Sokatchev, Eden Korchemsky Sokatchev PH, Bourjaily Tran PH]
- The bosonised superspace is $4+n+k = 2n$ dimensional

$$G^{(0)}_{n;n-4} = \langle X_1 X_2 \dots X_n \rangle^4 \times f^{(n-4)}(\langle X_i X_j Y_0 \rangle)$$

- Crucial hidden permutation symmetry is manifest: permutation symmetry of $\langle X_1 X_2 \dots X_n \rangle^4$!
- $f^{(n-4)}(x_{ij}^2) \to f^{(n-4)}(\langle X_i X_j Y_0 \rangle)$: f-graphs, graphical operations
- the analogue of $\overline{\text{MHV}}$ amplitudes BUT contains a lot of non-trivial info eg 4- and 5-point amplitudes to 10, 9 loops !
- Only one other correlator is known explicitly, $G^{(0)}_{6;1}$
  [Chicherin Doobary Korchemsky Sokatchev PH]

superinvariants in bosonised superspace

- Bosonised superspace provides very useful new way to consider superspace (nilpotent) invariants (even if it wasn't accompanied with the geometrical aspect)
- Clarifies non-trivial identities and symmetries
- Consider $G^{(0)}_{6;1}$ found originally in analytic superspace
- The relevant superspace structures were found (eventually) to have the form $\mathcal{I}^{ijkl;\alpha\beta\gamma\delta}$
- satisfy an identity as a very non-trivial consequence of superconformal invariance:

$$\sum_{i=1}^{6} X_{i\alpha}\mathcal{I}^{ijkl;\alpha\beta\gamma\delta} = 0 \qquad (\text{for all } j, k, l, M, \beta, \gamma, \delta)\,,$$

$G^{(0)}_{6;1}$ in bosonised superspace

- bosonised superspace is $n+k+4$ = 11-dimensional, but we have 6 Xs therefore define

$$\langle \dots \rangle^{i\alpha} := \langle X_{11} X_{12} X_{21} \dots \widehat{X_{i\alpha}} \dots X_{62} \rangle (-1)^{\alpha} \ .$$

- Superspace structure means we always have four such brackets, so the most general structure is

$$\mathcal{I}^{ijkl;\alpha\beta\gamma\delta} = \langle \dots \rangle^{i\alpha} \langle \dots \rangle^{j\beta} \langle \dots \rangle^{k\gamma} \langle \dots \rangle^{l\delta}$$

- Further the non-trivial identity is a simple consequence of generalised Schouten identity in 11 dimensions

$$\sum_{i=1}^{6} X_{i\alpha} \langle \dots \rangle^{i\alpha} = 0 \ .$$

(can't antisymmetrise 12 objects in 11 dimensions)

The correlator itself $G^{(0)}_{6;1}$ (originally given in analytic superspace) can be directly transcribed to this bosonised superspace as

$$G^{(0)}_{6;1} = \frac{A_2 - 2\,A_1 - 8\,B_2}{\prod_{1\le i<j\le 6}\langle Y_0 X_i X_j\rangle}\,,$$

where

$$A_1 = \langle Y_0 X_{5\alpha} X_1 X_{6\gamma}\rangle\langle Y_0 X_{5\beta} X_2 X_{6\delta}\rangle\langle Y_0 X_3 X_5\rangle\langle Y_0 X_4 X_6\rangle \mathcal{I}^{5566;\alpha\beta\gamma\delta} + S_6 \text{ perm}$$

$$A_2 = \langle Y_0 X_{5\alpha} X_1 X_{6\gamma}\rangle\langle Y_0 X_{5\beta} X_2 X_{6\delta}\rangle\langle Y_0 X_3 X_4\rangle\langle Y_0 X_5 X_6\rangle \mathcal{I}^{5566;\alpha\beta\gamma\delta} + S_6 \text{ perm}$$

$$B_2 = \langle Y_0 X_{4\alpha} X_3 X_{6\gamma}\rangle\langle Y_0 X_{5\beta} X_2 X_{6\delta}\rangle\langle Y_0 X_1 X_6\rangle\langle Y_0 X_4 X_5\rangle \mathcal{I}^{4566;\alpha\beta\gamma\delta} + S_6 \text{ perm}$$

Note this is clearly much more complicated than the analagous 6 pnt NMHV amplitude $\Leftarrow$ no Yangian
(also no spurious poles though)

$Y_0 \to Y$

- NB Y_0 becomes a crucial player in the -hedron story.
- Here we saw it as a fixed object which projects the extended brackets to 4-brackets
- Note that $Y_0 \in Gr(n+k, n+k+4)$ and given the manifest $GL(n+k+4)$ symmetry of the problem it is useful to let Y_0 vary (and call it Y)
- so the amplituhedron and correlahedron naturally extend to functions of Y as well as the external data, Z_i or X_i
- also natural to multiply by a volume differential form factor on the Grassmanian $\prod_{i=1}^{n+k} \langle Y d^4 Y_i \rangle$
- NB this procedure gives a volume form on the Grassmanian
- integrate form over a delta function $\delta(Y; Y_0)$ to get back original

Lightlike limit in bosonised superspace

Question: what does the lightlike limit look like in bosonised superspace?

Answer: Geometric "freeze and project" procedure

Importantly: Act directly on $G_{n;k}$ (without needing to divide by the tree)

Light like limit

- Freeze: $\langle X_i X_{i+1} Y \rangle \to 0$ means Y intersects the 4-plane formed by the two 2-planes X_i, X_{i+1}. So Y is simultaneously frozen to intersect n 4-planes.
- Project: At the same time we project from all n of these intersection points (onto any co-dimension n plane that doesn't go through them)

- the second "project" step is necessary to reduce the dimension of the space down from $\mathbb{C}^{n+k+4} \to \mathbb{C}^{k+4}$ in which the amplitude lives
- corresponds to dividing by the additional 4n fermionic degrees of freedom in $G^{(0)}_{n;0}$

Freeze and Project, explicit procedure

- perform the freezing of Y as $Y = Y_1 \wedge .. \wedge Y_{n+k}$ with

$$Y_p = \sigma_i^\alpha X_{i\alpha} - \tau_i^\alpha X_{i+1\,\alpha} \qquad \text{for } p = i = 1 \dots n\,,$$
$$Y_p = \hat{Y}_{p'} \qquad p = n + p', \quad p' = 1 \dots k$$

  for some parameters σ_i^α, τ_i^α.
- project from $Y_1, \dots Y_n$. In practice we can pick a basis for $\mathbb{R}^{k+n+4}$

$$\text{basis} = \left\{ Y_1, \dots, Y_n, e_1, \dots, e_{4+k} \right\}\,,$$

  where $e_1, \dots e_{4+k}$ are any $4 + k$ vectors such that this yields an independent basis.
- Choose $\hat{Y}_{p'}$ to be a linear combination of the $e_{\mathcal{A}'}$
- Projection is then

$$X_{i\alpha} \to \hat{X}_{i\alpha} \quad \text{where} \quad \hat{X}^{\mathcal{A}}_{i\alpha} = \begin{cases} 0 & \mathcal{A} = 1, \dots, n \\ X^{\mathcal{A}}_{i\alpha} & \mathcal{A} = n+1, \dots, n+k+4 \end{cases}\,,$$

  in this basis

- define reduced brackets in the obvious way on the hyperplane spanned by $\{e_1, \ldots, e_{4+k}\}$ and it is clear that

$$\langle \hat{\mathcal{X}} \rangle := \langle Y_1 \ldots Y_n \mathcal{X} \rangle \ .$$

  Here $\mathcal{X}$ represents any collection of $4 + k$ independent vectors, and $\hat{\mathcal{X}}$ the same vectors projected onto the hyperplane.
- Defining $Z_i := \sigma_i.X_i = \tau_i.X_{i+1} + Y_i$ then after the projection $\hat{Z}_i := \sigma_i.\hat{X}_i = \tau_i.\hat{X}_{i+1}$ and the projected planes $\hat{X}_i$ intersect each other consecutively at $\hat{Z}_i$ in the projected space.
- Thus freezing and projection yields a k-plane $\hat{Y}$ living in the $4+k$ dimensional hyperplane spanned by $\{e_1, \ldots e_{4+k}\}$ and we have projected planes $\hat{X}_{i\alpha}$ in the same $4+k$ dimensional space.

EG. $G^{(0)}_{5;1} \to A^{(0)}_{5;1}$

Here we have $Y = Y_1 \wedge \cdots \wedge Y_6 \in Gr(6,10)$ and we freeze $Y_1, \ldots, Y_5$ as $Y_i = \sigma_i^\alpha X_{i\alpha} - \tau_i^\alpha X_{i+1\,\alpha}$, leaving $Y_6 = \hat{Y}$ orthogonal. Then

$$\prod_{i=1}^{6} \langle Y d^4 Y_i \rangle \frac{\langle X_1 X_2 X_3 X_4 X_5 \rangle^4}{\langle Y X_1 X_2 \rangle \ldots \langle Y X_4 X_5 \rangle} \xrightarrow[\text{project} X]{\text{freeze } Y} \left(\prod_{i=1}^{5} \frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2} \right) \frac{\langle Y d^4 \hat{Y} \rangle \langle Y_1 .. Y_5 \hat{Z}_1 .. \hat{Z}_5 \rangle^4}{\langle Y \hat{Z}_1 \hat{Z}_2 \hat{Z}_3 \hat{Z}_4 \rangle \ldots \langle Y \hat{Z}_5 \hat{Z}_1 \hat{Z}_2 \hat{Z}_3 \rangle}$$

$$= \left(\prod_{i=1}^{5} \frac{d^2\sigma_i d^2\tau_i}{(\tau_{i-1}.\sigma_i)^2} \right) \frac{\langle \hat{Y} d^4 \hat{Y} \rangle \langle \hat{Z}_1 .. \hat{Z}_5 \rangle^4}{\langle \hat{Y} \hat{Z}_1 \hat{Z}_2 \hat{Z}_3 \hat{Z}_4 \rangle \ldots \langle \hat{Y} \hat{Z}_5 \hat{Z}_1 \hat{Z}_2 \hat{Z}_3 \rangle}$$

$$\downarrow$$

$$\frac{\langle \hat{Y} d^4 \hat{Y} \rangle \langle \hat{Z}_1 .. \hat{Z}_5 \rangle^4}{\langle \hat{Y} \hat{Z}_1 \hat{Z}_2 \hat{Z}_3 \hat{Z}_4 \rangle \ldots \langle \hat{Y} \hat{Z}_5 \hat{Z}_1 \hat{Z}_2 \hat{Z}_3 \rangle}$$

Using

$$\langle X_1 X_2 X_3 X_4 X_5 \rangle = \langle Y_1 .. Y_5 \hat{Z}_1 .. \hat{Z}_5 \rangle \prod_{i=1}^{5} (\tau_i . \sigma_{i+1})^{-1} \, .$$

as well as

$$\langle Y X_i X_j \rangle = \langle Y \hat{Z}_{i-1} \hat{Z}_i \hat{Z}_{j-1} \hat{Z}_j \rangle \times (\tau_{i-1}.\sigma_i \, \tau_{j-1}.\sigma_j)^{-1} \, .$$

Non-maximal lightlike limit

- Similar procedure at loop-level
- In superspace: non-maximal lightlike limit of $G^{(0)}_{n;k}$ + integrate out the fermionic variables not associated with the limit.
- In bosonised superspace: additional projection from the planes corresponding to these variables
- The freeze and project procedure is simple to implement in practice algebraically using mathematica (much easier than in superspace where we have to essentially pick separate components)
- Non-trivial checks: We show that the $G^{(0)}_{6;1}$ correlator indeed reduces to $(\mathcal{A}^2)^{(0)}_{6;1}$ (6 point NMHV) as well as $(\mathcal{A}^2)^{(1)}_{5;0}$ (5 point MHV 1 loop parity even and odd) via this freeze and project procedure

Geometry: -hedrons

[Arkani-Hamed Trnka, Arkani-Hamed Thomas Trnka]

Amplituhedron: beautiful geometric picture giving amplitudes from pure geometry

(Tree) Amplituhedron

$$\text{amplituhedron}_{n;k}(Z) = \left\{ Y \in Gr(k, 4+k) : \ Y_p^{\mathcal{A}} = C_p^i Z_i^{\mathcal{A}} \text{ for } C \in Gr^+(k, n) \right\} .$$

Definition somewhat implicit $\Rightarrow$ difficult to obtain explicit results from.

(although see [Arkani-Hamed Thomas Trnka])

Squared (tree) amplituhedron

$$\text{squared amplituhedron}_{n;k}(Z) = \left\{ Y \in Gr(k, 4+k) : \ \langle Y Z_{i-1} Z_i Z_{j-1} Z_j \rangle > 0 \right\} .$$

Much more explicit, easier to compute from.

Loop level squared amplituhedron

$$\text{squared amplituhedron}^{(\ell)}_{n;k}(Z)$$
$$=\Big\{(Y,\mathcal{L}_1,..,\mathcal{L}_\ell):\ \langle YZ_{i-1}Z_iZ_{j-1}Z_j\rangle>0,\ \langle YZ_{i-1}Z_i\mathcal{L}_j\rangle>0,\ \langle Y\mathcal{L}_i\mathcal{L}_j\rangle>0\Big\}\,.$$

$$Y\in Gr(k,4+k),\qquad \mathcal{L}_i\in Gr(2,4+k)$$

Corrrelahedron

Correlahedron proposal

$$\left\{ Y \in Gr(n+k, n+k+4) : \langle Y X_i X_j \rangle > 0 \right\} .$$

Lives in a (very) large dimension but is conceptually very simple

- Further, performing exactly the same “freeze and project” procedure as detailed above reduces the geometry to (in the maximal lightlike limit)

$$\langle YX_iX_j\rangle = \begin{cases} 0 & |i-j| = 1 \mod n \\ \dfrac{\langle \hat{Y}\hat{Z}_{i-1}\hat{Z}_i\hat{Z}_{j-1}\hat{Z}_j\rangle}{\tau_{i-1}.\sigma_i\,\tau_{j-1}.\sigma_j} & \text{otherwise}\,. \end{cases}$$

So the correlahedron space reduces to the squared amplituhedron

- (up to signs from the denominator. This reflects the ambiguity in $X_i \to Z_{i-1} \wedge Z_i$ or $X_i \to Z_i \wedge Z_{i-1}$). This choice of signs either doesn’t seem to matter, or only 1 sign choice matters.

So the same geometric procedure (freeze and project):

correlator $\to$ (squared) amplitude

correlahedron $\to$ (squared) amplituhedron

Algebra (volume forms) from geometry

- Amplitude is the unique volume form with no divergences inside the amplituhedron and log divergences on its boundary [Arkani-Hamed Trnka]
- Given an explicit description of the geometry (as for the squared amplituhedron) there is a straightforward algorithm to obtain this differential form via "cylindrical decomposition" (see [Arkani-Hamed Lam] for related approaches)
- Start with the same procedure as for converting multiple integrals over regions to iterated single integrals, ie convert any region in $\mathbb{R}^n$ to a union of non-intersecting regions of the form

$$\left\{ (x_1, \ldots, x_n) : \begin{array}{rl} a < & x_1 < b, \\ a(x_1) < & x_2 < b(x_2), \\ a(x_1, x_2) < & x_3 < b(x_1, x_2), \\ \ldots, & \\ a(x_1, .., x_{n-1}) < & x_n < b(x_1, .., x_{n-1}) \end{array} \right\},$$

- Instead of integrating over this region one assigns a differential form to it by assigning to each inequality a dlog:

$$a(x_1,..,x_{i-1}) < x_i < b(x_1,..,x_{i-1}) \quad \rightarrow \quad d\log\left(\frac{x_i - b(x_1,..,x_{i-1})}{x_i - a(x_1,..,x_{i-1})}\right)$$

thus yielding the n-form

$$\prod_{i=1}^{n} \frac{dx_i\Big(b(x_1,..x_{i-1}) - a(x_1,..x_{i-1})\Big)}{\Big(x_i - b(x_1,..x_{i-1})\Big)\Big(x_i - a(x_1,..x_{i-1})\Big)} .$$

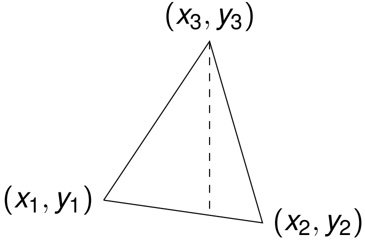
- One then simply adds together the contributions from each region.
- This gives a form with log divergences on each boundary and no divergences inside (as long as the original region is convex).
- Remarkably it is independent of the order in which you perform the cylindrical decomposition (for linear inequalities)

Simple example

- Consider a triangle in P^2 with vertices Z_1, Z_2, Z_3
- give them inhomogeneous coordinates $Z_i = (x_i, y_i, 1)$
- region (inside of the triangle) is the space of $Y \in P^2$ such that

$$\langle Y Z_1 Z_2 \rangle > 0, \quad \langle Y Z_2 Z_3 \rangle > 0, \quad \langle Y Z_3 Z_1 \rangle > 0 \ .$$

- also give Y inhomogeneous coordinates $Y = (x, y, 1)$



$$\frac{xy_1 - x_2y_1 - xy_2 + x_1y_2}{x_1 - x_2} < y < \frac{xy_1 - x_3y_1 - xy_3 + x_1y_3}{x_1 - x_3} \text{ and } x_1 < x < x_3$$
$$\frac{xy_1 - x_2y_1 - xy_2 + x_1y_2}{x_1 - x_2} < y < \frac{xy_2 - x_3y_2 - xy_3 + x_2y_3}{x_2 - x_3} \text{ and } x_3 < x < x_2 \ .$$

So the differential form corresponding to the above region becomes

$$d\log\left(\frac{y-\frac{xy_1-x_3y_1-xy_3+x_1y_3}{x_1-x_3}}{y-\frac{xy_1-x_2y_1-xy_2+x_1y_2}{x_1-x_2}}\right)\wedge d\log\left(\frac{x-x_3}{x-x_1}\right)+d\log\left(\frac{y-\frac{xy_2-x_3y_2-xy_3+x_2y_3}{x_2-x_3}}{y-\frac{xy_1-x_2y_1-xy_2+x_1y_2}{x_1-x_2}}\right)\wedge d\log\left(\frac{x-x_2}{x-x_3}\right)$$

$$=\frac{dxdy\,(x_2y_1-x_3y_1-x_1y_2+x_3y_2+x_1y_3-x_2y_3)^2}{(x_1y-x_1y_2-x_2y-xy_1+x_2y_1+xy_2)(x_1y-x_1y_3-x_3y-xy_1+x_3y_1+xy_3)(x_2y-x_2y_3-x_3y-xy_2+x_3y_2+xy_3)}$$

$$=\frac{\langle Yd^2Y\rangle\langle Z_1Z_2Z_3\rangle^2}{\langle YZ_1Z_2\rangle\langle YZ_2Z_3\rangle\langle YZ_3Z_1\rangle}$$

- The procedure is very straightforward to implement in mathematica (which has a very powerful Cylindrical Decomposition algorithm)
- Unfortunately it scales badly with the number of variables so is only useful in fairly small examples
- active area of computational research to improve speed
- Using this procedure verified the squared amplituhedron gives the square of the amplitude in a number of cases.

eg. $(\mathcal{A}^2)^{(0)}_{7;3}$

- this should give the combination $2N^3MHV_7 + 2NMHV_7N^2MHV_7$
- squared amplituhedron = subset of $Y = Y_1 \wedge Y_2 \wedge Y_3 \subset Gr(3,7)$ such that $\langle Yi\, i{+}1\, j\, j{+}1\rangle > 0$
- We coordinatise $Gr(3,7)$ as

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & a & b & 0 & c & d & 0 \\ 0 & e & f & 1 & g & h & 0 \\ 0 & i & j & 0 & k & l & 1 \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_7 \end{pmatrix}$$

- Set the Z_i as basis elements, then the inequalities are written in the variables $a, \ldots, l$.
- performing a cylindrical decomposition, converting the result into a differential form and covariantising yields:

$$\langle Yd^4Y_1\rangle\langle Yd^4Y_2\rangle\langle Yd^4Y_3\rangle\langle 1234567\rangle^4 \times \left(\frac{\langle Y7123\rangle}{\langle Y1234\rangle\langle Y1267\rangle\langle Y2345\rangle\langle Y2356\rangle\langle Y2367\rangle\langle Y7134\rangle\langle Y7145\rangle\langle Y7156\rangle} + \ldots\right) .$$

- precisely the lightlike limit of the 7 point correlator, or equivalently the square of the amplitude $2N^3MHV_7 + 2NMHV_7N^2MHV_7$

5-point 1 loop NMHV

Here we have external twistors $Z_i \in P^4$, the loop 2-plane $\mathcal{L} = \mathcal{L}_1 \wedge \mathcal{L}_2 \in Gr(2,5)$ as well as $Y \in P^4$. Y and $\mathcal{L}$ satisfy the following inequalities

$$\langle \mathcal{L} Y 12 \rangle > 0,\ \langle \mathcal{L} Y 23 \rangle > 0,\ \langle \mathcal{L} Y 34 \rangle > 0,\ \langle \mathcal{L} Y 45 \rangle > 0,\ \langle \mathcal{L} Y 51 \rangle > 0$$
$$\langle Y 1234 \rangle > 0,\ \langle Y 2345 \rangle > 0,\ \langle Y 3451 \rangle > 0,\ \langle Y 4512 \rangle > 0,\ \langle Y 5123 \rangle > 0$$

Putting coordinates for $\mathcal{L}$ and Y as

$$\begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & c & d & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_5 \end{pmatrix}, \qquad Y = \begin{pmatrix} e & f & 1 & g & h \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_5 \end{pmatrix},$$

inequalities lead (via cyl. decomp.)

$$-\frac{2adef - 2aeg - 2bcef + be - cfg + df + 2g}{defgh(ad - bc)(ae + cf - 1)(adf - ag + b(-c)f + b)} da \wedge db \wedge .. \wedge dh$$

This lifts to the co-ordinate independent form

$$\frac{\langle\mathcal{L}Yd^2\mathcal{L}_1\rangle\langle\mathcal{L}Yd^2\mathcal{L}_2\rangle\langle Yd^4Y\rangle\langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$
$$\times\Bigg(\frac{\langle 1234Y\rangle\langle 2345Y\rangle}{\langle\mathcal{L}Y12\rangle\langle\mathcal{L}Y23\rangle\langle\mathcal{L}Y34\rangle\langle\mathcal{L}Y45\rangle}+\frac{\langle 5134Y\rangle\langle 2345Y\rangle}{\langle\mathcal{L}Y23\rangle\langle\mathcal{L}Y34\rangle\langle\mathcal{L}Y45\rangle\langle\mathcal{L}Y51\rangle}$$
$$+\frac{\langle 1234Y\rangle\langle 5123Y\rangle}{\langle\mathcal{L}Y12\rangle\langle\mathcal{L}Y23\rangle\langle\mathcal{L}Y34\rangle\langle\mathcal{L}Y51\rangle}+\frac{\langle 1245Y\rangle\langle 5123Y\rangle}{\langle\mathcal{L}Y12\rangle\langle\mathcal{L}Y23\rangle\langle\mathcal{L}Y45\rangle\langle\mathcal{L}Y51\rangle}$$
$$+\frac{\langle 1245Y\rangle\langle 5134Y\rangle}{\langle\mathcal{L}Y12\rangle\langle\mathcal{L}Y34\rangle\langle\mathcal{L}Y45\rangle\langle\mathcal{L}Y51\rangle}\Bigg)\ .$$

- Recognise the sum of five box functions (parity even part of the one loop amplitude) multiplied by the tree-level NMHV amplitude.
- precisely what we expect: the square of the superamplitude at first non-trivial order in both coupling and the Grassmann odd variable expansion is

$$\left(\frac{A^{(0)}_{\text{MHV}}+A^{(0)}_{\text{NMHV}}+aA^{(1)}_{\text{MHV}}+aA^{(1)}_{\text{NMHV}}+\dots}{A^{(0)}_{\text{MHV}}}\right)^2\Big|_{a^1,\chi^4}=\frac{2A^{(0)}_{\text{MHV}}A^{(1)}_{\text{NMHV}}+A^{(0)}_{\text{NMHV}}A^{(1)}_{\text{MHV}}}{\left(A^{(0)}_{\text{MHV}}\right)^2}$$
$$=2\frac{A^{(0)}_{\text{NMHV}}}{A^{(0)}_{\text{MHV}}}\left(\overline{M}^{(1)}_{\text{MHV}}+M^{(1)}_{\text{MHV}}\right)\ ,$$

Examples checked:

- Tree level:
  - $(\mathcal{A}^2)^{(0)}_{5;1}$ (5 point NMHV)
  - $(\mathcal{A}^2)^{(0)}_{6;2}$ (6 point N^2MHV) Here we needed to sum two orientations
  - $(\mathcal{A}^2)^{(0)}_{7;3}$ (7 point N^3MHV)
- Loop level:
  - $(\mathcal{A}^2)^{(1)}_{4;0}$ (4 point 1-loop)
  - $(\mathcal{A}^2)^{(2)}_{4;0}$ (4 point 2-loop)
  - $(\mathcal{A}^2)^{(1)}_{5;1}$ (5 point 1-loop NMHV)

Direct Correlahedron Check

- Unfortunately the smallest example of the correlator $G_{5;1}^{(0)}$ is already far too big for cylindrical decomposition to be helpful!
- $Y \in Gr(6, 10)$ is 24 dimensional!
- Worse: evidence that a naive implementation of the above procedure can not work. We know the singularity structure of the correlator, contains eg $1/(\tau_{i-1}.\sigma_i)^2$ and more generally Parke-Taylor-like singularities $1/(\tau.\sigma\,\sigma.\nu\,\nu.\tau)$ Singularity structure “wraps around”.
- Use additional local $GL(2)$ symmetries of each X_i.
- Using this we can put coordinates on Y as follows

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & a & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & b \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & c & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & e & f \end{pmatrix}$$

Correlahedron inequalities $\langle Y X_i X_j \rangle > 0$ become

- rewriting as a cylindrical decomposition and converting to a differential form gives

$$\frac{(a-b)^2 da\, db\, dc\, de\, df}{(a-1)a(b-1)b(e-cf)(c(-f)+c+e+f-1)(ab-af-bc+cf-e)}$$

- The known answer in these coordinates becomes

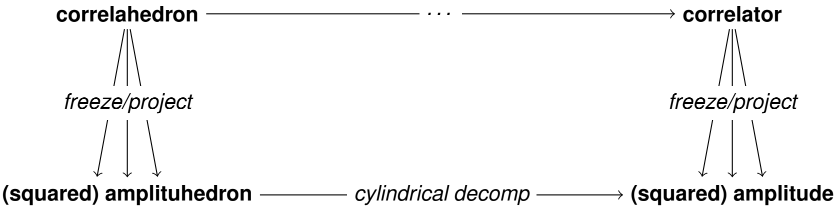
$$\frac{d\mu(a,b,c,e,f)}{(a-1)a(b-1)b(e-cf)(c(-f)+c+e+f-1)(ab-af-bc+cf-e)}$$

 where $d\mu(a,b,c,e,f)$ is the measure, $\langle Yd^4 Y_1\rangle \ldots \langle Yd^4 Y_6\rangle$ reduced to these variables.

- Complete agreement on identifying $d\mu(a,b,c,e,f) = (a-b)^2 da\, db\, dc\, de\, df$. Note that the term $(a-b)^2$ is indeed the natural measure factor, the Vandermonde determinant squared, one obtains when writing an integral measure on $GL(2)$ invariant under conjugation in terms of its eigenvalues.

Conclusions and further directions

- Proposed a conceptually simple geometric object the "correlahedron" equivalent to stress-tensor multiplet correlators



- Further examples both of the squared amplituhedron and especially the correlahedron
- Clarify subtleties, especially the "..." above
- Take bosonised superspace more seriously: understand how to extract components directly rather than going via superspace
- Generalisations: Higher charge correlators [Chicherin Drummond Sokatchev PH]
- Obtaining amplitudes from the $k = n - 4$ squared amplitude (limit of 4-pnt correlator)