Monodromy Relations in Higher-Loop String Amplitudes

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Scattering Amplitudes: from Gauge Theory to Gravity
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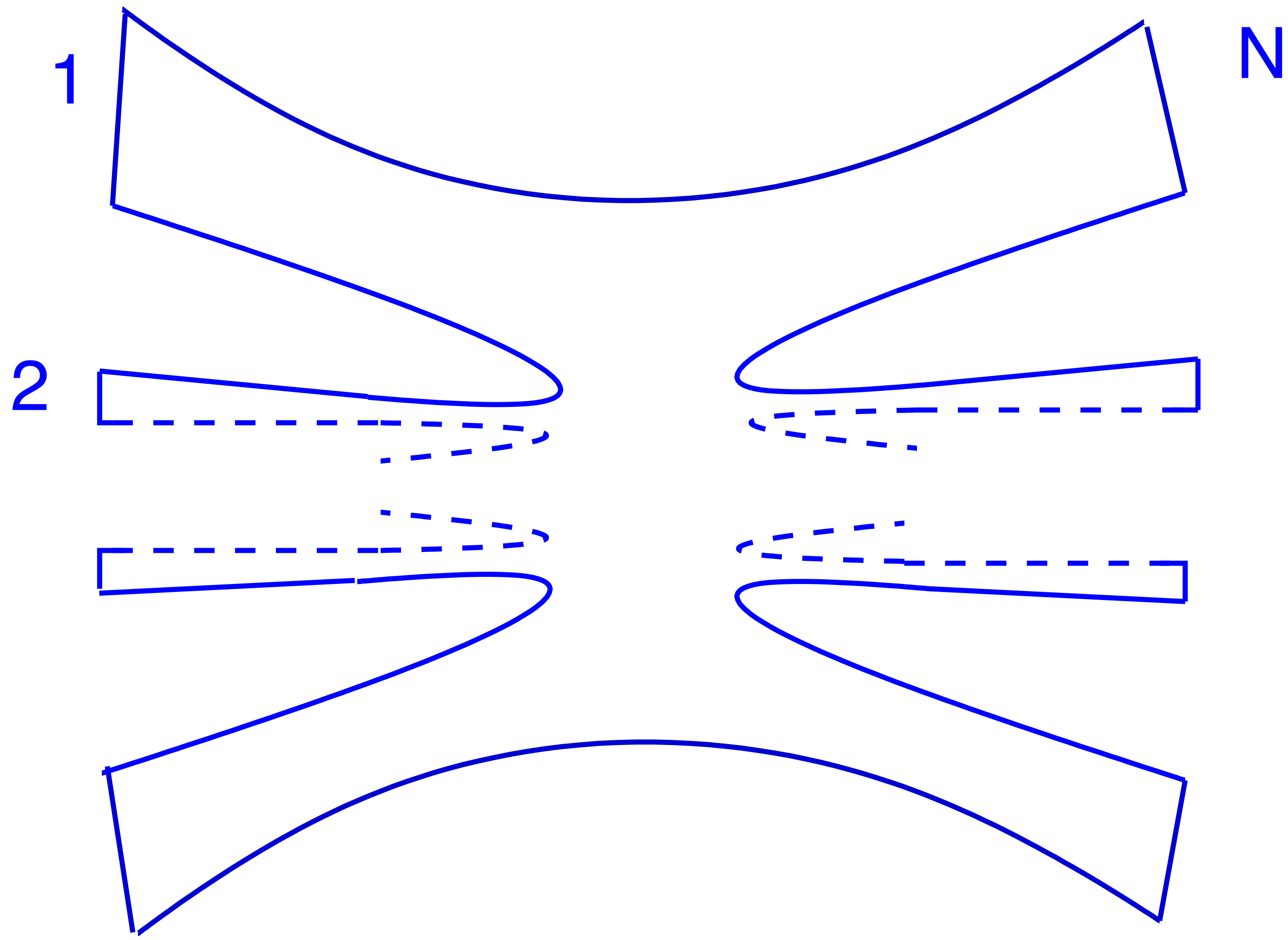
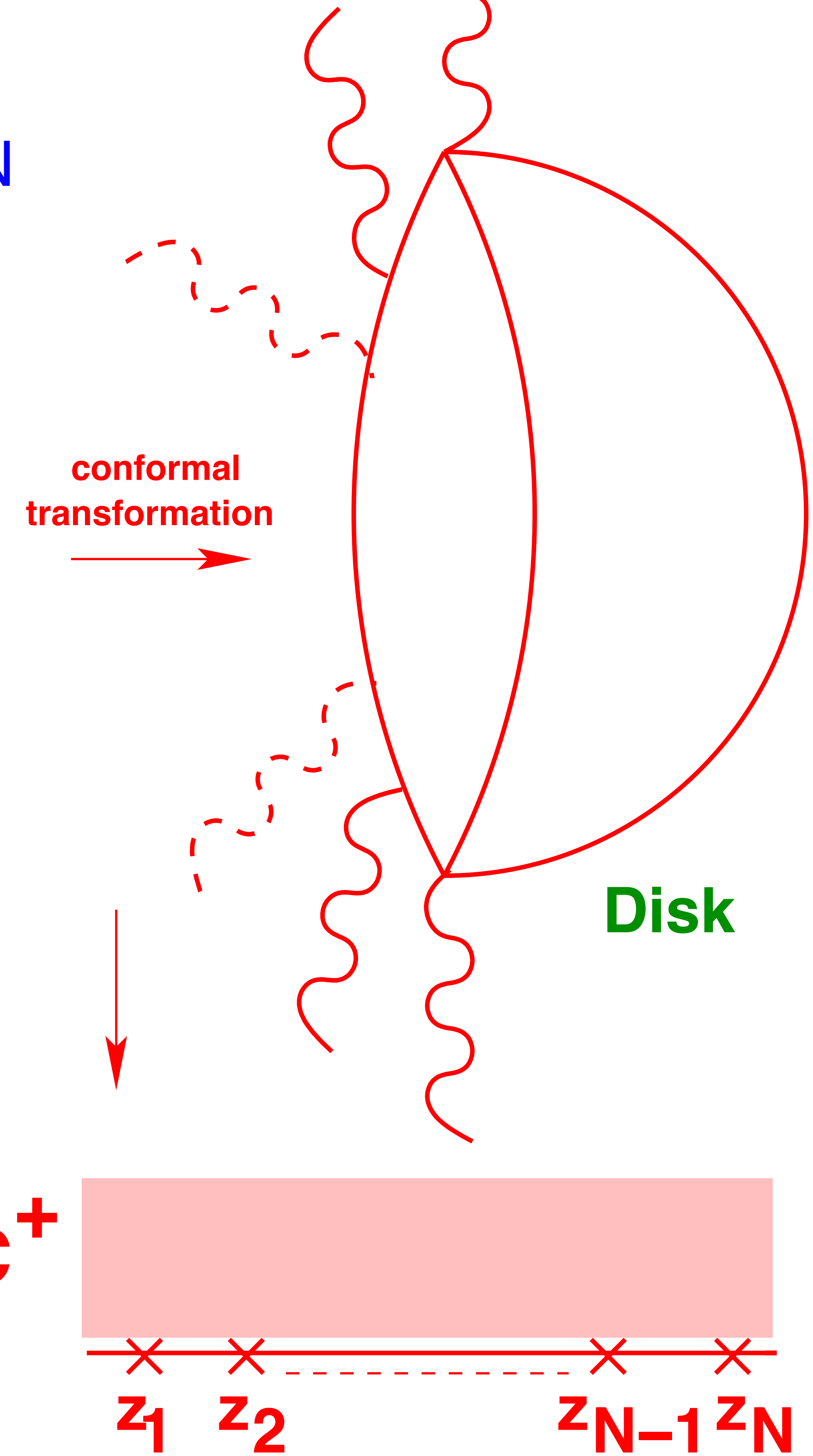
based on:

S. Hohenegger, St.St.:

- **Monodromy Relations in Higher-Loop String Amplitudes,** [arXiv:1702.04963]
- **work in progress**

Tree-level disk world-sheet:





$$s_{ij} = \alpha'(k_i + k_j)^2$$

$$A(1,\ldots,N) = V_{\text{CKG}}^{-1} \int\limits_{z_1<\ldots<z_N} \left(\prod_{j=1}^{N} dz_j \right) \sum_{\mathcal{K}_I} \mathcal{K}_I \prod_{i<j}^{N} |z_i - z_j|^{s_{ij}} \, (z_i - z_j)^{n_{ij}^I}$$

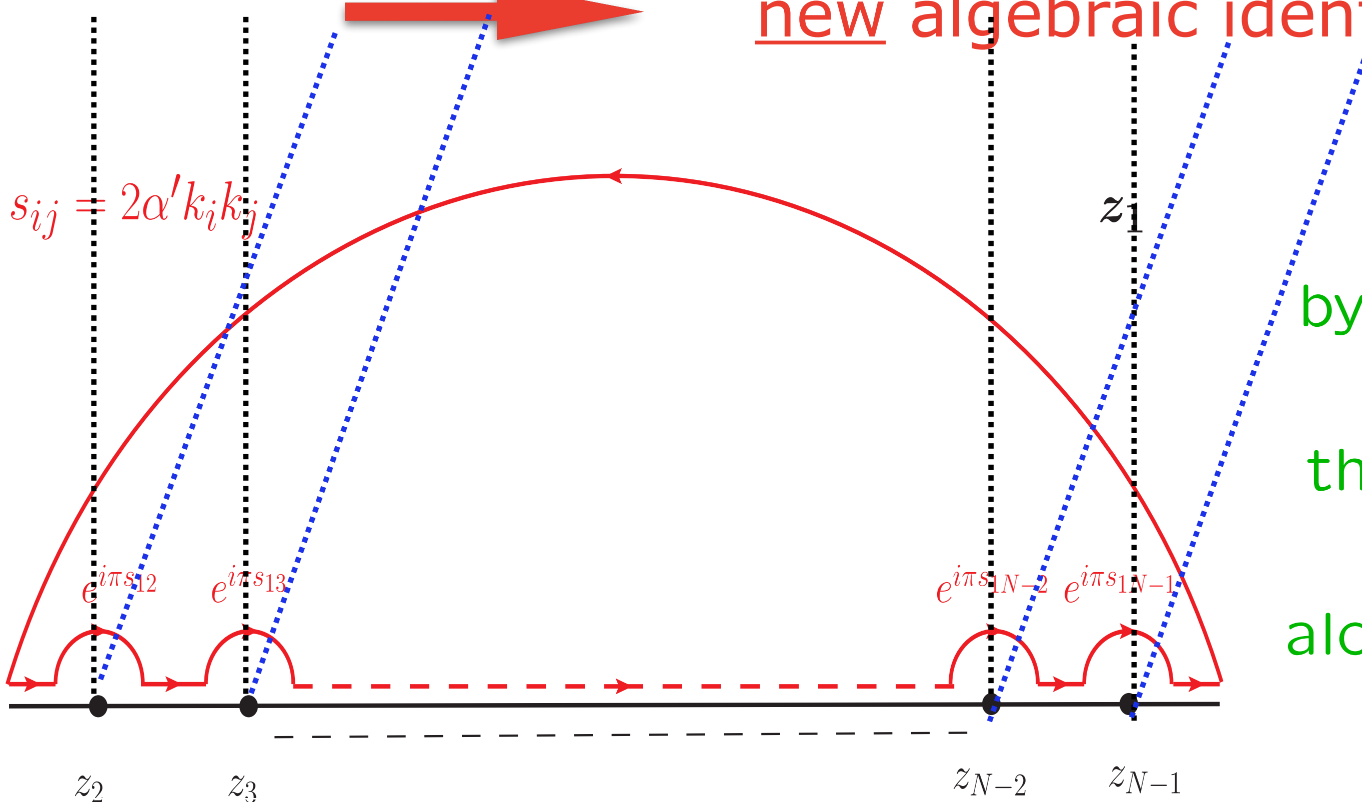
I. Tree-level amplitude relations

$$A(1,\dots,N) = V_{\text{CKG}}^{-1} \int_{z_1<\dots<z_N} \left(\prod_{j=1}^{N} dz_j \right) \sum_{\mathcal{K}_I} \mathcal{K}_I \prod_{i<j}^{N} |z_i - z_j|^{s_{ij}} \, (z_i - z_j)^{n_{ij}^I}$$

By applying world-sheet string techniques

→ <u>new</u> algebraic identities

integrate along single-valued function



by analytically continuing
the z_1-integration to
the whole complex plane
and integrating z_1
along the contour integral

$$A(1,2,\dots,N) + e^{i\pi s_{12}} A(2,1,3,\dots,N-1,N) + e^{i\pi(s_{12}+s_{13})} A(2,3,1,\dots,N-1,N)$$
$$+ \dots + e^{i\pi(s_{12}+s_{13}+\dots+s_{1N-1})} A(2,3,\dots,N-1,1,N) = 0$$

→ *monodromy relations*

- proof does not rely on **any kinematic properties** of subamplitudes
- for any open string state: **boson or fermion**
- these relations hold in **any space–time dimensions** D
- for **any amount of supersymmetry**

Take $\alpha' \to 0$ limit: $e^{\pi i\ s_{ij}} = 1 + \pi i\ s_{ij} + \mathcal{O}(\alpha'^2)$

(real part) field–theory relations (Kleiss–Kuijf relations):

$$A_{YM}(1,2,\ldots,N) + A_{YM}(2,1,3,\ldots,N-1,N) + \ldots + A_{YM}(2,3,\ldots,N-1,1,N) = 0$$

(imaginary part) field–theory relations (BCJ relations):

$$s_{12}\ A_{YM}(2,1,3,\ldots,N-1,N) + \ldots + (s_{12}+s_{13}+\ldots+s_{1N-1})\ A_{YM}(2,3,\ldots,N-1,1,N) = 0$$

→ proof of BCJ relations from string theory !

St.St., arXiv:0907.2211
Bjerrum-Bohr, Damgaard, Vanhove, arXiv:0907.1425

subamplitude relations in string theory

E.g. $N=4$:

$$\frac{A(1,2,4,3)}{A(1,2,3,4)}=\frac{\sin(\pi u)}{\sin(\pi t)} \quad , \quad \frac{A(1,3,2,4)}{A(1,2,3,4)}=\frac{\sin(\pi s)}{\sin(\pi t)}$$

As a result these relations allow to express all six partial amplitudes in terms of **one**, say $A(1,2,3,4)$:

$$\begin{aligned} A(1,4,3,2) &= A(1,2,3,4) \ , \\ A(1,2,4,3) &= A(1,3,4,2)=\frac{\sin(\pi u)}{\sin(\pi t)}\ A(1,2,3,4) \ , \\ A(1,3,2,4) &= A(1,4,2,3)=\frac{\sin(\pi s)}{\sin(\pi t)}\ A(1,2,3,4) \ . \end{aligned}$$

generic N:

these relations allow for a **complete reduction** of the full string subamplitudes to a **minimal basis** of **(N-3)! dimensional basis** of subamplitudes

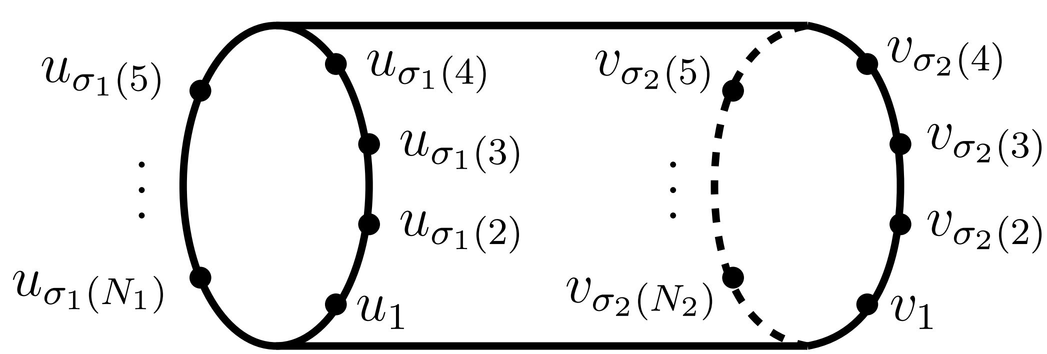
II. One-loop amplitude relations

$$\mathfrak{A}_N^{(1)} = g_{YM}^N \Big\{ N_c \sum_{\sigma \in S_{N-1}} \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(N)}})\ A^{(1)}(\sigma(1), \ldots, \sigma(N))$$

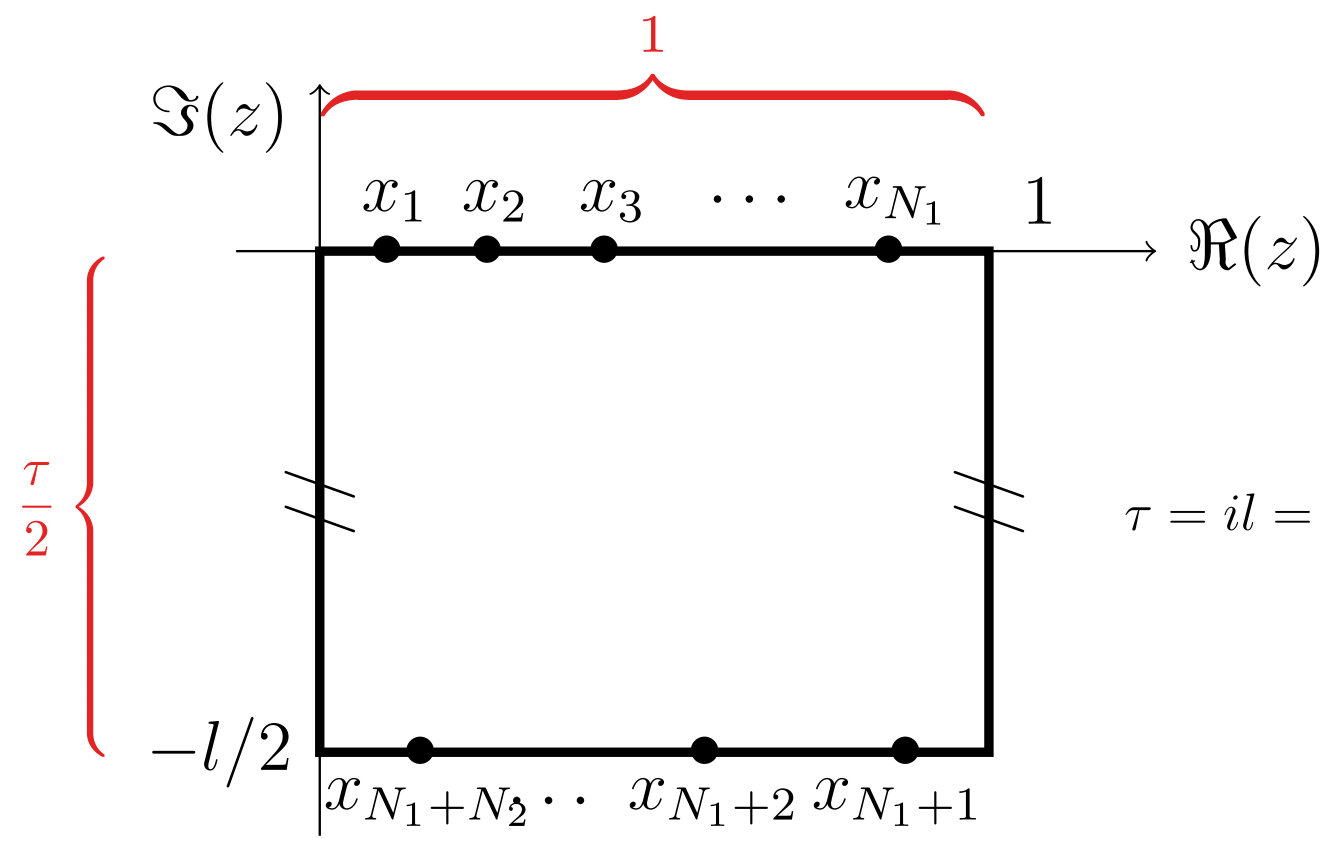
$$+ \sum_{c=2}^{\lfloor \frac{N}{2} \rfloor + 1} \sum_{\sigma \in S_N / S_{N;c}} \mathrm{Tr}(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(c-1)}})\ \mathrm{Tr}(T^{a_{\sigma(c)}} \ldots T^{a_{\sigma(N)}})$$

$$\times A^{(1)}(\sigma(1), \ldots, \sigma(c-1) | \sigma(c), \ldots, \sigma(N)) \Big\}$$

One-loop cylinder world-sheet:



dual (closed string) channel:



$$\tau = il = \frac{i}{t} = -\frac{1}{\tilde{\tau}} \ ,$$

bosonic Green functions:

$$G(z,\tau) = \ln \left| \frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)} \right|^2$$

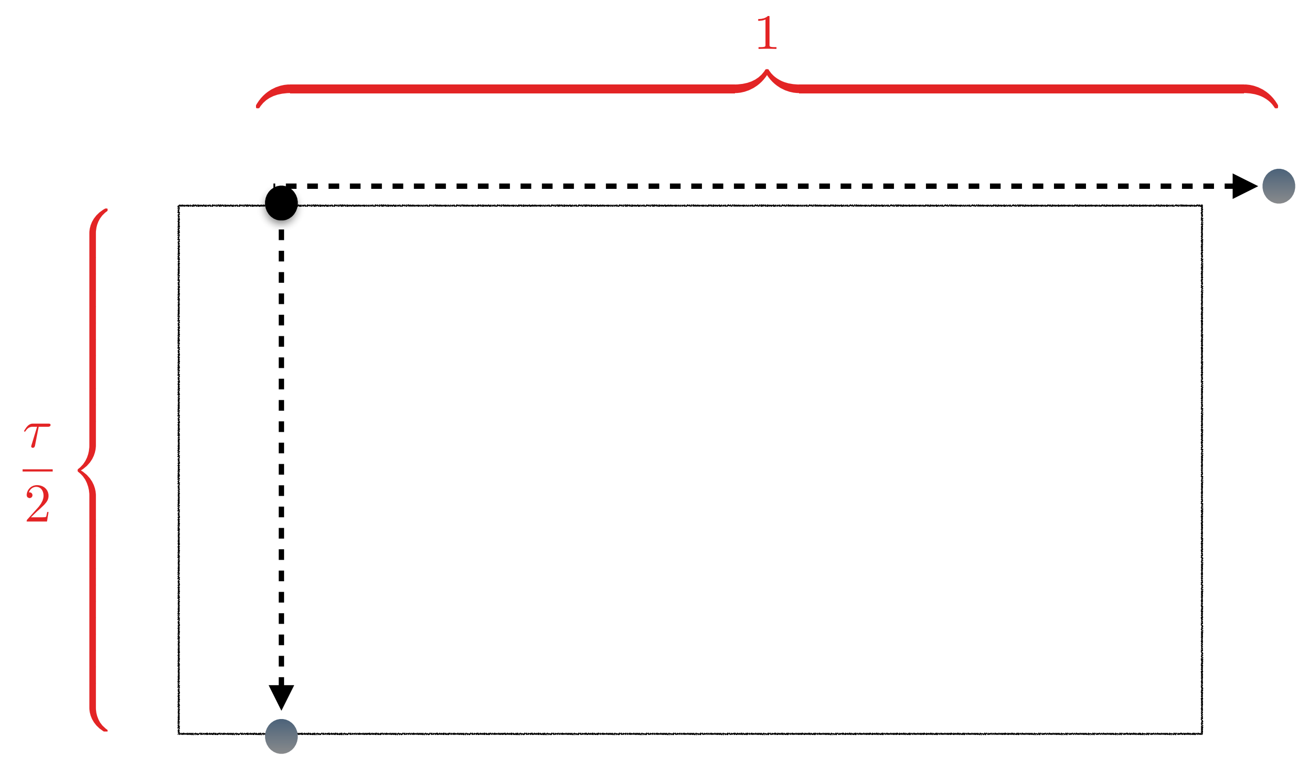
$$G_T(z,\tau) = \ln \left| \frac{\theta_4(z,\tau)}{\theta_1'(0,\tau)} \right|^2$$

$$A^{(1)}(1,\ldots,N) = \delta(k_1 + \ldots k_N) \ \int_0^\infty dl \ V_{\rm CKG}^{-1} \int_{\mathcal{I}_1} \prod_{i=1}^N dz_i \ P_N(z_1,\ldots,z_N,\tau) \ \exp\left\{ \frac{1}{2} \sum_{1\leq i<j\leq N} s_{ij} \ G(z_{ji},\tau) \right\}$$

elliptic functions (Jacobi theta functions):

$$\theta_1(z,\tau) = \theta\left[{1 \atop 1}\right](z,\tau) \quad , \quad \theta_4(z,\tau) = \theta\left[{0 \atop 1}\right](z,\tau)$$

$$\theta\left[{a \atop b}\right]\left(z + \frac{\epsilon_1}{2}\,\tau + \frac{\epsilon_2}{2}, \tau\right) = e^{-\frac{i\pi\tau}{4}\,\epsilon_1^2 - \frac{i\pi\epsilon_1}{2}(2z-b) - \frac{i\pi}{2}\,\epsilon_1\epsilon_2}\,\theta\left[{a-\epsilon_1 \atop b-\epsilon_2}\right](z,\tau)$$



$$\ln \frac{\theta_1(z+1,\tau)}{\theta_1'(0,\tau)} = \ln \frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)} + i\pi$$

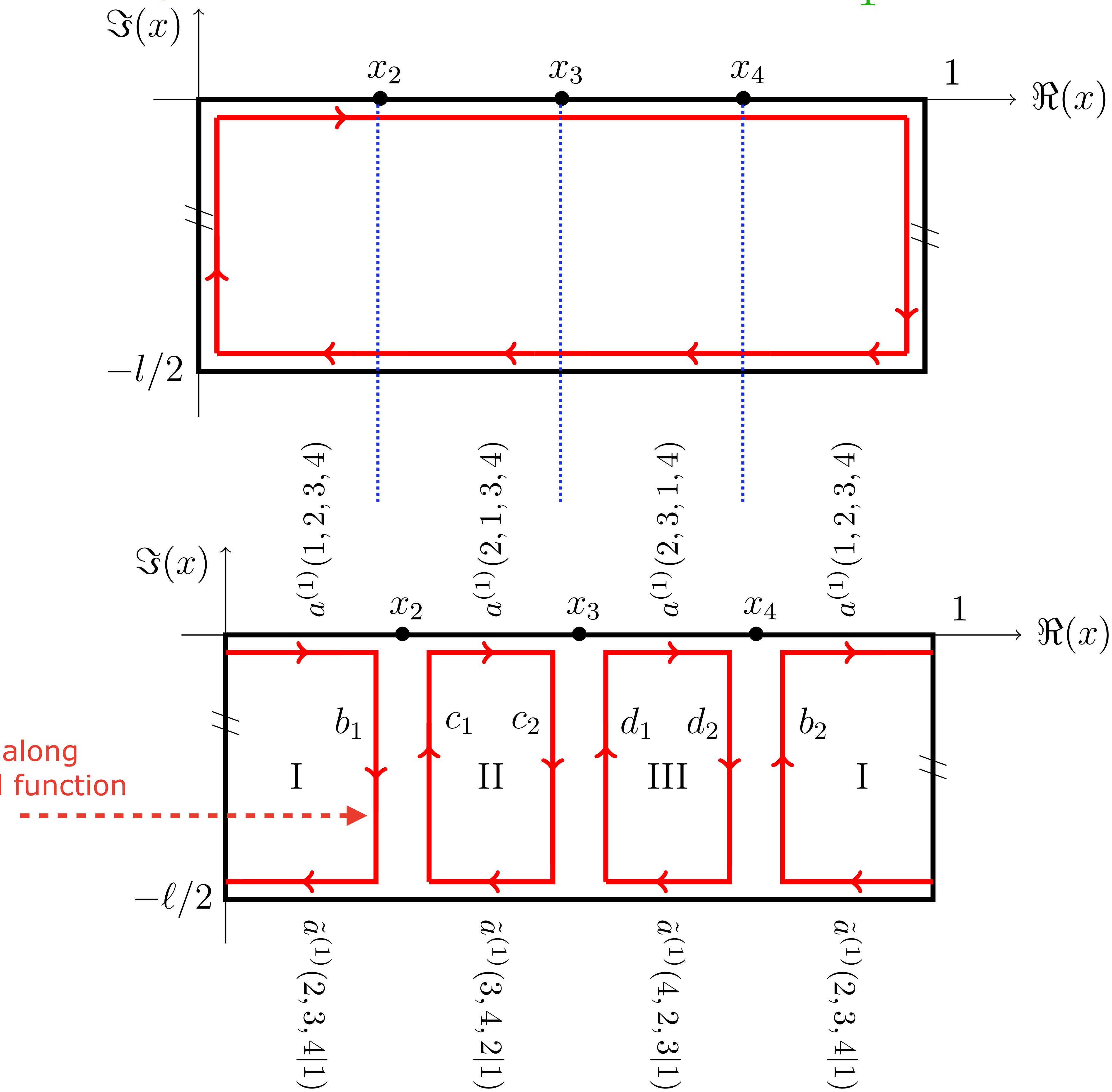
$$\ln \frac{\theta_4(z+1,\tau)}{\theta_1'(0,\tau)} = \ln \frac{\theta_4(z,\tau)}{\theta_1'(0,\tau)}$$

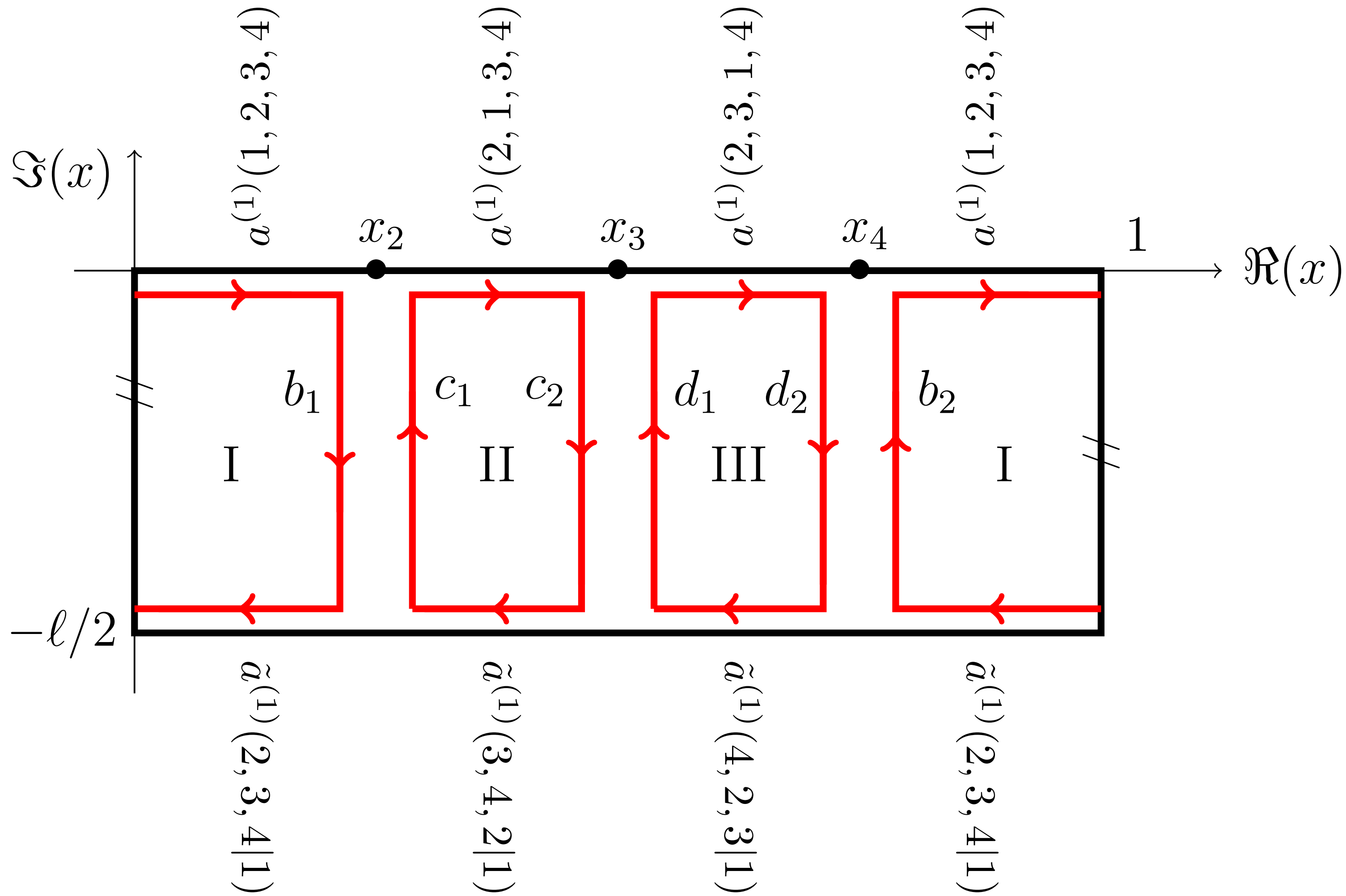
$$\ln \frac{\theta_1(z+\frac{\tau}{2},\tau)}{\theta_1'(0,\tau)} = \ln \frac{\theta_4(z,\tau)}{\theta_1'(0,\tau)} - \frac{1}{4}\,i\pi\tau + \frac{1}{2}\,i\pi - i\pi\,z$$

$$\ln \frac{\theta_4(z+\frac{\tau}{2},\tau)}{\theta_1'(0,\tau)} = \ln \frac{\theta_1(z,\tau)}{\theta_1'(0,\tau)} - \frac{1}{4}\,i\pi\tau + \frac{1}{2}\,i\pi - i\pi\,z$$

planar monodromy relation:

consider contour integral w.r.t. holomorphic coordinate x_1:





$$a^{(1)}(i_1,i_2,i_3,i_4) = V_{\mathrm{CKG}}^{-1}\; P_4 \int_0^1 dx_{i_4} \int_0^{x_{i_4}} dx_{i_3} \int_0^{x_{i_3}} dx_{i_2} \int_0^{x_{i_2}} dx_{i_1} \prod_{1\le a<b\le 4} \left(\frac{\theta_1(x_{i_b i_a}, i\ell)}{\theta_1'(0,i\ell)}\right)^{s_{i_a i_b}}$$

$$\tilde{a}^{(1)}(i_1,i_2,i_3|j) = V_{\mathrm{CKG}}^{-1}\; P_4 \int_0^1 dx_{i_3} \int_0^{x_{i_3}} dx_{i_2} \int_0^{x_{i_2}} dx_{i_1} \int_0^{x_{i_1}} dx_j \,\exp\left\{ i\pi \sum_{l=1}^{3} s_{j i_l} x_{j i_l} \right\}$$
$$\times \prod_{a=1}^{3} \left(\frac{\theta_4(x_{j i_a}, il)}{\theta_1'(0,il)}\right)^{s_{j i_a}} \prod_{1\le a<b\le 3} \left(\frac{\theta_1(x_{i_b i_a}, il)}{\theta_1'(0,il)}\right)^{s_{i_a i_b}}$$

Monodromy relations

planar monodromy relation:

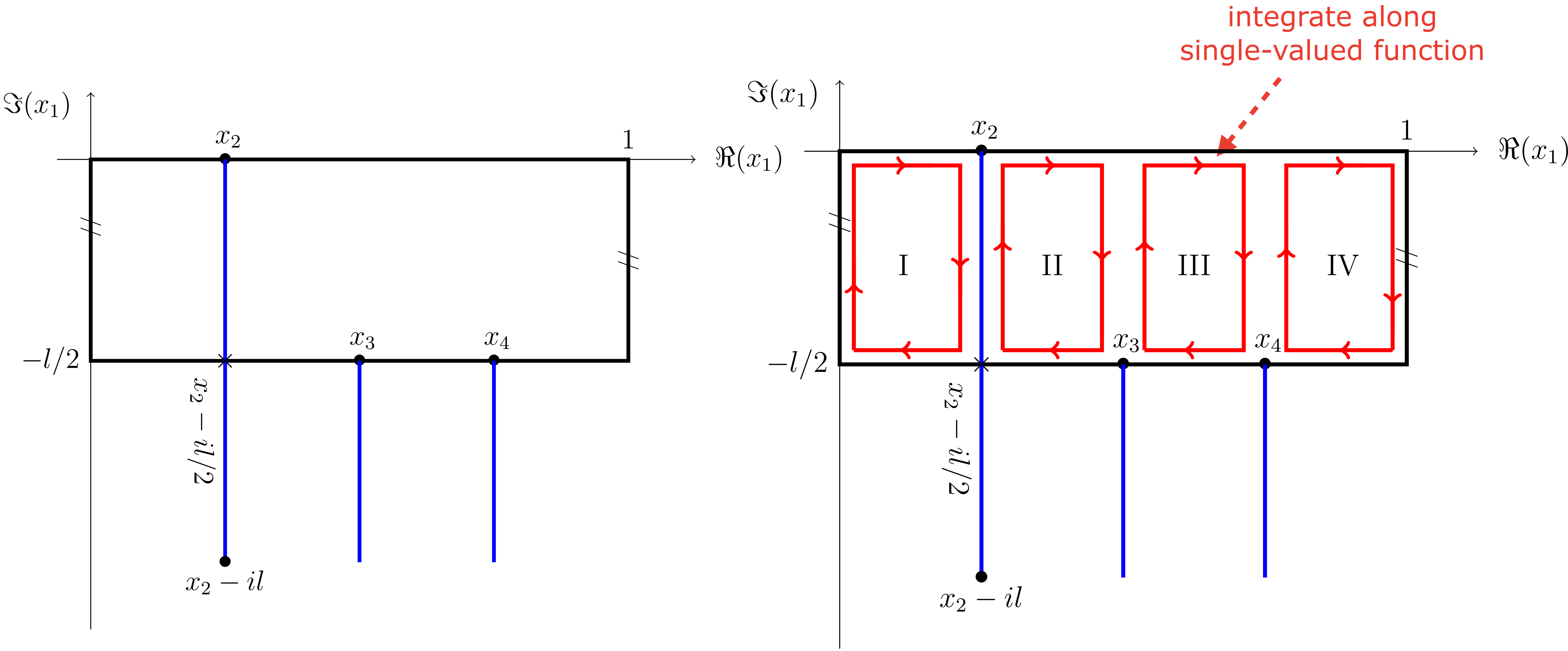
$$A^{(1)}(1,2,3,4) + e^{i\pi s_{12}}\, A^{(1)}(2,1,3,4) + e^{i\pi(s_{12}+s_{13})}\, A^{(1)}(2,3,1,4)$$
$$= \tilde{A}^{(1)}(2,3,4|1) + e^{i\pi s_{12}}\, \tilde{A}^{(1)}(3,4,2|1) + e^{i\pi(s_{12}+s_{13})}\, \tilde{A}^{(1)}(4,2,3|1)$$

generalization to arbitrary N is straightforward

see also: Tourkine, Vanhove, arXiv:1608.01665
for related, but differing findings

non-planar monodromy relation:

consider contour integral w.r.t. holomorphic coordinate x_1:



$$\frac{1}{2}\ A^{(1)}(2,1|3,4) + \hat{A}^{(1)}(2|1,3,4) - (1 - e^{-i\pi s})\ B^{(1)}(2,1|3,4) = 0$$

$\alpha' \ \ln q$ $\qquad$ $\alpha' \ \ln q$

generalization to arbitrary N is straightforward

field-theory limit (I):

$$A^{(1)}_{YM}(1|2,3,4) = -A^{(1)}_{YM}(1,2,3,4) - A^{(1)}_{YM}(2,1,3,4) - A^{(1)}_{YM}(2,3,1,4)$$

$$A^{(1)}_{YM}(1,2|3,4) = A^{(1)}_{YM}(1,2,3,4) + A^{(1)}_{YM}(1,3,2,4) + A^{(1)}_{YM}(2,1,3,4) + A^{(1)}_{YM}(2,3,1,4) + A^{(1)}_{YM}(3,1,2,4) + A^{(1)}_{YM}(3,2,1,4)$$

corresponds to leading order in field-theory (real part or KK like relations)

agrees with Bern, Dixon, Dunbar, Kosower (1994)

→ there are also relations from imaginary part (BCJ like relations)

We have performed various *checks* by computing α'- expansions in *two* regimes:

- field-theory limit: non-analytic terms, branch cuts in kinematic invariants

  stemming from boundaries of moduli space of Riemann surface

  effects can be decoupled in the limit $\tau \to i\infty$

  lowest order yields N=4 SYM

  Brink, Green, Schwarz 1982

- finite τ : perform α'- expansion and get analytic terms

$$\exp\left\{ \frac{1}{2} \sum_{1\leq i<j\leq N} s_{ij}\ G(z_j - z_i, \tau) \right\} = 1 + \frac{1}{2} \sum_{1\leq i<j\leq N} s_{ij}\ G(x_j - x_i, \tau) + \mathcal{O}({\alpha'}^2)\ ,$$

$$\exp\left\{ \frac{1}{2} \sum_{\substack{1\leq i\leq N_1 \\ N_1+1\leq j\leq N_2}} s_{ij}\ G_T(z_j - z_i + \tfrac{il}{2}, \tau) \right\} = 1 + \frac{1}{2} \sum_{\substack{1\leq i\leq N_1 \\ N_1+1\leq j\leq N_2}} s_{ij}\ G_T(x_j - x_i, \tau) + \mathcal{O}({\alpha'}^2)$$

$$x_l \in [0,1]$$

yields iterated integrals over elliptic polylogarithms (elliptic iterated integrals)

$$\int_0^z \omega^{(n_1)}(z_1) \int_0^{z_1} \omega^{(n_2)}(z_2) \ \ldots \int_0^{z_{r-1}} \omega^{(n_r)}(z_r)$$

$\omega^{(k)}$ = family of one-forms on $E_\tau^\times$

may be integrated over A and B-cycle $\begin{cases} \text{Enriquez A-elliptic multiple zeta values} \\ \text{Enriquez B-elliptic multiple zeta values} \end{cases}$

This program has been accomplished for

planar-amplitude in: Broedel, Mafra, Matthes, Schlotterer, arXiv:1412.5535

non-planar amplitude in: Hohenegger, St.St, arXiv:1702.04963
Broedel, Matthes, Richter, Schlotterer, arXiv:1704.03449

Example:

$$A^{(1)}(2,3,4|1) = (s_{12}s_{14})\ A^{(0)}_{YM}(1,2,3,4)\ \int_0^\infty dl\,[\ g(s,u)+g(t,s)+g(u,t)\ ]$$

$$g(s,u)=\int_0^1 dx_4\int_0^{x_4} dx_3\int_0^{x_3} dx_2\ \exp\left\{\tfrac{1}{2}\sum_{2\leq i<j\leq 4} s_{ij}\ G(x_j-x_i,\tau)\right\}\exp\left\{\tfrac{1}{2}\sum_{j=2}^{4} s_{1j}\ G_T(x_j-x_1,\tau)\right\}$$

$$\exp\left\{\tfrac{1}{2}\sum_{2\leq i<j\leq 4} s_{ij}\ G(x_j-x_i,\tau)\right\}\ \exp\left\{\tfrac{1}{2}\sum_{j=2}^{4} s_{1j}\ G_T(x_j-x_1,\tau)\right\}$$

$$=1+\tfrac{1}{2}\ s\ [\ G_T(x_{21})+G(x_{43})-G_T(x_{31})-G(x_{42})\ ]$$

$$+\tfrac{1}{2}\ u\ [\ G_T(x_{41})+G(x_{32})-G_T(x_{31})-G(x_{42})\ ]+\mathcal{O}(\alpha'^2)$$

e.g.: $$\frac{1}{2}\int_0^1 dz_4\int_0^{z_4} dz_3\int_0^{z_3} dz_2\ G_T(z_2) = -\frac{1}{6}\ln(2\pi)-\frac{1}{48}\ln q+\sum_{m\geq 1}\frac{q^{m/2}}{1-q^m}\left(\frac{1}{3m}-\frac{1}{2\pi^2 m^3}\right)+\frac{1}{6}Q_3$$

$$g(s,u)=\frac{1}{6}-(s+u)\ \left\{\frac{3}{4\pi^2}\ \zeta_3+\frac{3}{2\pi^2}\ \sum_{n=1}^{\infty}[\mathcal{L}i_3(q^n)+\mathcal{L}i_3(q^{n-1/2})]\right\}+\mathcal{O}(\alpha'^2)$$

field-theory limit (II):

$$A^{(1)}(1,2,3,4) + e^{i\pi s_{12}}\, A^{(1)}(2,1,3,4) + e^{i\pi(s_{12}+s_{13})}\, A^{(1)}(2,3,1,4)$$
$$= \tilde{A}^{(1)}(2,3,4|1) + e^{i\pi s_{12}}\, \tilde{A}^{(1)}(3,4,2|1) + e^{i\pi(s_{12}+s_{13})}\, \tilde{A}^{(1)}(4,2,3|1)$$

consider field-theory expansion:

use:
$$A^{(1)}(1,2,3,4) = A^{(1)}_{YM}(1,2,3,4) + \mathcal{O}(\alpha') \ ,$$
$$\tilde{A}^{(1)}(2,3,4|1) = \tilde{A}^{(1)}_{YM}(2,3,4|1) + i\pi\ \tilde{A}^{(1)}_{YM}(1,2,3,4)[k_1] + \mathcal{O}(\alpha')$$

with:
$$A^{(1)}_{YM}(2,3,4|1) = \tilde{A}^{(1)}_{YM}(2,3,4|1) + \tilde{A}^{(1)}_{YM}(3,4,2|1) + \tilde{A}^{(1)}_{YM}(4,2,3|1)$$

$$\tilde{A}^{(1)}_{YM}(1,2,3,4)[k_1] = s_{12}s_{23}\ A^{(0)}_{YM}(1,2,3,4)\ \tilde{g}_{YM}(s_{12},s_{23})$$

with field-theory object:

$$\tilde{g}_{YM}(s,u) = \int_0^\infty \frac{d\lambda}{\lambda^{D/2-3}} \int_0^1 \left(\prod_{i=1}^4 d\eta_i\right) \delta\left(1-\sum_{i=1}^4 \eta_i\right) (\eta_3 s - \eta_4 u)\ e^{-\lambda\ (s\eta_1\eta_3 + u\eta_2\eta_4)}$$
$$= \frac{1}{2}\ \frac{(1+\gamma)\ \Gamma(1+\gamma)^2\ \Gamma(-\gamma)}{(2+\gamma)\ \Gamma(2\gamma+4)}\ \left(s^{\gamma+1} - u^{\gamma+1}\right) \quad , \quad \gamma = \frac{D}{2} - 4$$

altogether:

$$\tilde{A}^{(1)}_{YM}(1,2,3,4)[k_1] = s_{12}s_{23}\ A^{(0)}_{YM}(1,2,3,4)\ \tilde{g}_{YM}(s_{12},s_{23})\ ,$$
$$\tilde{A}^{(1)}_{YM}(1,3,4,2)[k_1] = s_{12}s_{23}\ A^{(0)}_{YM}(1,2,3,4)\ \tilde{g}_{YM}(s_{13},s_{34})\ ,$$
$$\tilde{A}^{(1)}_{YM}(1,4,2,3)[k_1] = s_{12}s_{23}\ A^{(0)}_{YM}(1,2,3,4)\ \tilde{g}_{YM}(s_{14},s_{24})\ .$$

"new" field-theory relation:

$$\tilde{A}^{(1)}_{YM}(1,2,3,4)[k_1] + \tilde{A}^{(1)}_{YM}(1,3,4,2)[k_1] + \tilde{A}^{(1)}_{YM}(1,4,2,3)[k_1] = 0$$

$$\Rightarrow \quad \tilde{g}_{YM}(s,u) + \tilde{g}_{YM}(t,s) + \tilde{g}_{YM}(u,t) = 0$$

let us understand, what this relation does mean in field-theory

box diagram with momentum insertion lk_i:

$$\Box_i(s,u) := \left(\frac{\alpha'}{2\pi}\right)^{\gamma} \int d^Dl\ \frac{lk_i}{l^2(l+k_1)^2\ (l+k_1+k_2)^2\ (l-k_4)^2} = \frac{1}{2}\int_0^\infty \frac{d\lambda}{\lambda^{D/2-3}} \int_0^1 \left(\prod_{j=1}^4 d\eta_j\right)$$
$$\times\ \delta\left(1-\sum_{i=1}^4 \eta_i\right)\ [\ \eta_4 s_{4i} - \eta_3(s_{1i}+s_{2i}) - \eta_2 s_{1i}\ ]\ e^{-\lambda(s\eta_1\eta_3+u\eta_2\eta_4)}\ ,\ \ i=1,\ldots,4\ .$$

actually we have:

$$\begin{aligned}
\tilde{g}_{YM}(s,u) &= -2\,\Box_1(s,u) && = -\Delta_1(s,u) + \Delta_4(s,u)\ , \\
\tilde{g}_{YM}(t,s) &= -2\,\Box_2(s,t) - s\,\Box(s,t) && = -\Delta_2(s,t) + \Delta_1(s,t)\ , \\
\tilde{g}_{YM}(u,t) &= -2\,\Box_3(u,t) + u\,\Box(u,t) && = -\Delta_3(u,t) + \Delta_2(u,t)\ .
\end{aligned}$$

with:

$$\Delta_i = \int d^D l\ \frac{d_i}{d_1 d_2 d_3 d_4} \quad , \qquad i = 1, \ldots, 4\ , \qquad d_i = (l + q_i)^2$$

i.e.: $$\tilde{g}_{YM}(s,u) + \tilde{g}_{YM}(t,s) + \tilde{g}_{YM}(u,t) = 0$$

turns into:

$$\Delta_1(s,u) - \Delta_4(s,u) + \Delta_2(s,t) - \Delta_1(s,t) + \Delta_3(u,t) - \Delta_2(u,t) = 0$$

identity between triangles

this is just the (integrated) integrand relation:

$$2\,\square_1(s,u) + 2\,\square_2(s,t) + s\,\square(s,t) + 2\,\square_3(u,t) - u\,\square(u,t) = 0$$

$$s_{1l}\;I(1,2,3,4) + (s_{12}+s_{1l})\;I(2,1,3,4) + (s_{12}+s_{13}+s_{1l})\;I(2,3,1,4) = 0$$

$$I(1,2,3,4) = \frac{1}{l^2\;(l+k_1)^2\;(l+k_1+k_2)^2\;(l-k_4)^2} = \frac{1}{d_1 d_2 d_3 d_4}$$

Boels, Isermann (2011)
Du, Luo (2012)

III. Amplitudes in non-commutative background

introduce constant background B-field (metric g)

open string anti-symmetric tensor: $\Theta^{\mu\nu} = (-2\pi\alpha') \left(\frac{1}{g+B} \, B \, \frac{1}{g-B} \right)^{\mu\nu}$

open string metric: $G^{\mu\nu} = \left(\frac{1}{g+B} \, g \, \frac{1}{g-B} \right)^{\mu\nu}$

Seiberg, Witten (1999)

$$[\ x^\mu, x^\nu\ ] = i\ \Theta^{\mu\nu} \quad , \quad \mu, \nu = 0, \ldots, p$$

$$A^{(0)}_{NC}(1, \ldots, N) = A^{(0)}(1, \ldots, N) \ \exp\left\{ -\frac{i}{2} \sum_{1 \le i < j \le N} k_i \times k_j \right\}$$

Filk (1996)

$$k \cdot p := k_\mu G^{\mu\nu} p_\nu \qquad\qquad k_i \times k_j = k_{i\mu} \Theta^{\mu\nu} k_{j\nu}$$

one-loop open string theory:

$$A^{(1)}(2,3,4|1) = -i\sqrt{\det G}\,\frac{g_o^4}{4\alpha'^2}\,(2\alpha')^4(2\pi)^{p-3}\,\delta^{(p+1)}\left(\sum_{q=1}^{4} k_q\right)\,\mathcal{K}\int_0^\infty \frac{dl}{2}\,l^{-5}$$

$$\times\left(\frac{8\pi^2\alpha'}{l}\right)^{-\frac{p+1}{2}}\ \exp\left\{\frac{k_\mu(\Theta G\Theta)^{\mu\nu}k_\nu\, l}{8\pi\alpha'}\right\}\ \left(\prod_{q=1}^{4}\int_0^1 dx_q\right)\ \prod_{2\le i<j\le 4}\left|\frac{\theta_1\left(x_{ij}, il\right)}{\theta_1'(0,il)}\right|^{2\alpha' k_i\cdot k_j}$$

$$\times\prod_{i=2}^{4}\left|\frac{\theta_4\left(x_{1i}, il\right)}{\theta_1'(0,il)}\right|^{2\alpha' k_1\cdot k_i}\ \exp\left\{-i\sum_{2\le i<j\le 4}(k_i\times k_j)\left[x_{ij}-\frac{1}{2}\right]\right\}$$

Liu, Michelson (2001)

$k = k_2 + k_3 + k_4 = -k_1$ = non-planar momentum equate: $$-i\pi \sum_{i=2}^{4} s_{1i}\ x_i \stackrel{!}{=} -i \sum_{2\leq i<j\leq 4} (k_i \times k_j)\ (x_i - x_j)$$

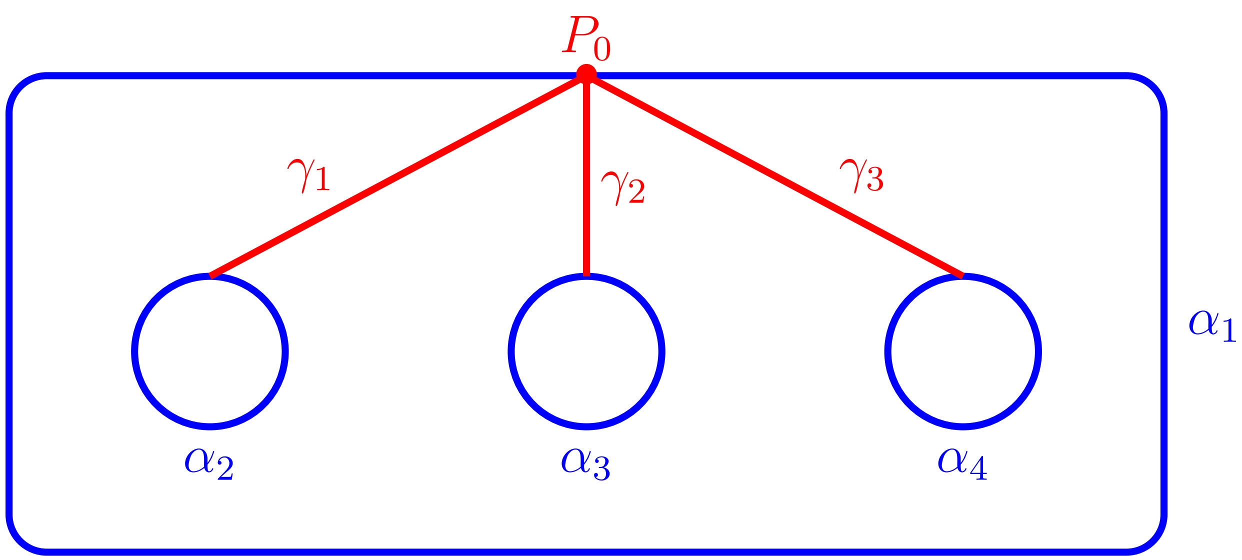
yields: $$\boxed{k_{1\mu}\ \left(\ 2\pi\alpha' G^{\mu\nu} - \Theta^{\mu\nu}\ \right) = 0}$$

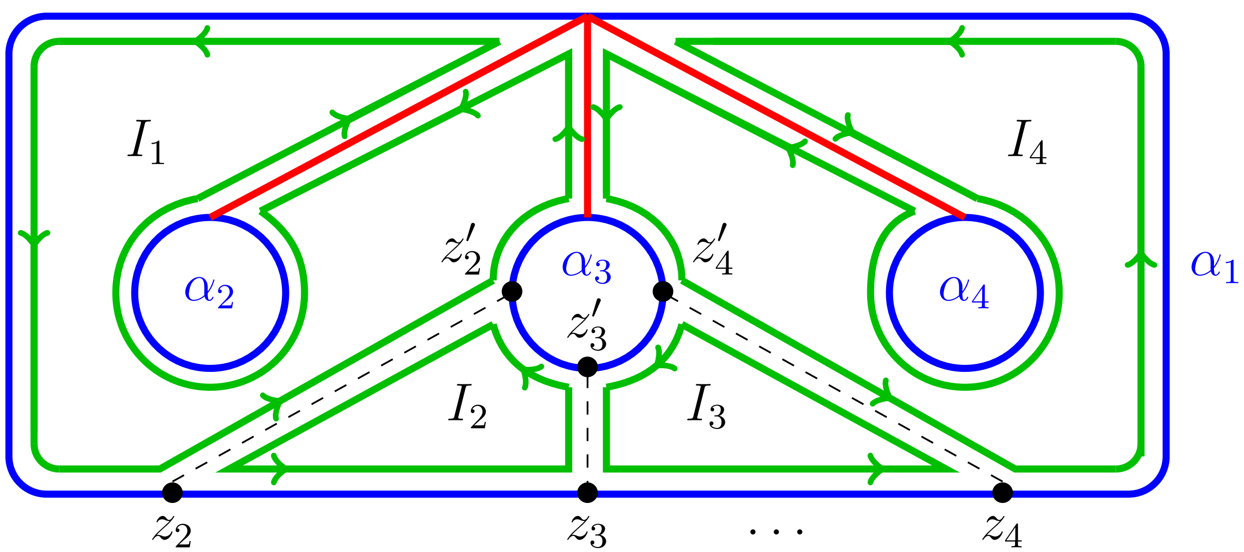
has solution: $$\Theta^{\mu\nu} = 2\pi\alpha' \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & \vartheta^{mn} \end{pmatrix}$$

for the choice: $k_{1\mu} = (k, k, 0, \ldots, 0)\,,$ $G^{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1)$

IV. Monodromy relations at g-loop

E.g. only n boundaries involved: $\mathbf{Z}_2$ involution of Riemann surface of genus g=n-1





n=4

$$A^{(g)}(1,2,\ldots,N)+e^{i\pi s_{12}}A^{(g)}(2,1,\ldots,N)+\ldots+e^{i\pi\sum_{j=2}^{N-1}s_{1j}}A^{(g)}(2,\ldots,1,N)$$

$$=\tilde{A}^{(g)}(2,3,\ldots,N|1)+e^{i\pi s_{12}}\tilde{A}^{(g)}(3,4,\ldots,N,2|1)+\ldots+e^{i\pi\sum_{j=2}^{N-1}s_{1j}}\tilde{A}^{(g)}(N,2,\ldots,N-1|1)$$

Concluding remarks

- *correct* monodromy relations for N-point at g-loop
- additional "boundary" terms necessary
  to cancel tachyonic poles
- expansion in terms of elliptic multiple zeta values
- interpretation in terms of
  scattering in non-commutative background
- subamplitude relations play crucial role
  for KLT at higher loops !
- basis of independent subamplitudes ?
- include closed strings ?