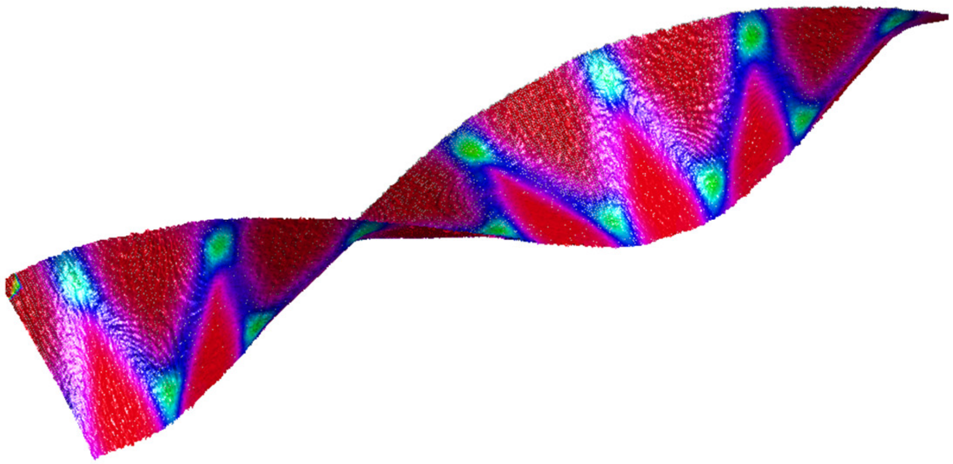
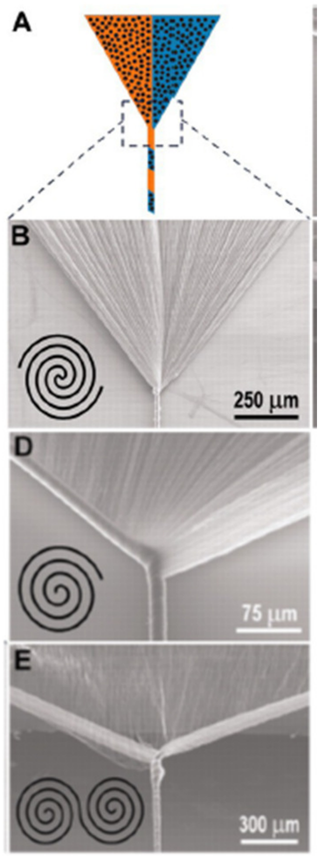
Experiments on ribbons with a twist



Arshad Kudrolli
Julien Chopin (ESPCI)
Department of Physics, Clark University
Worcester, Massachusetts
Supported by National Science Foundation

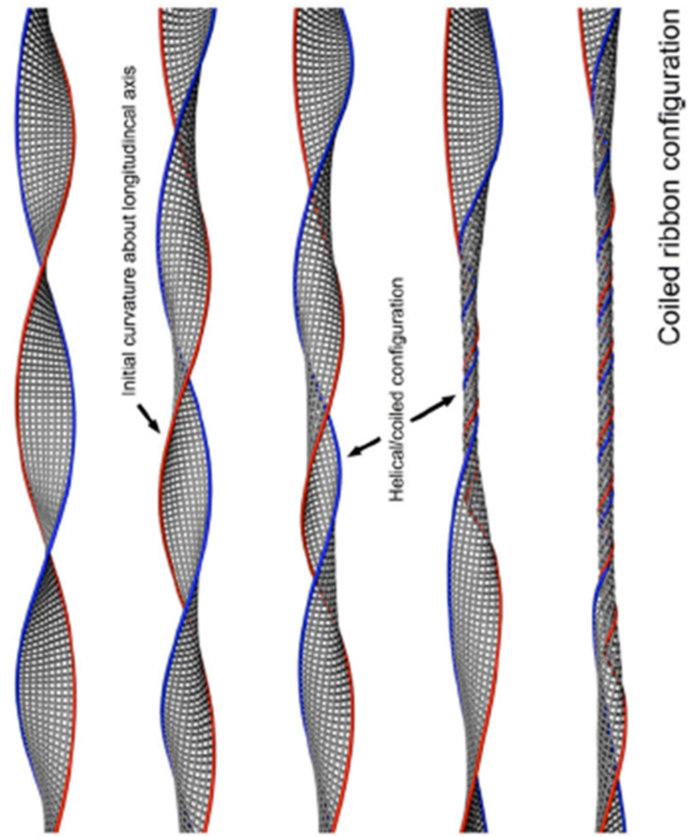
Twisted ribbons and novel materials

Yarn fabrication



Lima et al. Science 331, 51 (2011)

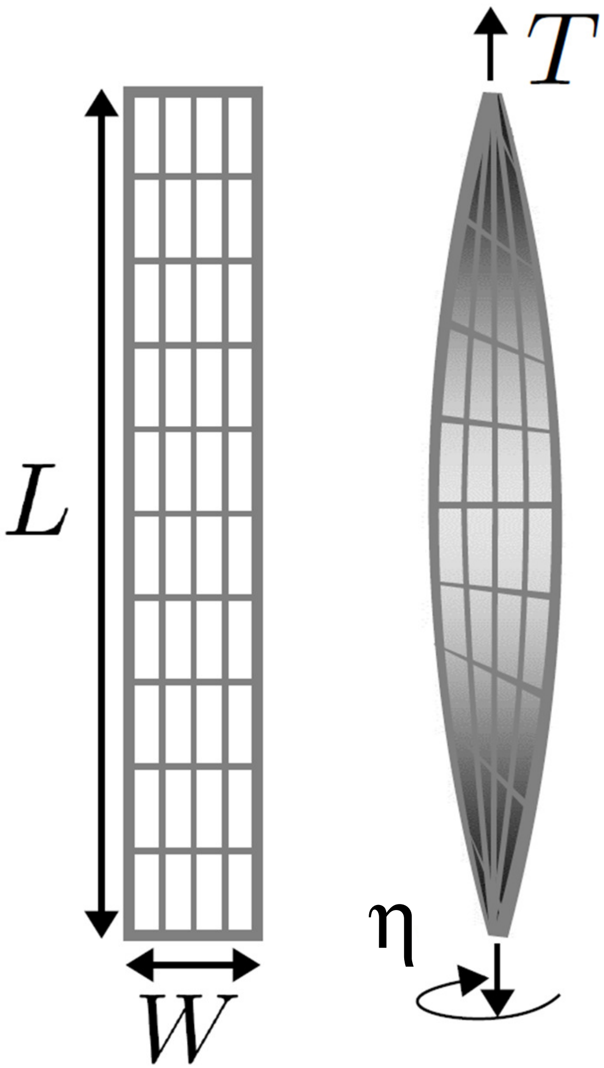
Graphene nanotube fabrication



S. Cranford and M. Buehler,
Mod. Simul. Mater. Sci. Eng., 19, 054003 (2011)

Geometry and Loading

Thin elastic ribbon where the short edge is clamped and twisted through an angle η under a tension T



Control parameters

Tension : $T = F / (E\, h\, W)$

Twist angle : $\eta = \alpha / (L/W)$

Typical values :

Thickness :	$h = 100\mu m$
Anisotropy :	$t = h/W < 0.02$
Slenderness :	$L/W > 10$
Young's modulus :	$E = 3.4 GPa$,
Tension :	$T \sim 10^{-3}$
Twist angle :	$\eta \sim 0.5$

Depending on:

1- the loading parameters η and T

2- the geometric parameters h/W and L/W

What are the equilibrium configurations of the ribbon?

First addressed by A.E. Green (1936):

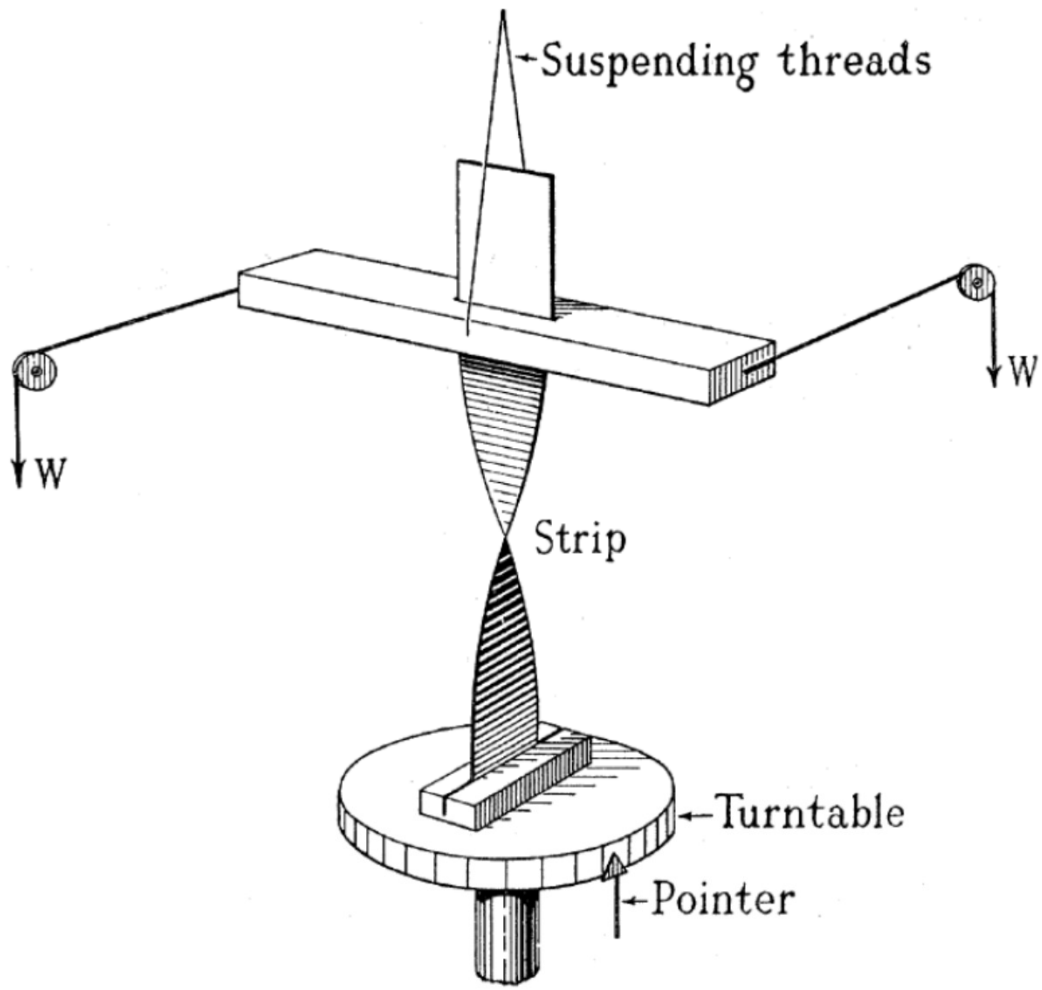


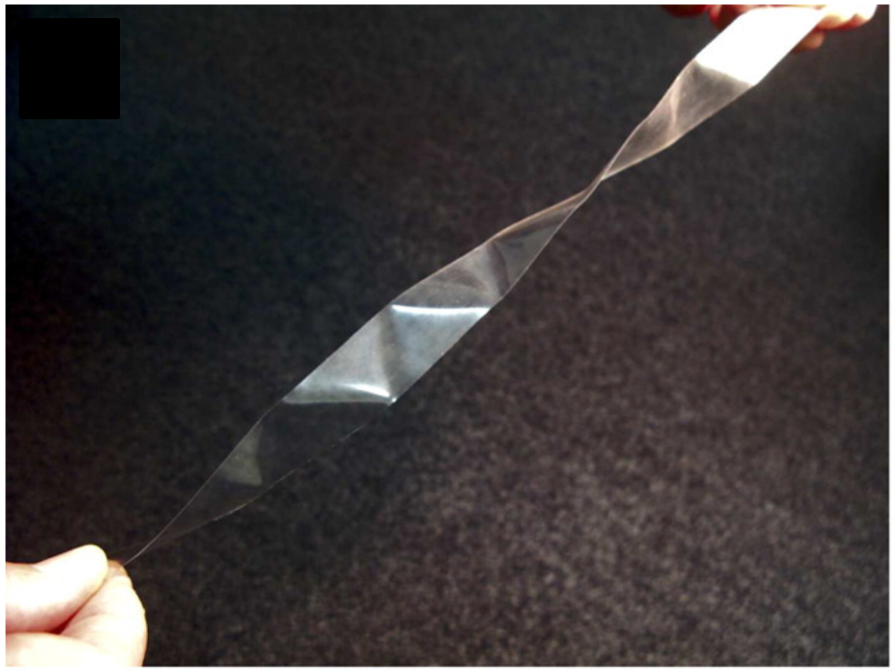
FIG. 2.

- Ribbon will buckle longitudinally

A.E. Green (1936, 1937)
Coman & Bassom (2008)

Triangular buckling patterns of twisted inextensible strips

A. P. Korte, E. L. Starostin, G. H. M. van der Heijden, Proc. Roy. Soc. A (2010)



- Twisted strip of acetate or paper
- Constructed solution assuming isometric deformation

Buckling Modes of Twisted Ribbons

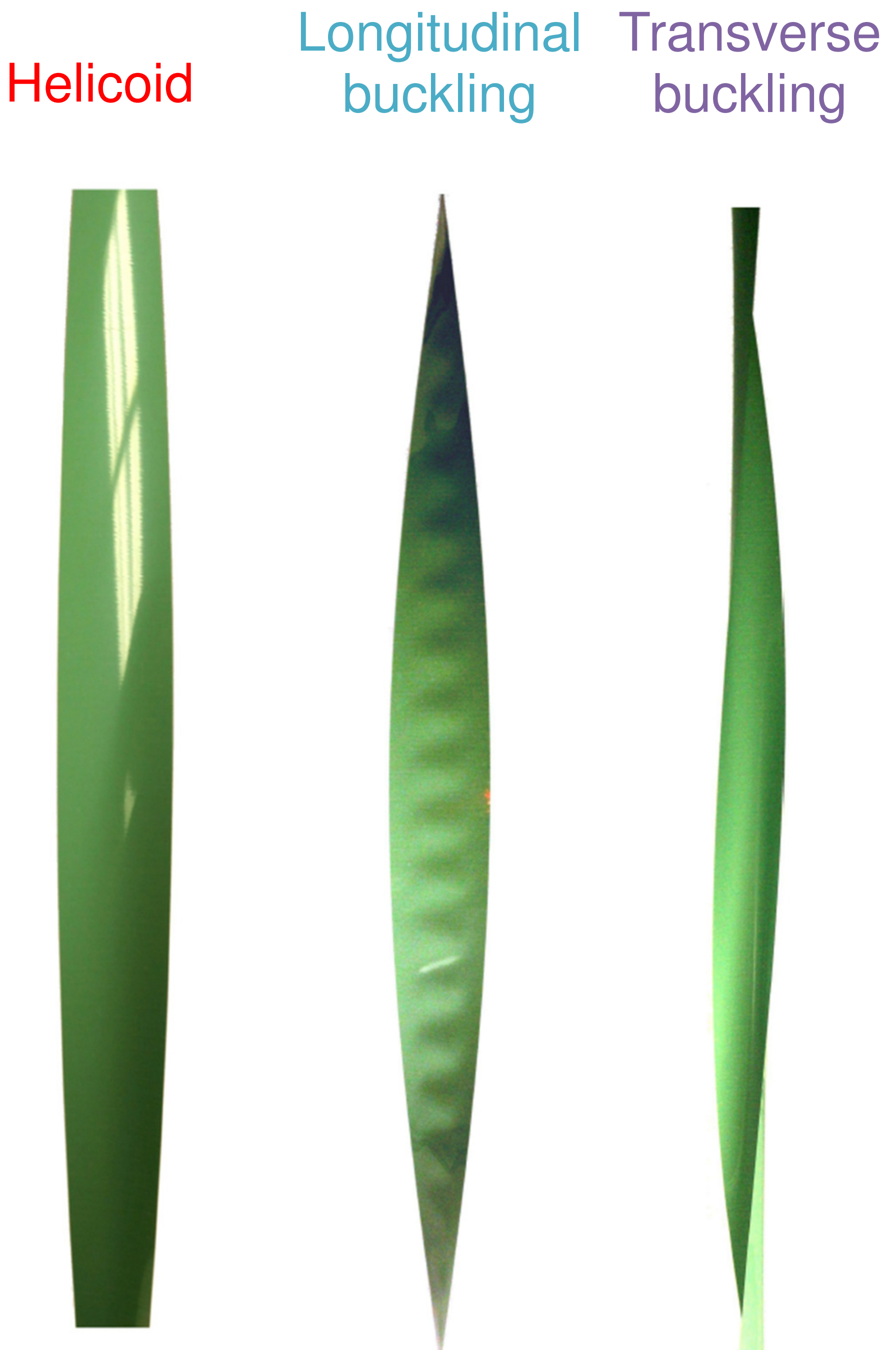
Helicoid

Buckling Modes of Twisted Ribbons

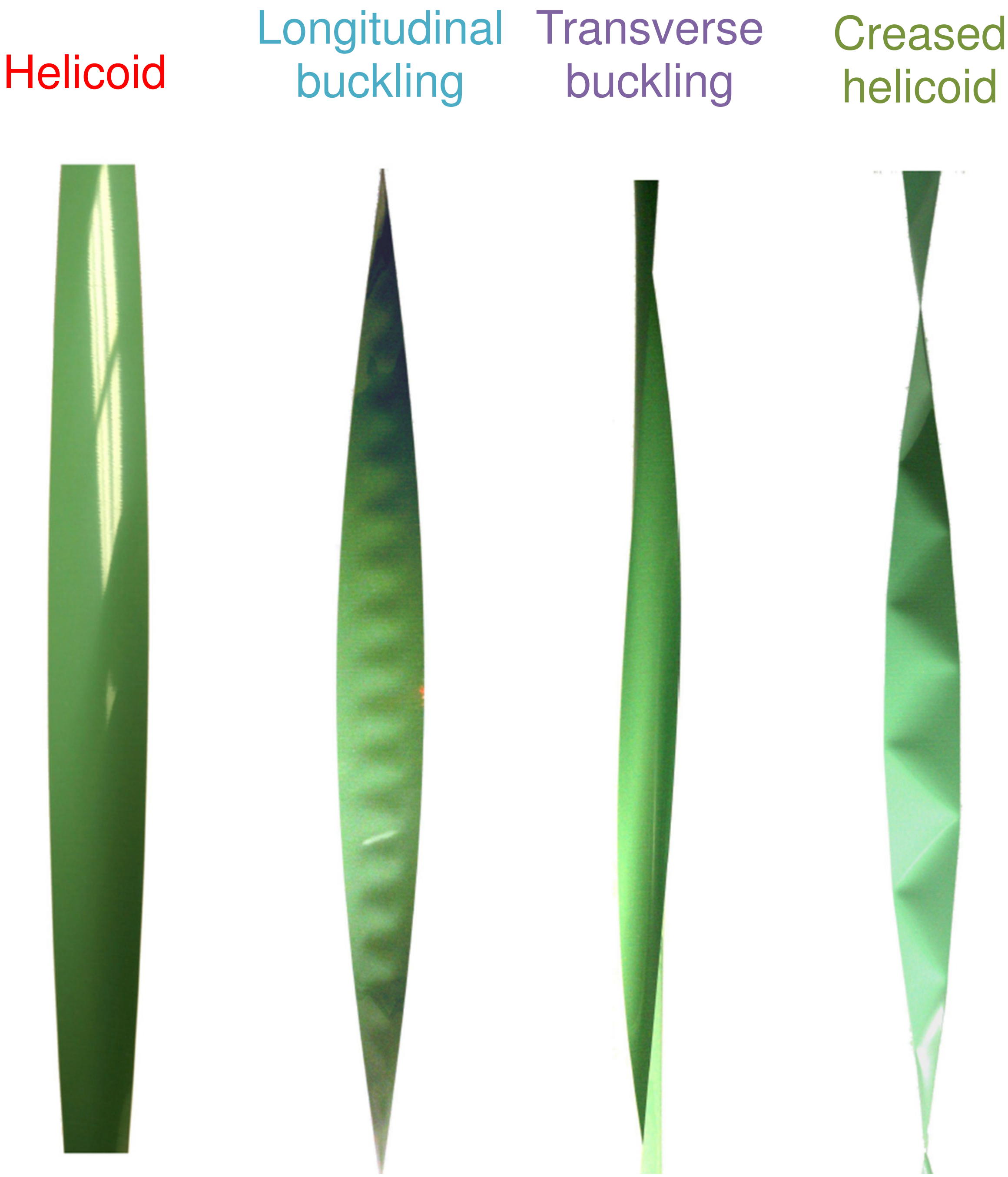
Helicoid

Longitudinal buckling

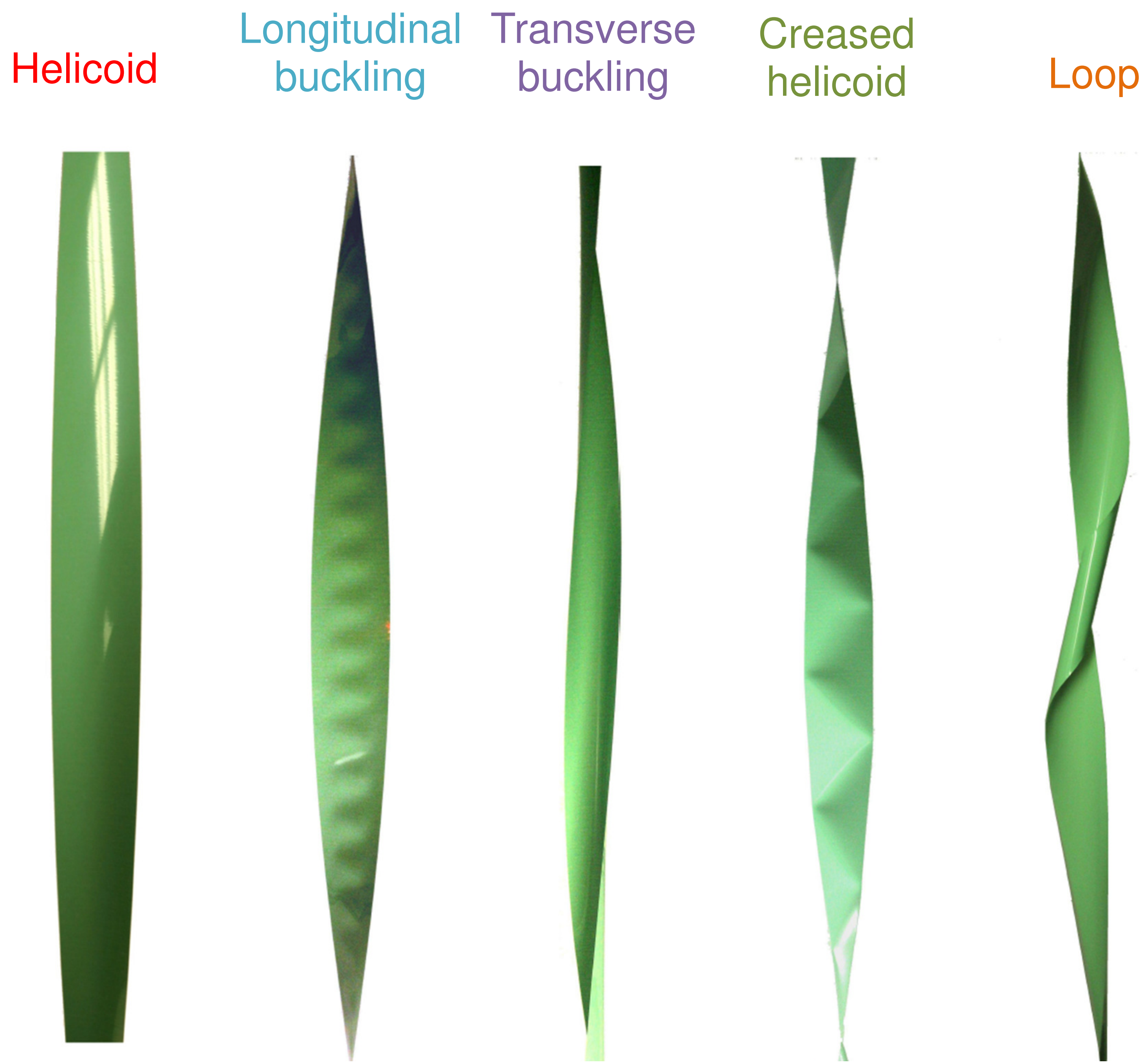
Buckling Modes of Twisted Ribbons



Buckling Modes of Twisted Ribbons

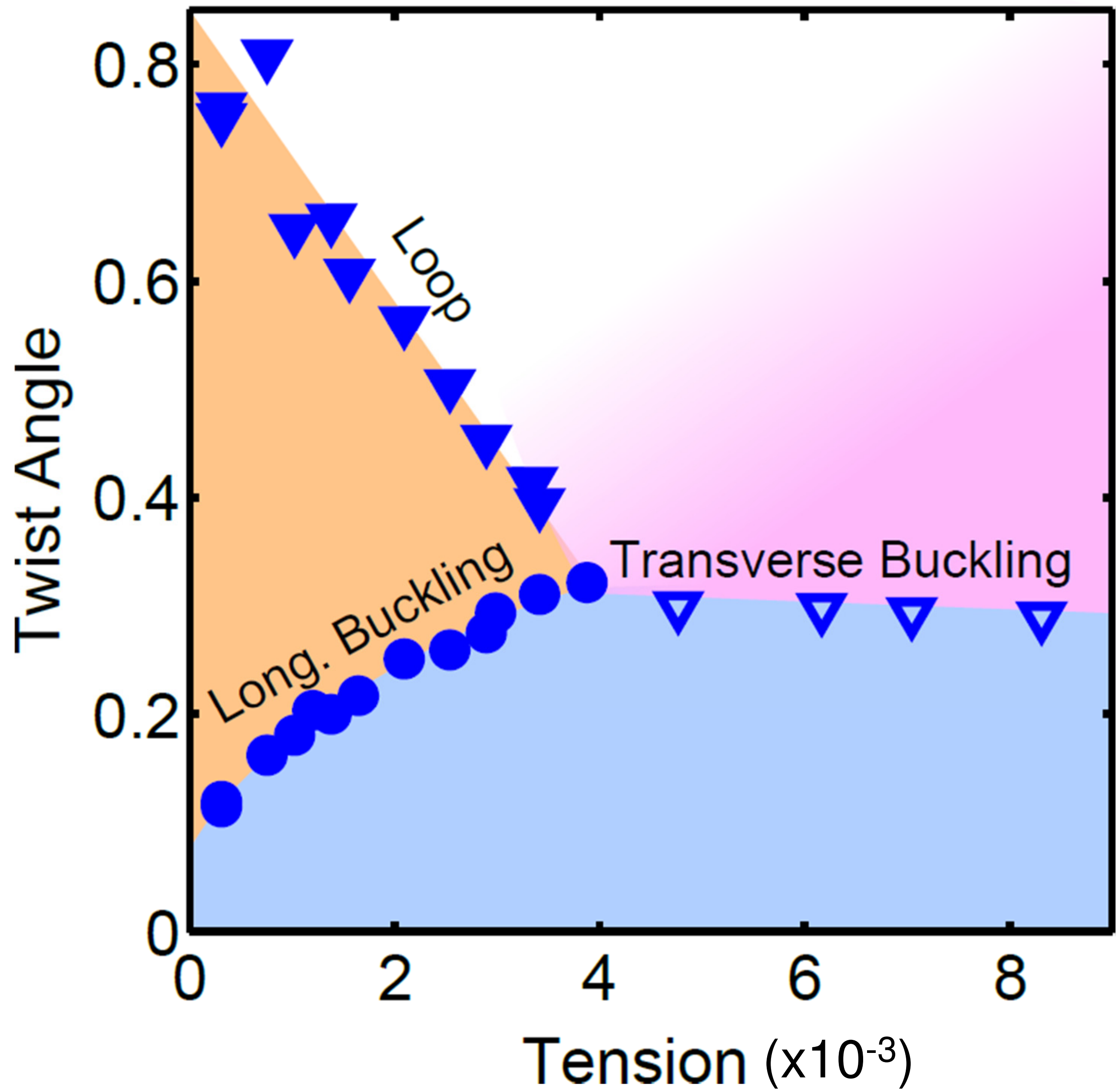


Buckling Modes of Twisted Ribbons

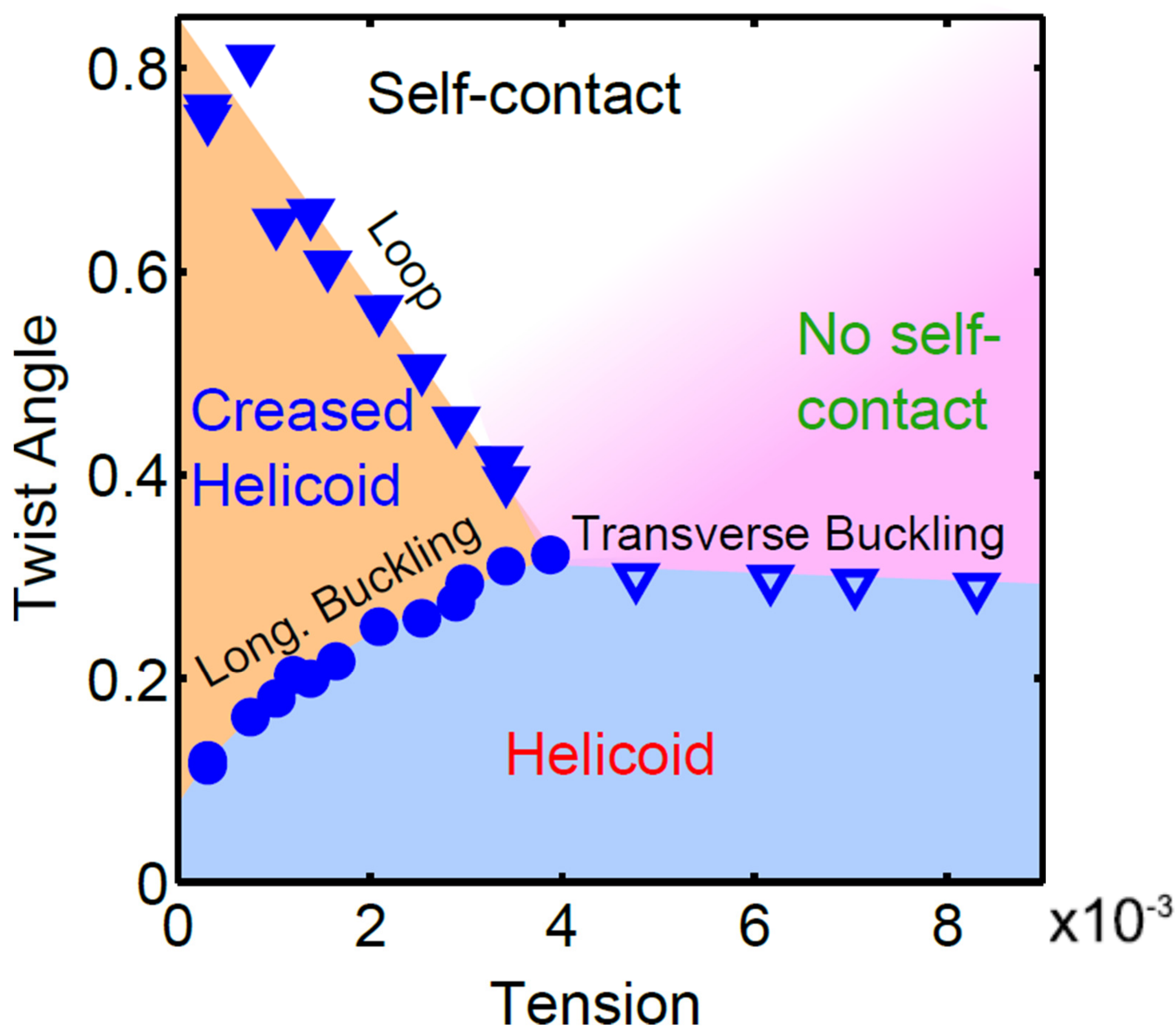


Phase diagram

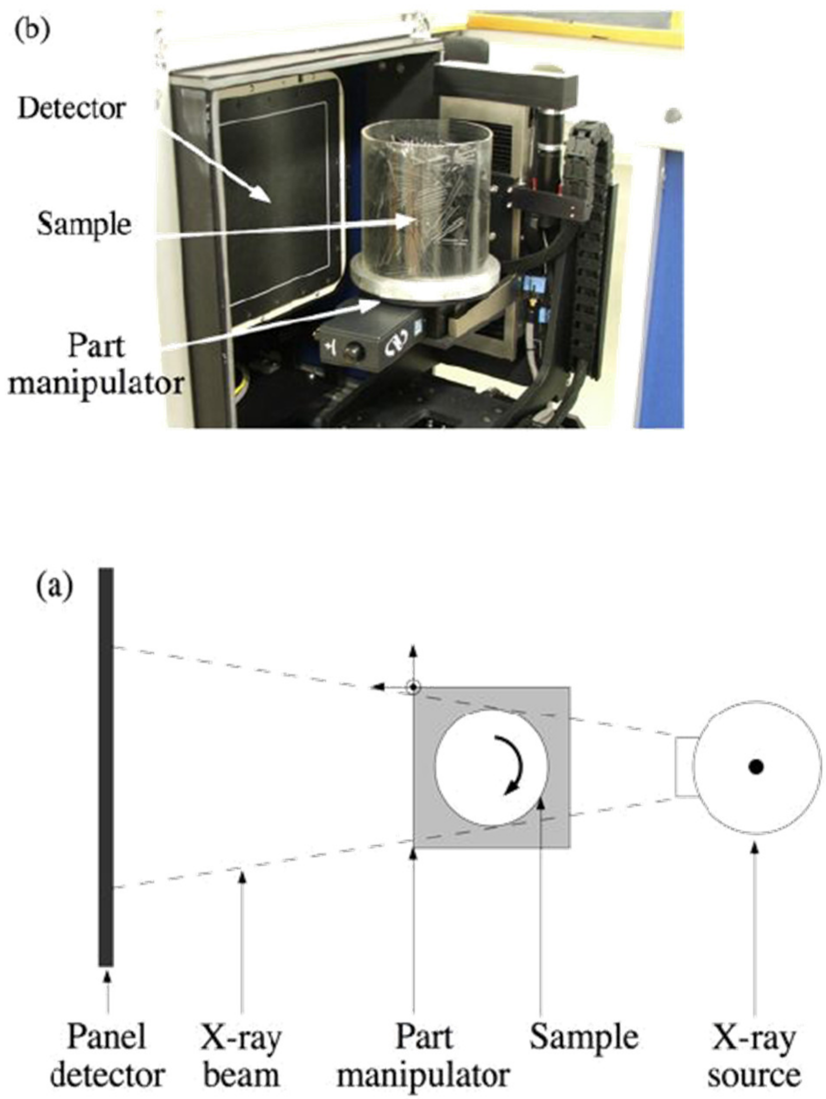
(L = 45 cm, W = 25.4 mm, and h = 127 μm)

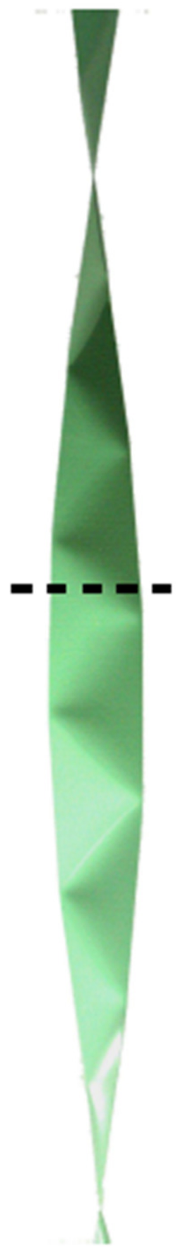


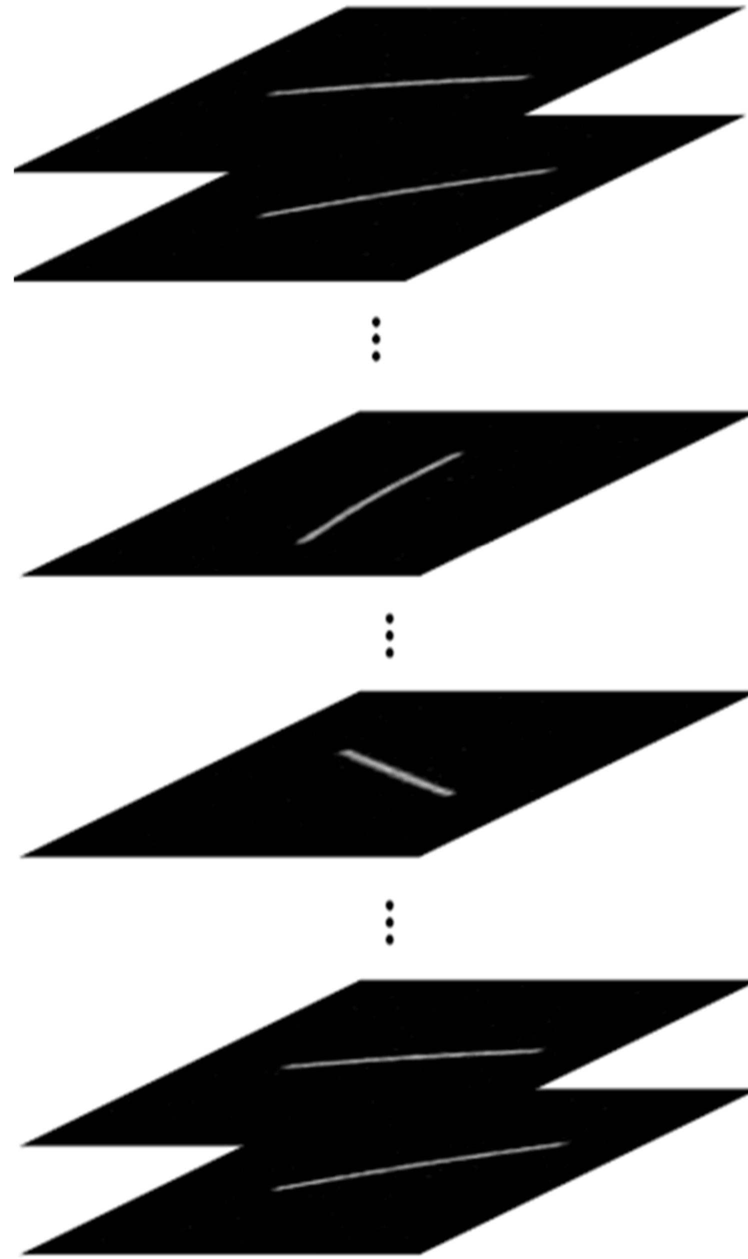
Phase diagram



Xray computed tomography

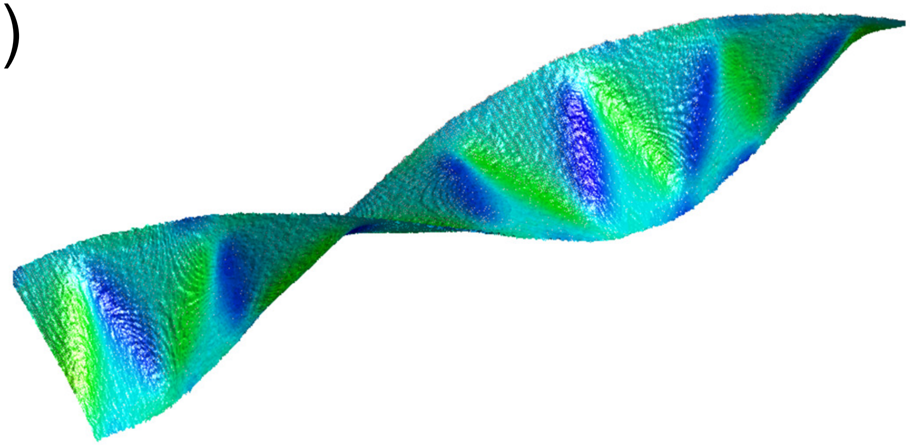




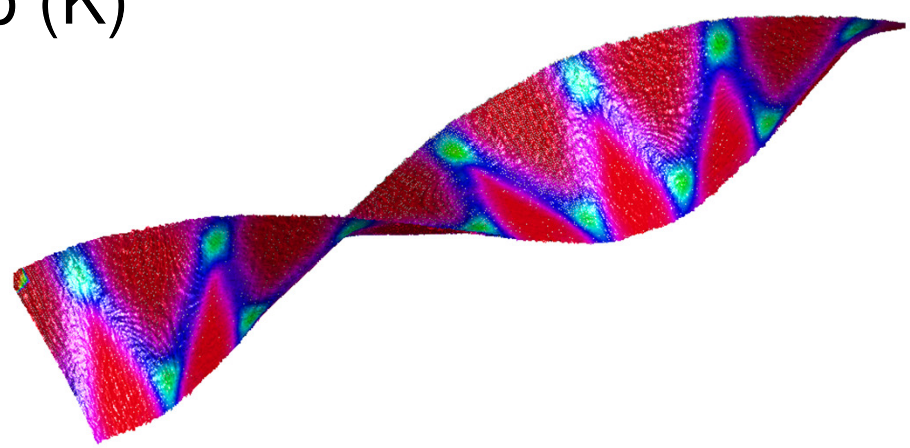


Xray computed tomography

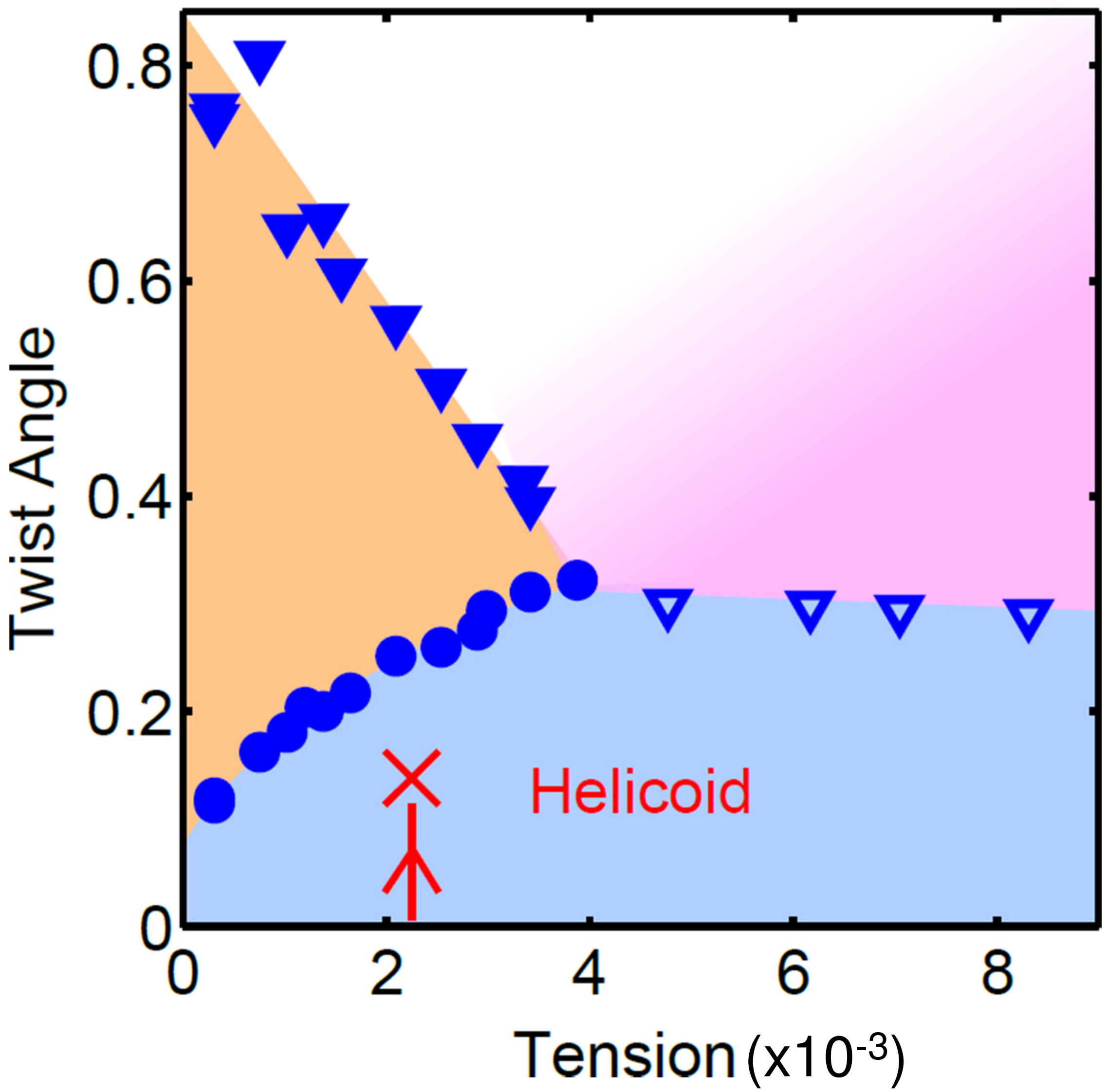
Mean curvature map (H)

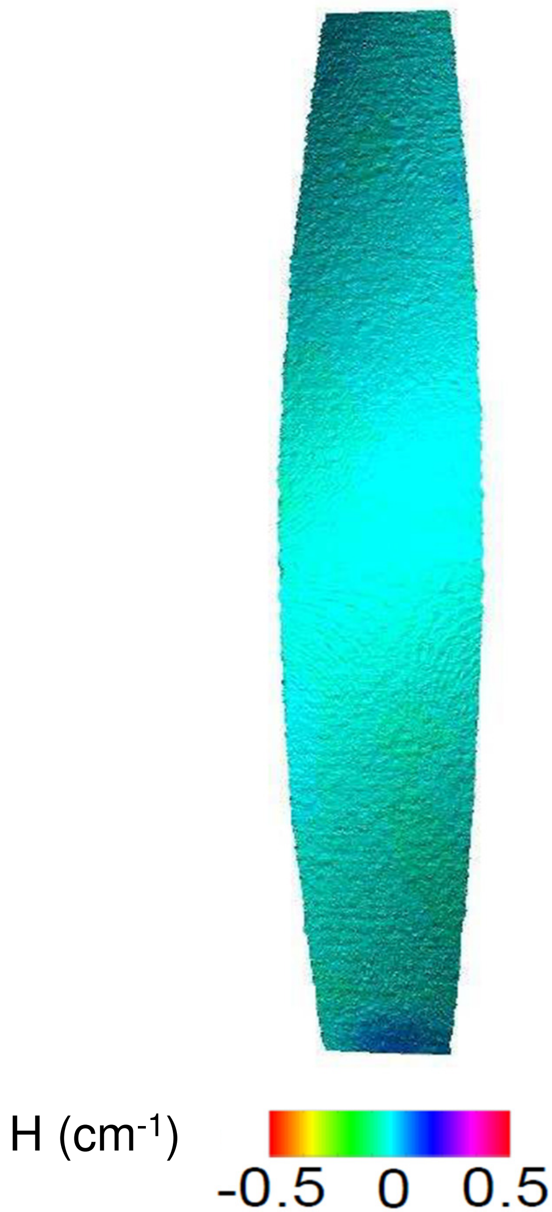


Gaussian curvature map (K)

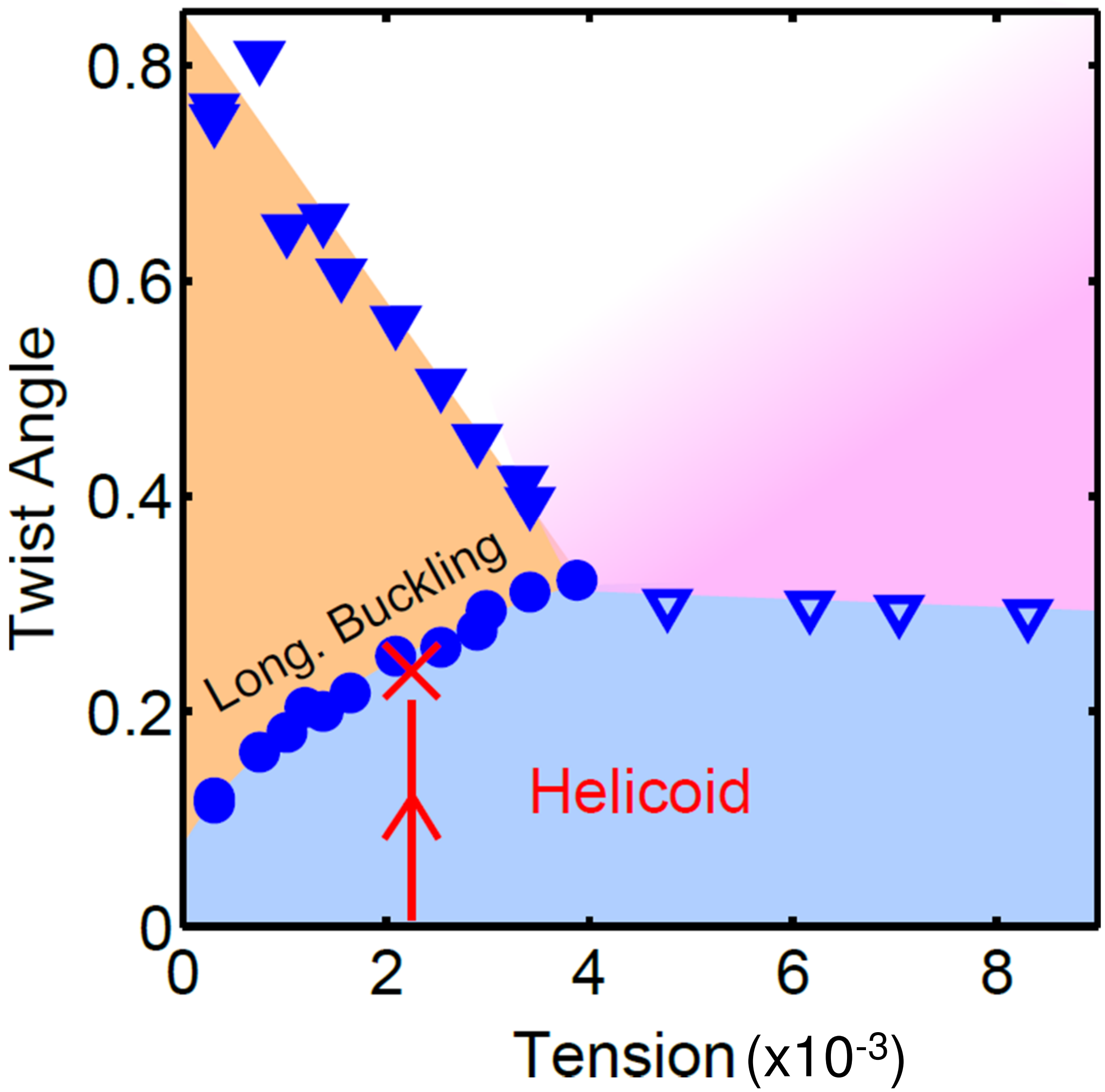


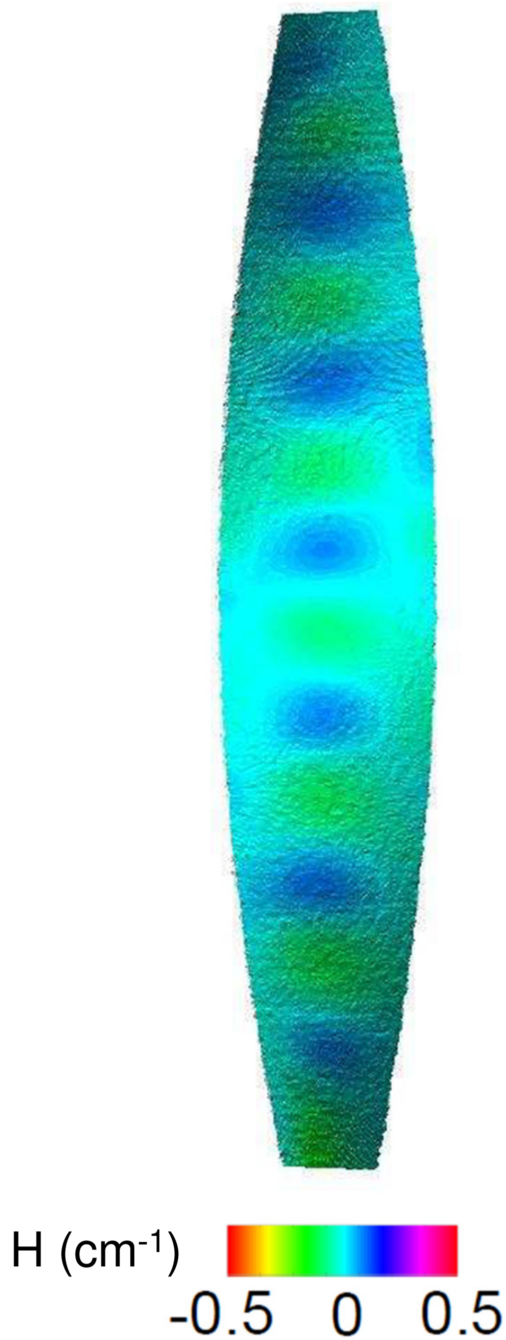
Helicoid


Twist Angle
0.8
0.6
0.4
0.2
0
0
2
4
6
8
Tension (x10-3)
Helicoid

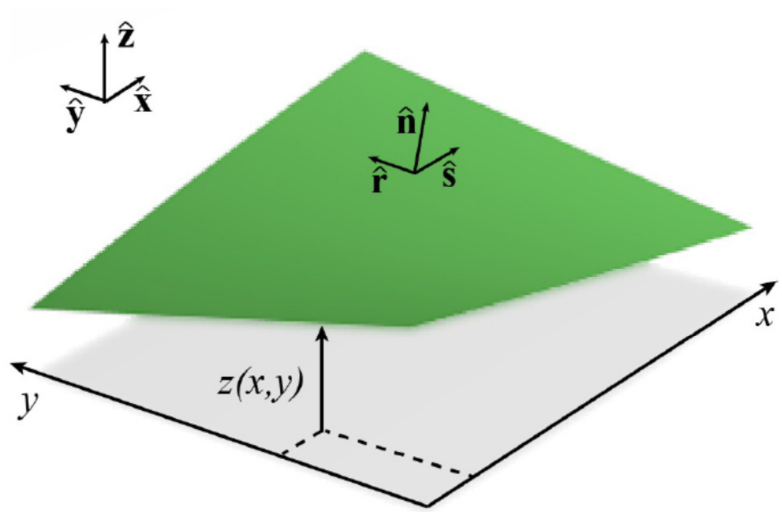

H (cm-1)
-0.5
0
0.5

Longitudinal buckling





Mechanical Equilibrium (Small-Slope)



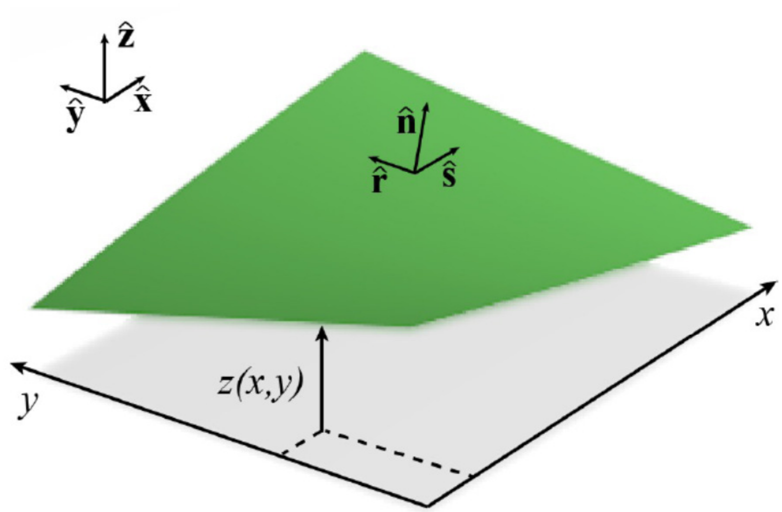
Mechanical equilibrium of the ribbon using the Föppl – von Kàrmàn (FvK) equations

$$\sigma^{ss}\partial_{ss}z + \sigma^{rr}\partial_{rr}z + 2\sigma^{rs}\partial_{rs}z = B\Delta^2 z,$$
$$\partial_s\sigma^{ss} + \partial_r\sigma^{sr} = 0,$$
$$\partial_s\sigma^{rs} + \partial_r\sigma^{rr} = 0.$$

where $B = t^2/[12(1-\nu^2)]$ is the bending modulus.

Mechanical Equilibrium (Small-Slope)

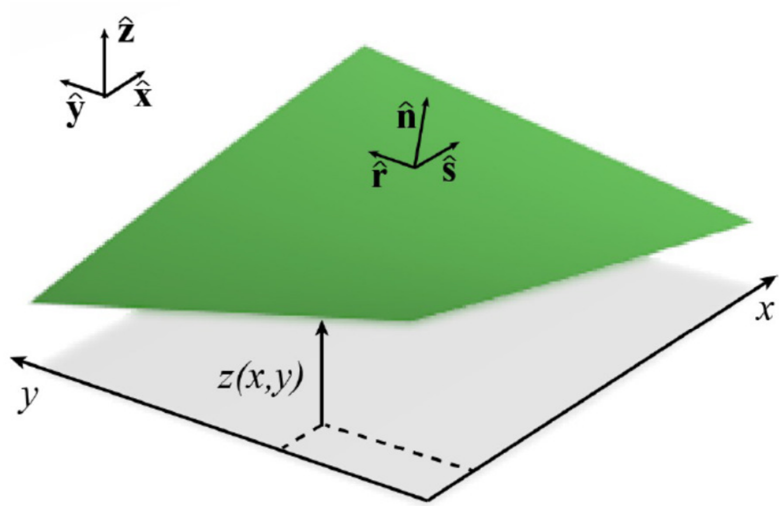
$z\,(s,r) = \eta\, s\, r$



$$\eta \sigma^{sr} = 0,$$
$$\partial_s \sigma^{ss} = 0,$$
$$\partial_r \sigma^{rr} = 0.$$

Mechanical Equilibrium (Small-Slope)

$$z\,(s,r) = \eta\, s\, r$$



$$\eta \sigma^{sr} = 0,$$
$$\partial_s \sigma^{ss} = 0,$$
$$\partial_r \sigma^{rr} = 0.$$

Boundary conditions :

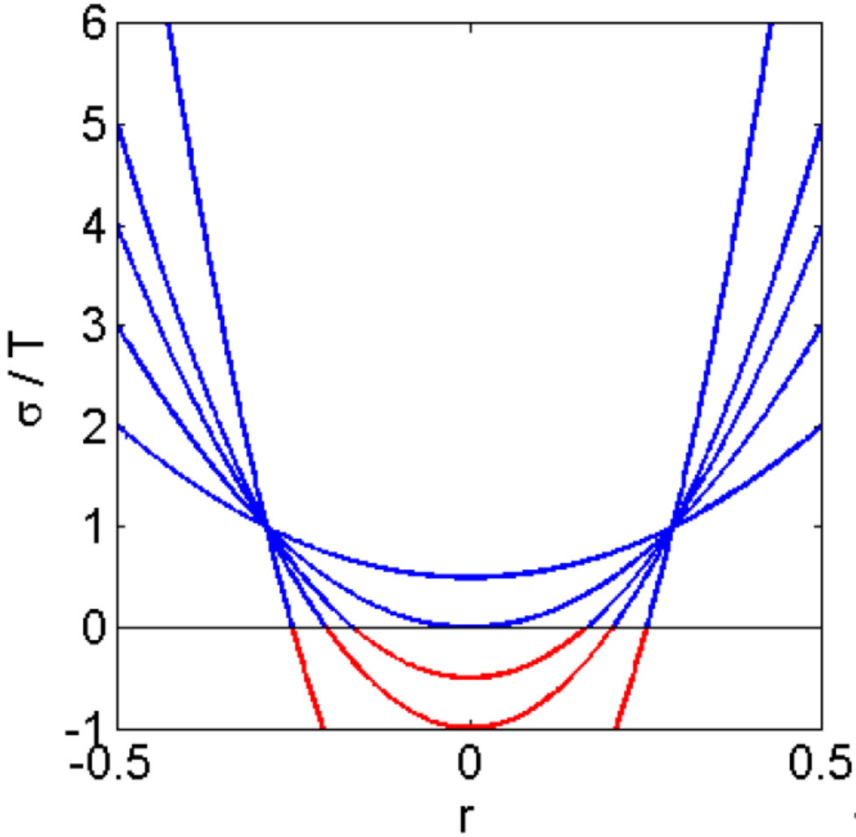
At r $= \pm$W/2 : $\sigma^{rr} = 0$

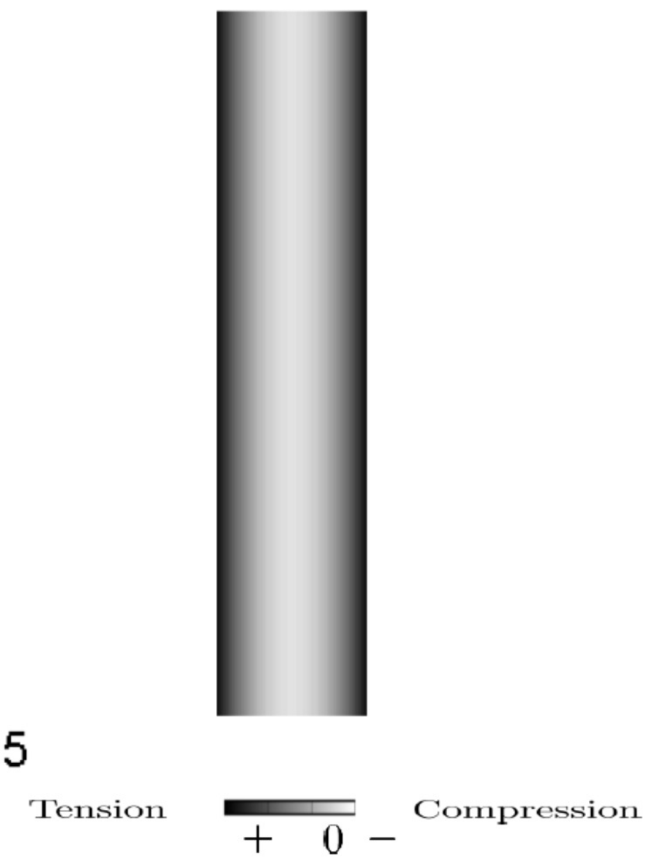
Constant tension : $T = \int \sigma^{ss} dr$

$$\sigma^{sr}(r) = 0,$$
$$\sigma^{ss}(r) = T + \frac{\eta^2}{2}\left(r^2 - \frac{1}{12}\right),$$
$$\sigma^{rr}(r) = 0.$$

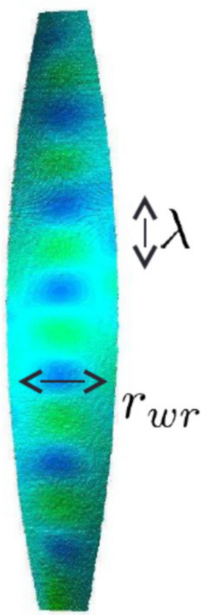
$$\sigma^{ss} = \left[T - \frac{1}{24}\eta^2\right] + \frac{1}{2}\eta^2 r^2$$

$\eta^2/T = 100$





Scaling of wavelength



The change in the elastic energy density due to longitudinal buckling ΔU_L is the sum of three contributions :

Stretching : $$\Delta U_S \sim \sigma^{ss} \left(\frac{A}{\lambda}\right)^2,$$

Bending(longitudinal) : $$\Delta U_B^{||} \sim B \left(\frac{A}{\lambda^2}\right)^2,$$

Bending(orthogonal) : $$\Delta U_B^{\perp} \sim B \left(\frac{A}{r_{wr}^2}\right)^2.$$

Using $\Delta U_S \sim \Delta U_B^{||} \sim \Delta U_B^{\perp}$, we have at threshold :

$$\lambda_{lon} \sim r_{wr}$$

$$\sigma^{ss} \sim \left(\frac{t}{r_{wr}}\right)^2$$

$\Rightarrow$

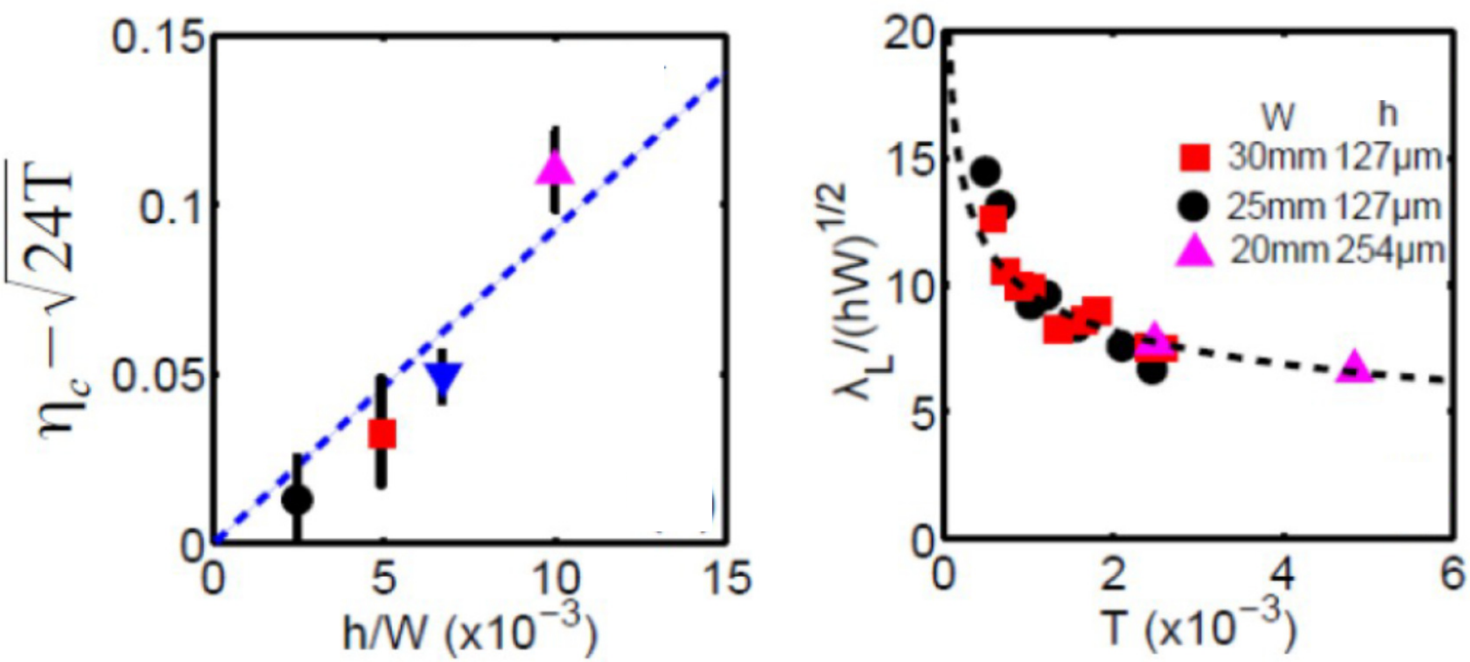
$$\lambda_{lon} \sim \frac{\sqrt{t}}{T^{1/4}}$$

$$\eta_{lon} - \sqrt{24T} \sim t$$

Coman and Bassom, Acta Mechanica **200**, 59 (2008)

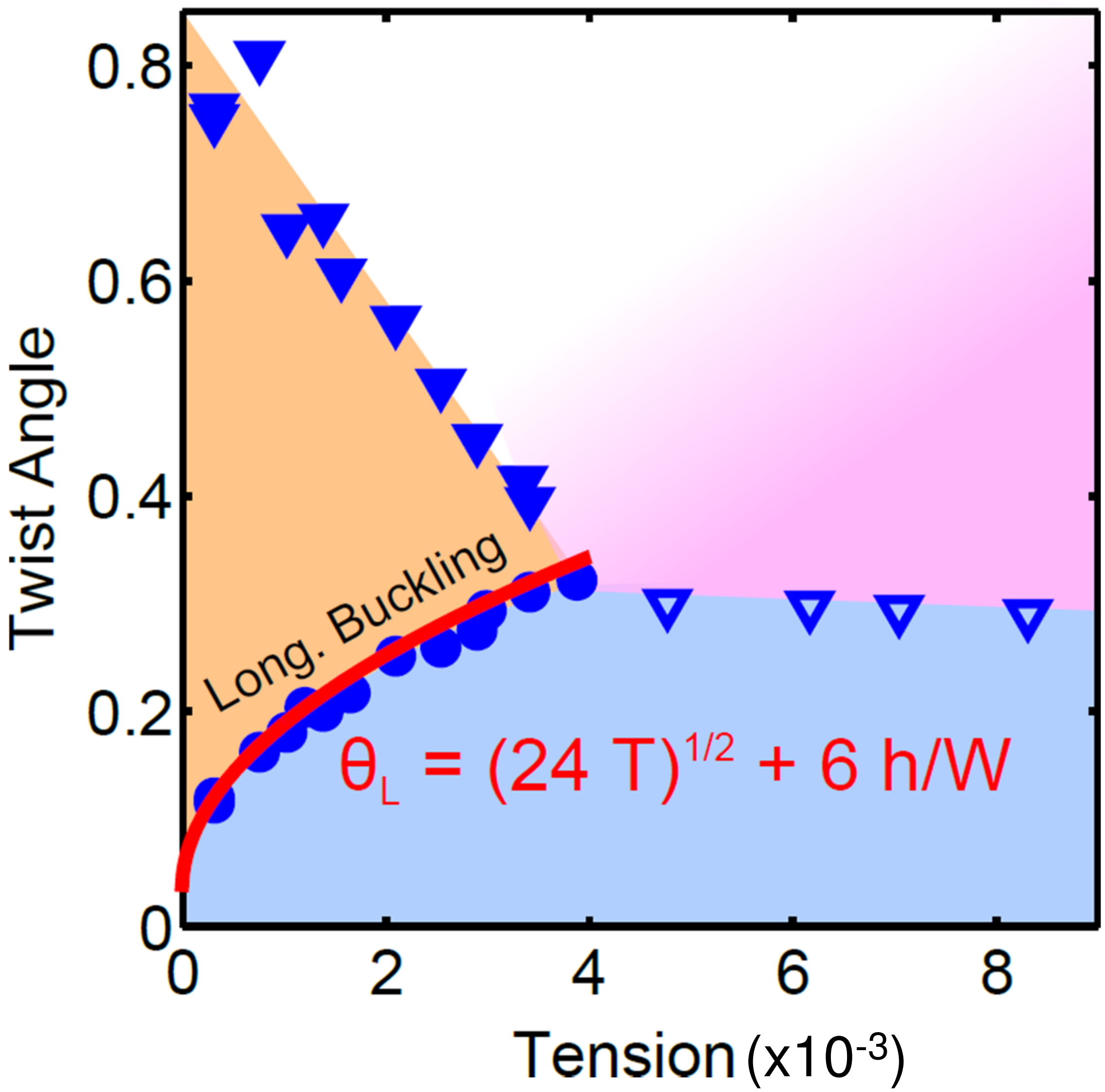
Chopin and Kudrolli, PRL (2013)

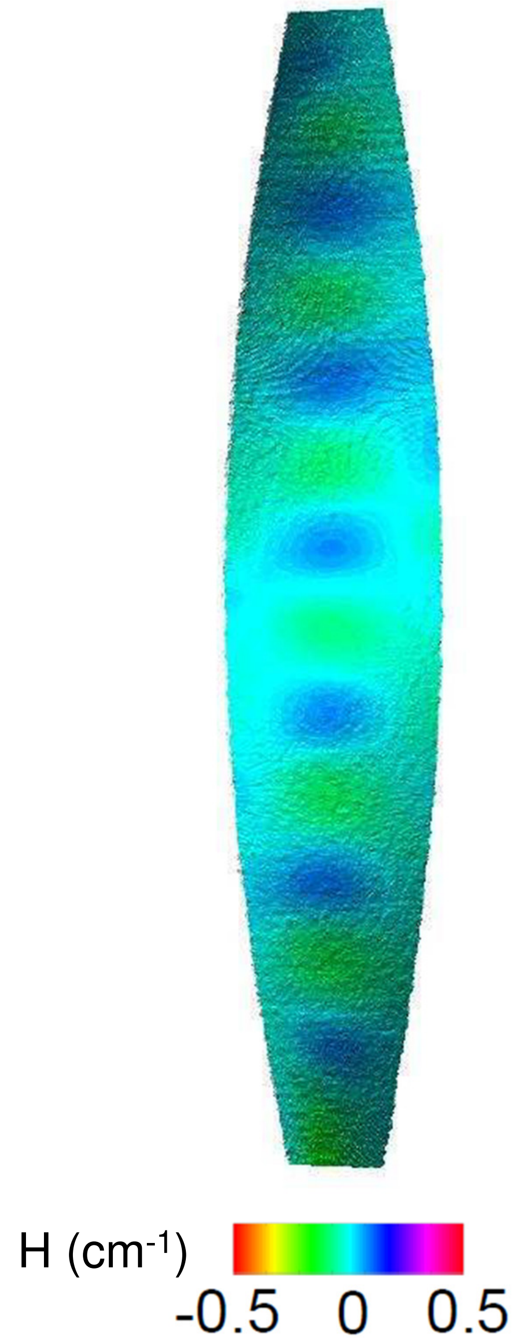
Comparison with experiments



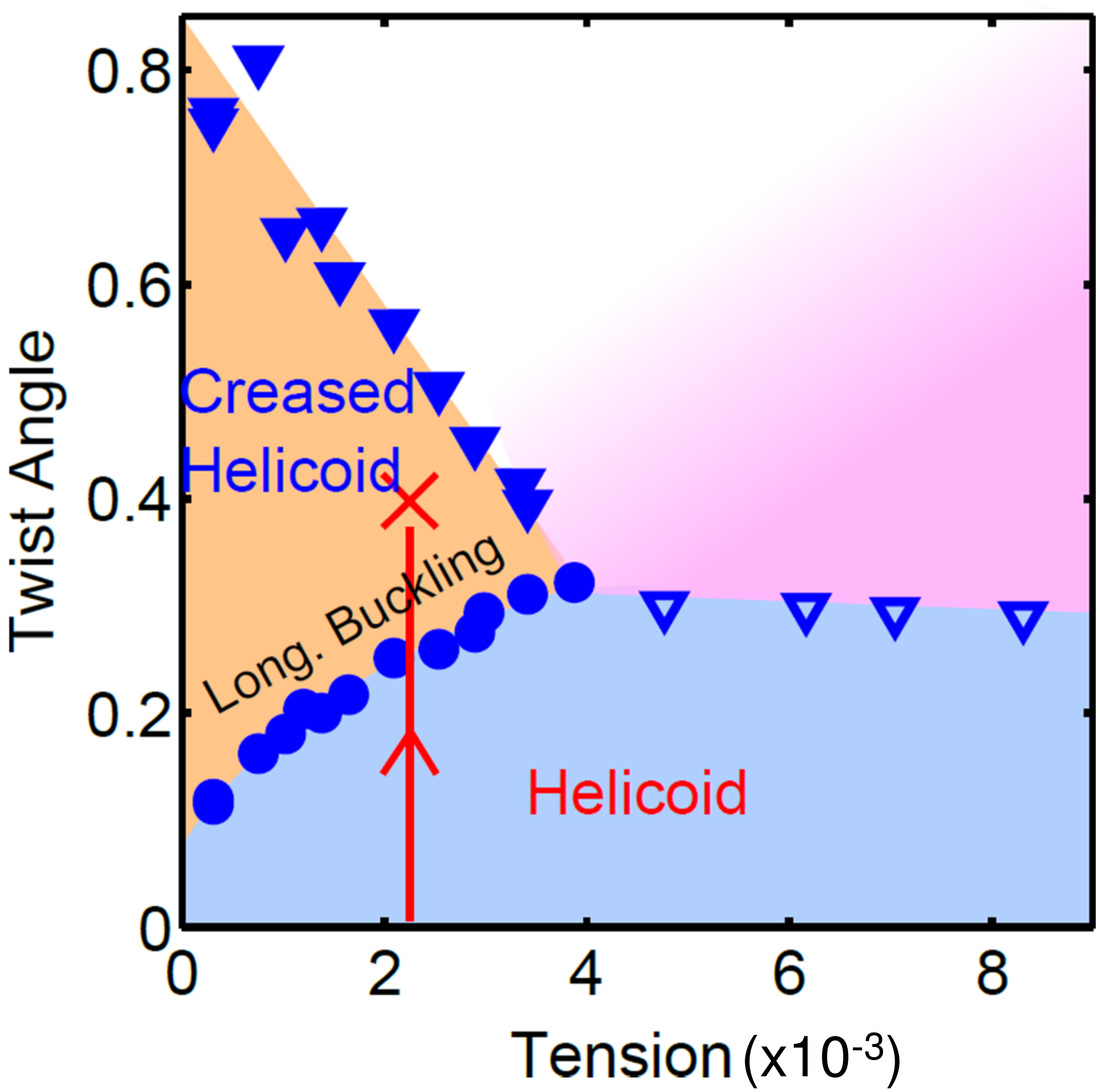
- Excess twist is observed increase linearly with h/W
- The wavelength is observed to decrease consistent with scaling analysis

Longitudinal buckling

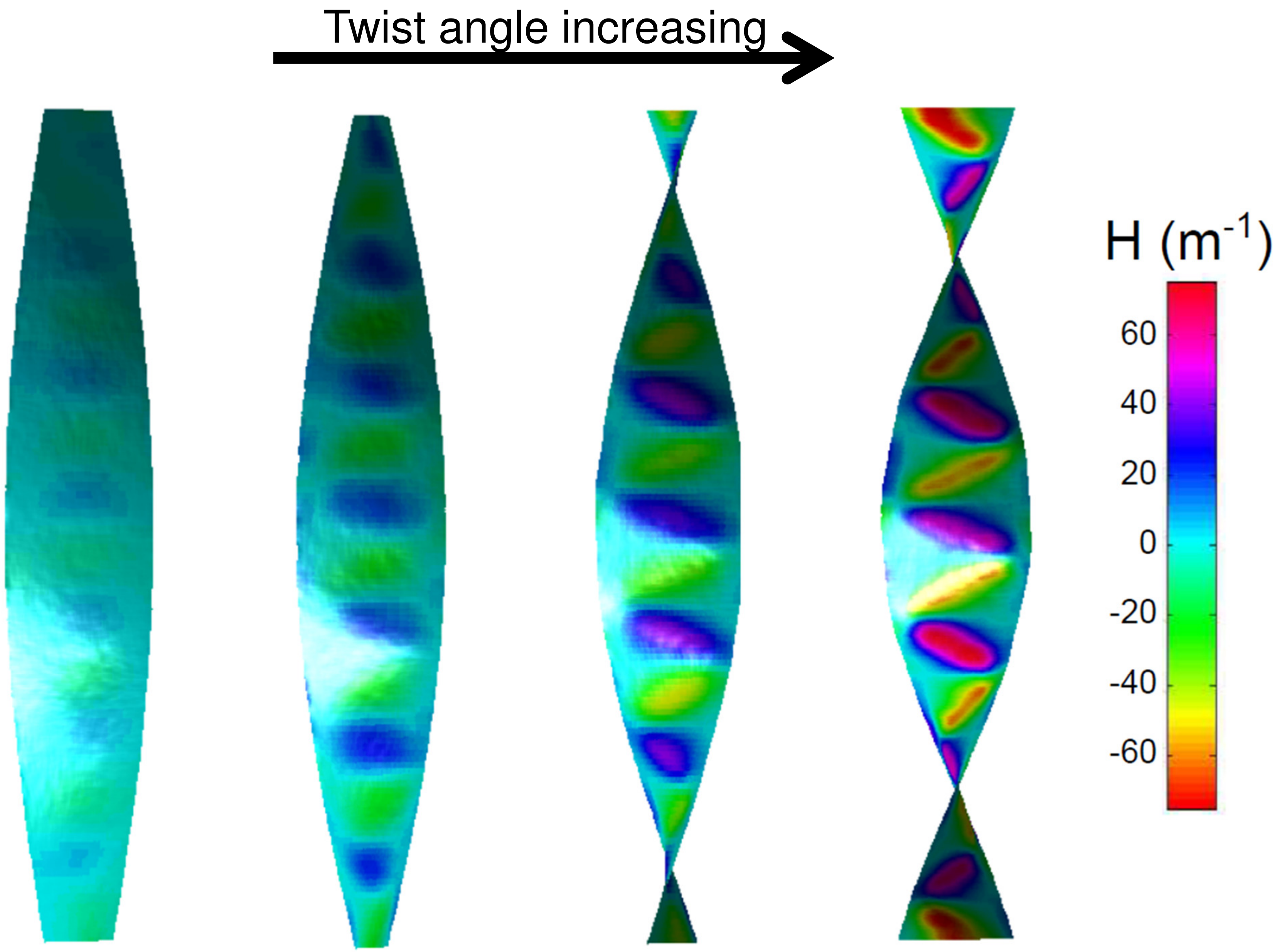




Creased helicoid



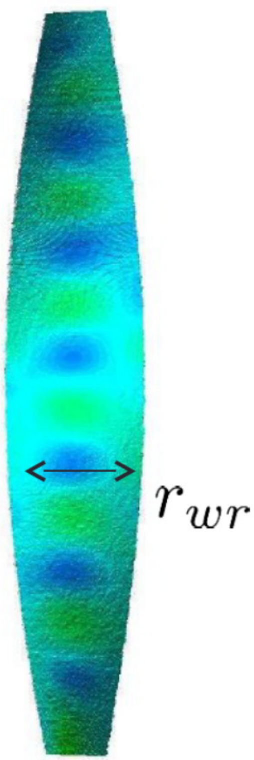
Creased helicoid?



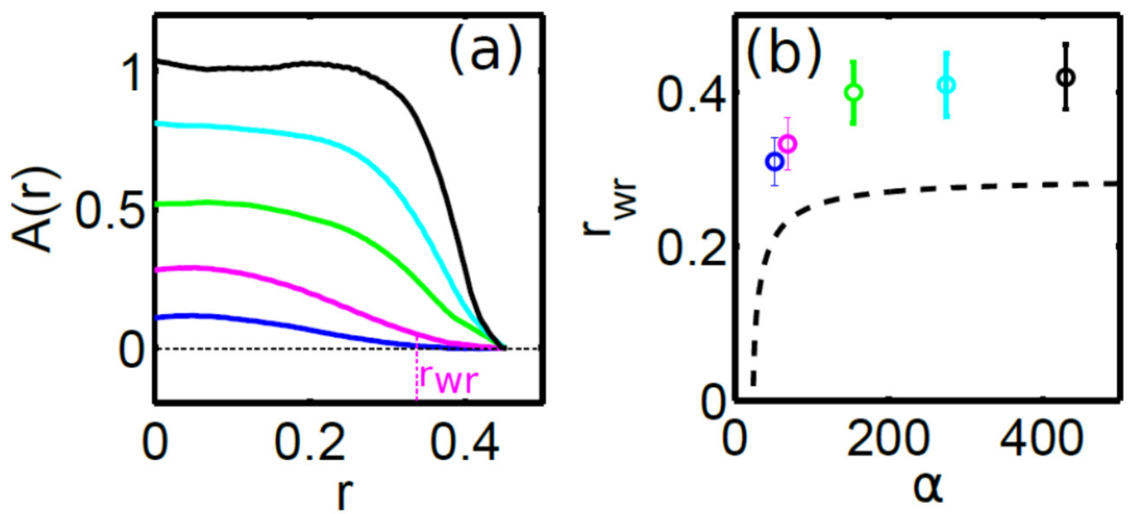
Far from threshold analysis

J. Chopin et al., J. Elasticity, **119**, 137 (2015)
B. Davidovitch *et al.*, PNAS **108**,18227 (2011)

Longitudinal buckling:



Amplitude of the wrinkling mode $A(r) = \sqrt{\langle H^2 \rangle_s}$

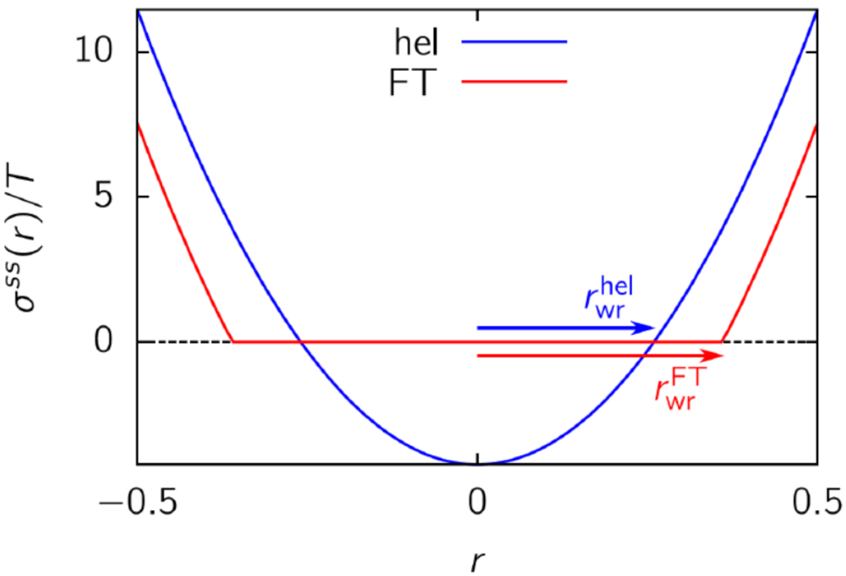


$r_{\rm wr} \sim \sqrt{\alpha - 24}.$

Far from threshold analysis

J. Chopin et al., J. Elasticity, **119**, 137 (2015)
B. Davidovitch *et al*., PNAS **108**,18227 (2011)

Longitudinal buckling:



Compression free stress field

$$\sigma_{\mathrm{FT}}^{ss}(r) = \begin{cases} 0 & \text{for} \quad |r| < r_{\mathrm{wr}}, \\ \dfrac{\eta^2}{2}\left(r^2 - r_{\mathrm{wr}}^2\right) & \text{for} \quad |r| > r_{\mathrm{wr}}. \end{cases}$$

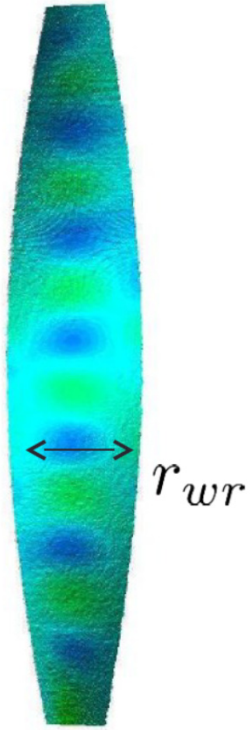
Far from threshold analysis

J. Chopin et al., J. Elasticity, **119**, 137 (2015)
B. Davidovitch *et al.*, PNAS **108**,18227 (2011)

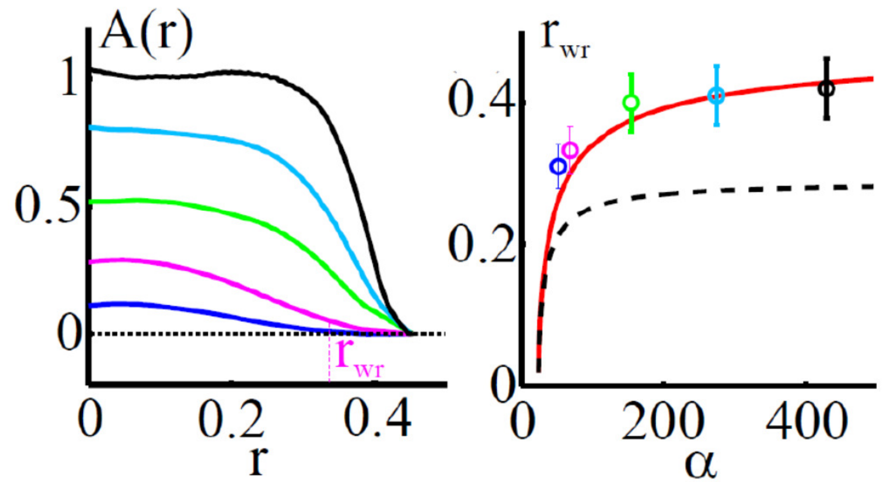
Longitudinal buckling:

Vertical mechanical equilibrium :

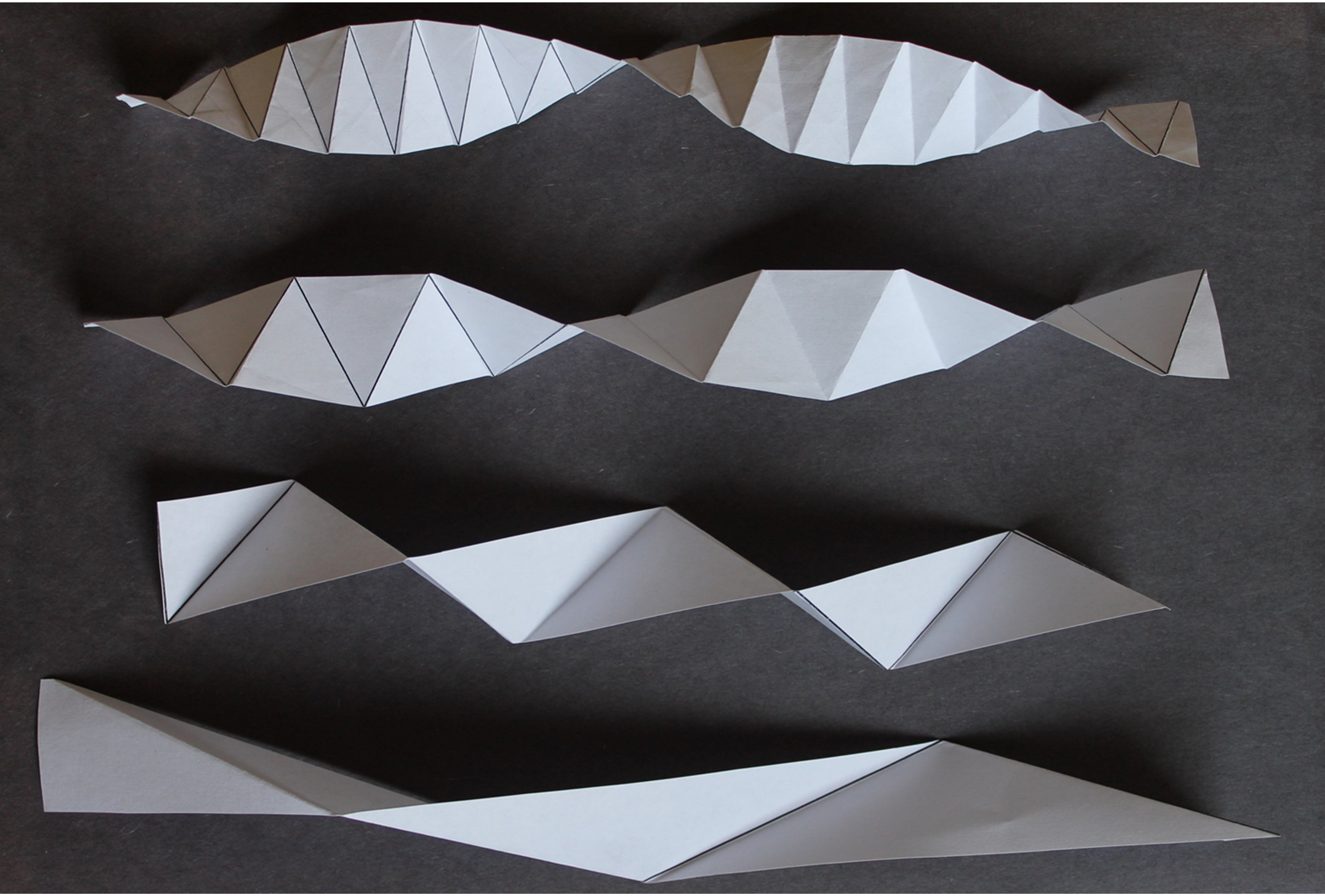
$$(1-2r_{\rm wr})^2(1+4r_{\rm wr}) = \frac{24}{\alpha}.$$



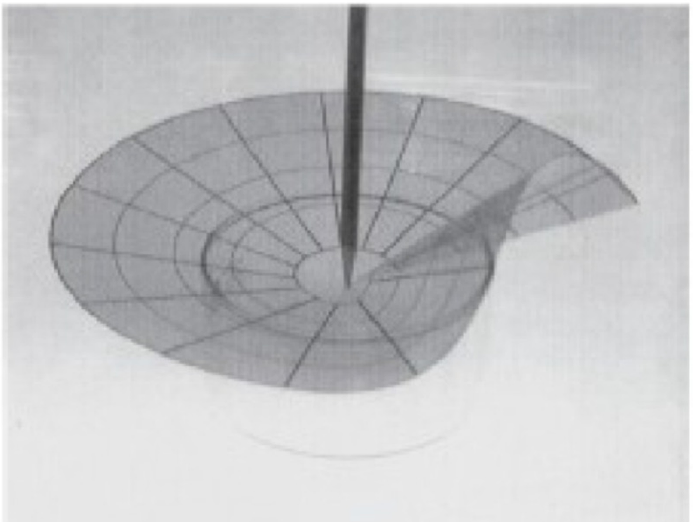
Amplitude of the wrinkling mode $A(r) = \sqrt{\langle H^2 \rangle_s}$



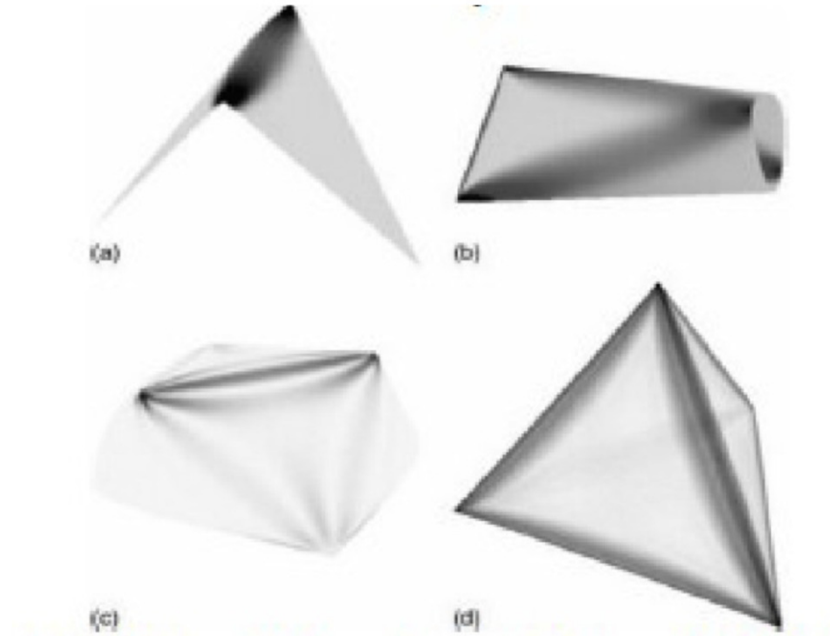
Creased Helicoid and Origami



Conical defects and ridges



Ben Amar & Pomeau (1997)
Cerda & Mahadevan (2003)

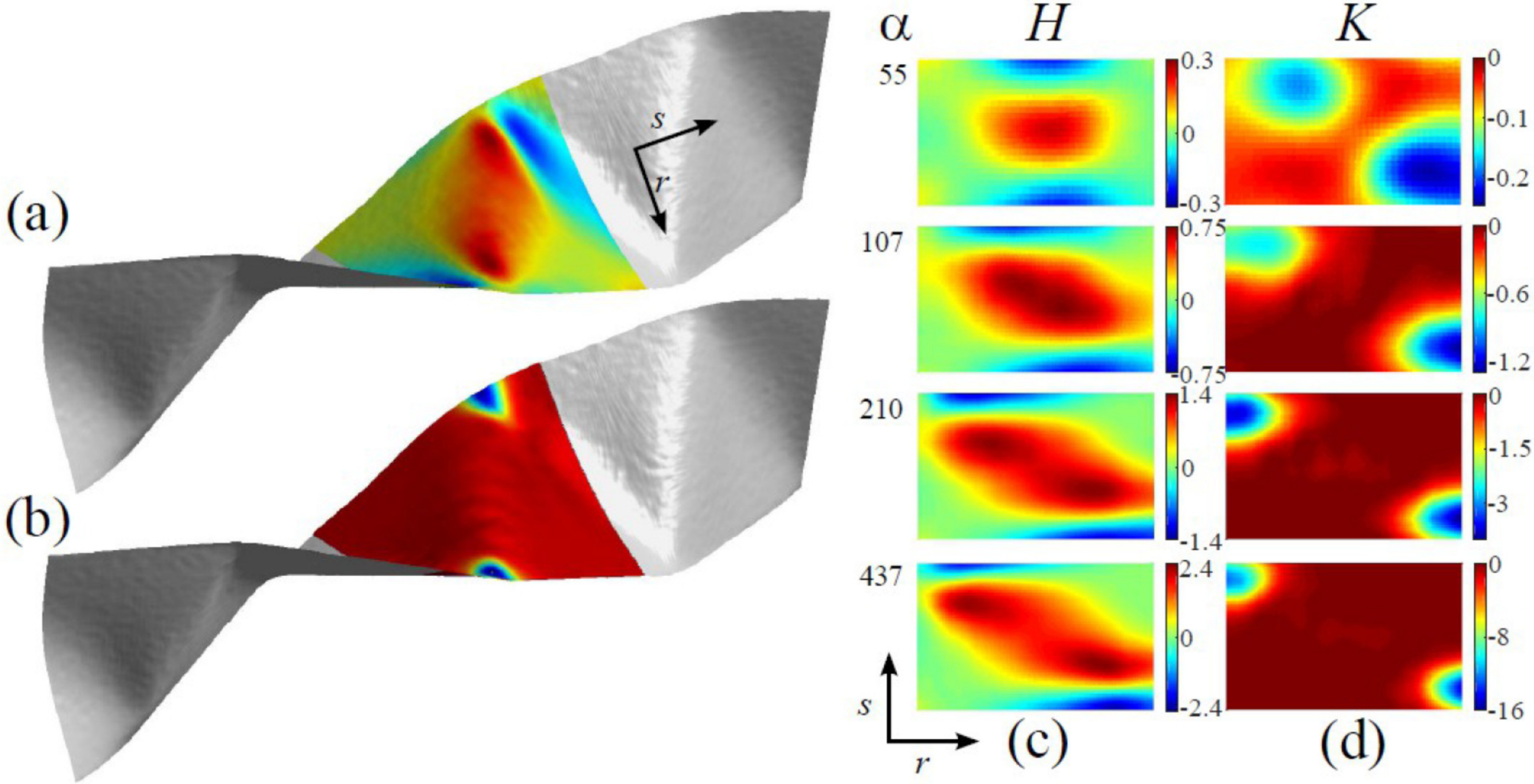


Witten (2007)

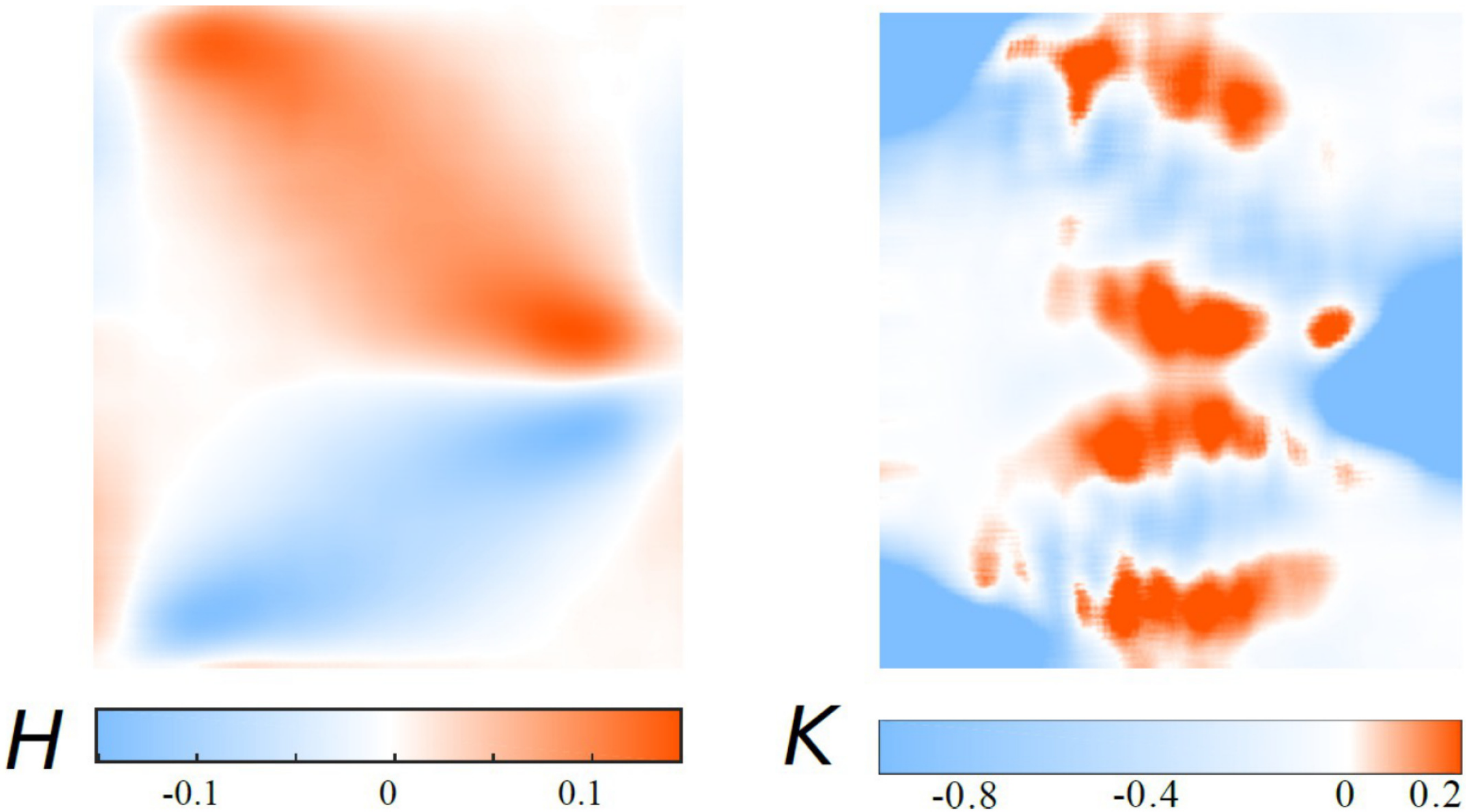
Venkataramani (2003)

- Can one decompose the triangular pattern into minimal ridges?

Ridges and Cones



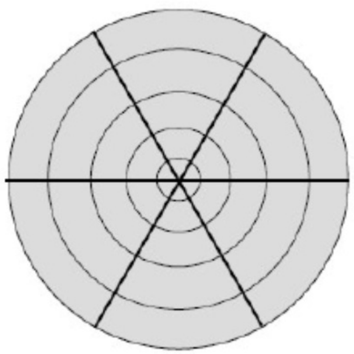
Magnified view



> -1.0 shown

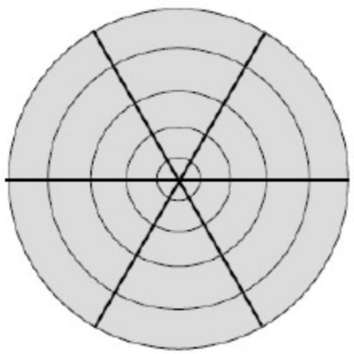
Developable cone

d-cone

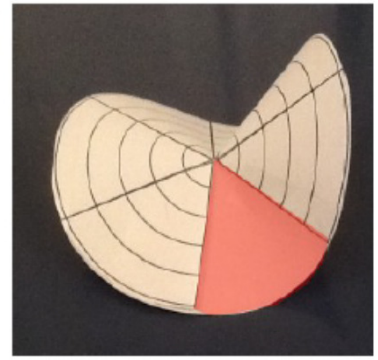


Developable cone

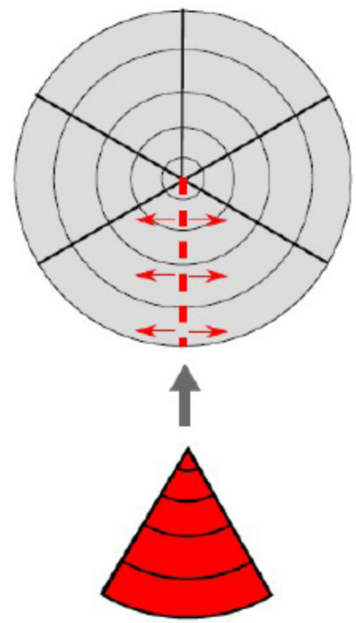
d-cone



Excess cone

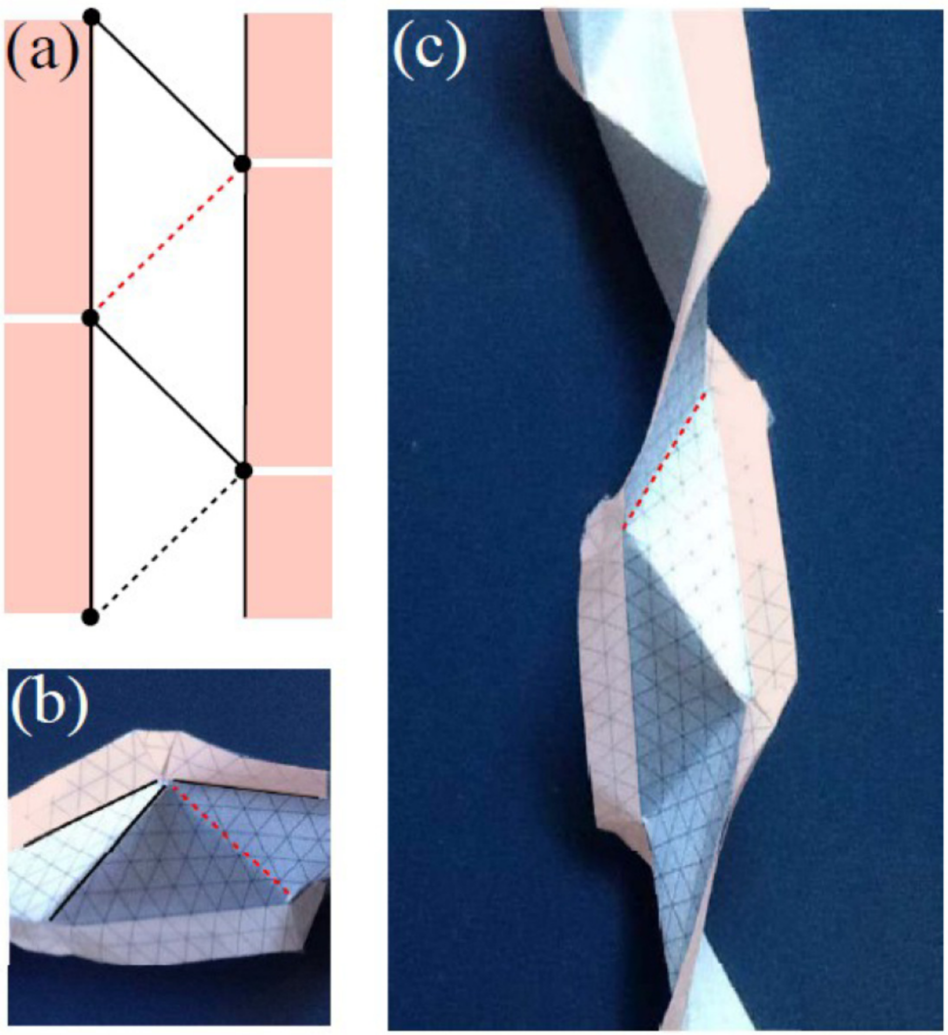


e-cone

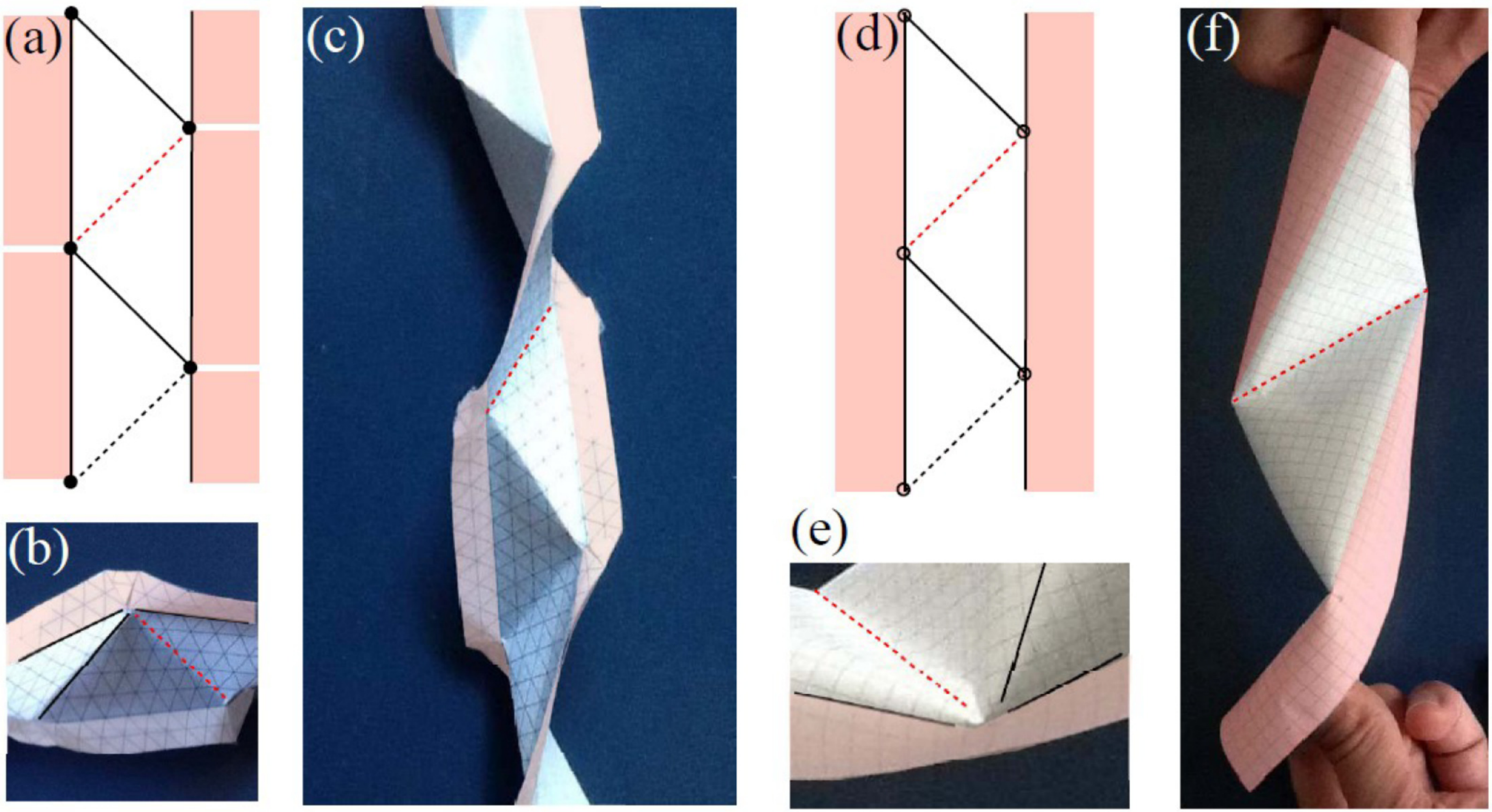


M. M. Müller, M. Ben Amar, and J. Guven, PRL (2008)

Paper Model



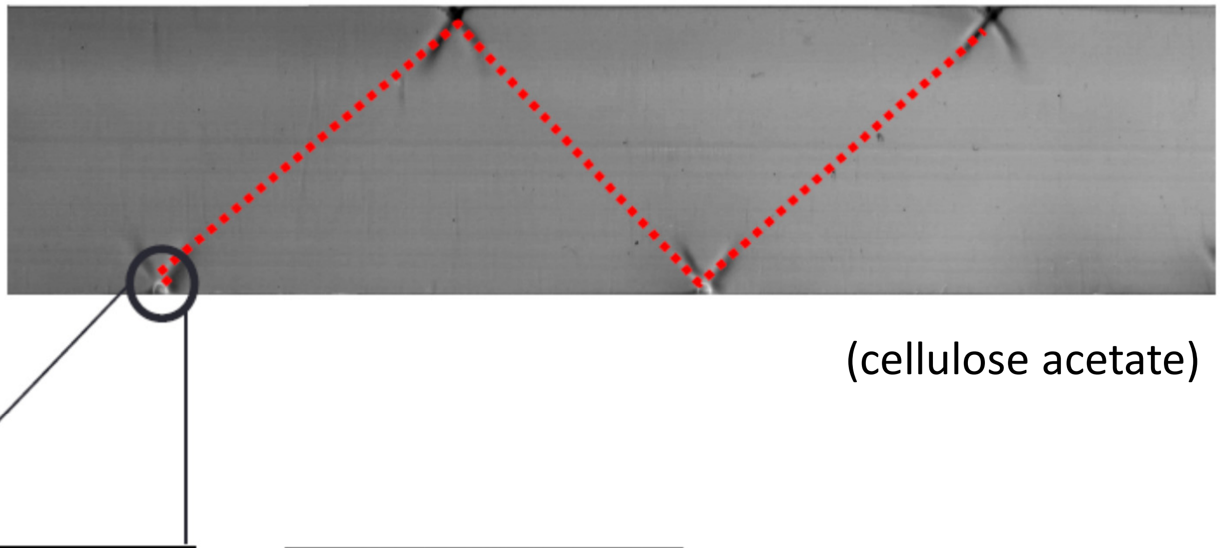
Paper Model



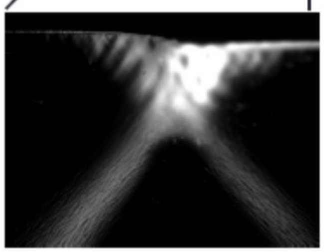
e-helicoid

d-helicoid

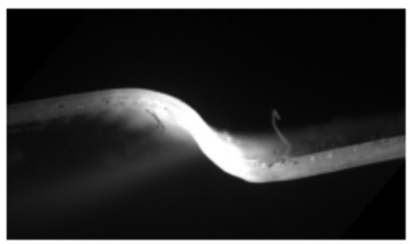
Observed plastic deformation



(cellulose acetate)



Front view



Side view

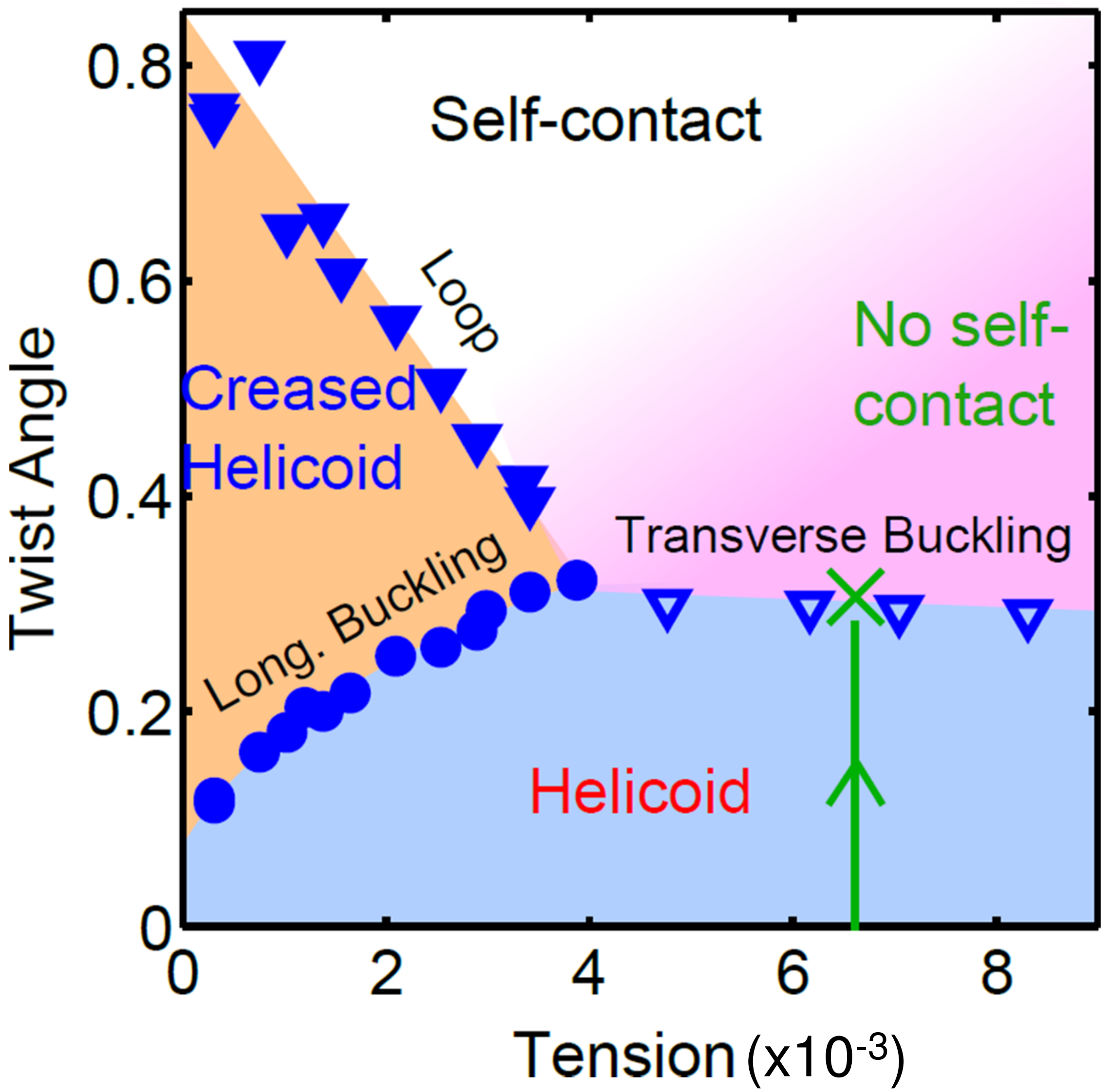
- Plastic deformation leads to e-cones centered inside the edge of the ribbon

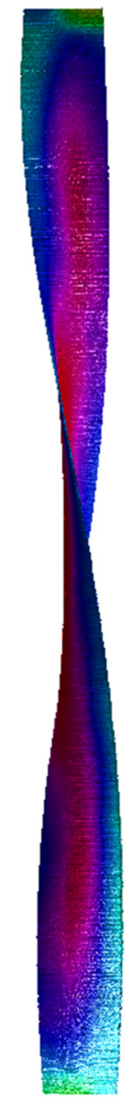
Extensible sheets

- Triangular lattice pattern in fact consists of ridges connecting e-cones

Disclinations, e-cones, and their interactions in extensible sheets
J. Chopin and A. Kudrolli, arXiv:1601.00575

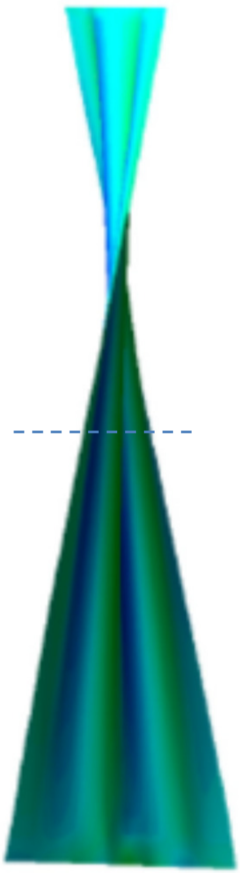
Transverse buckling

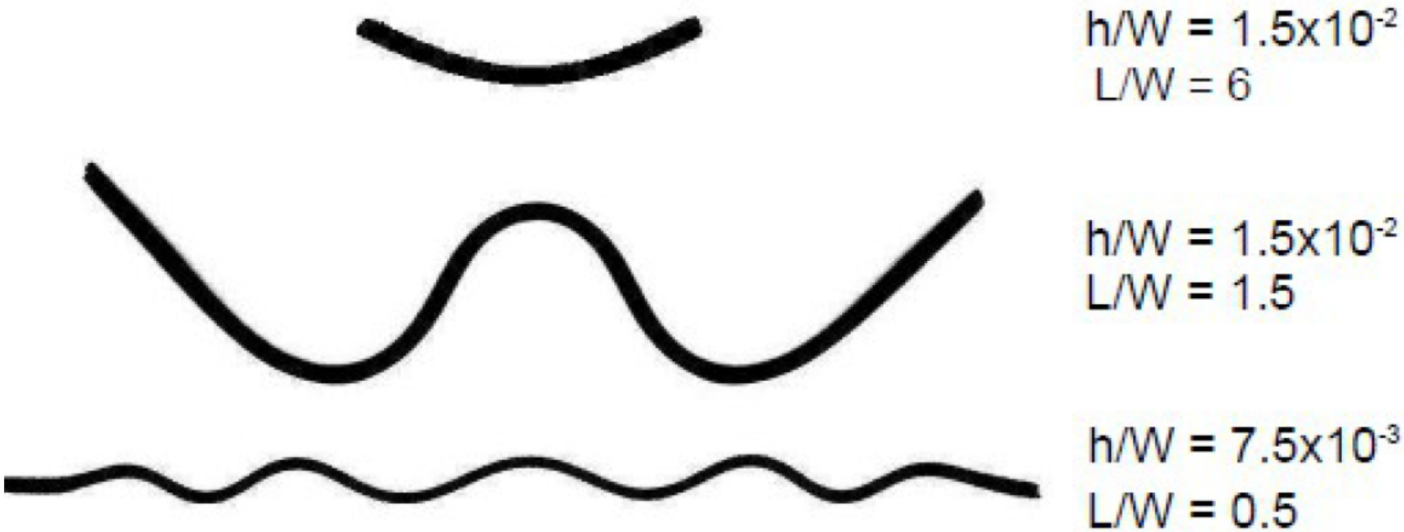




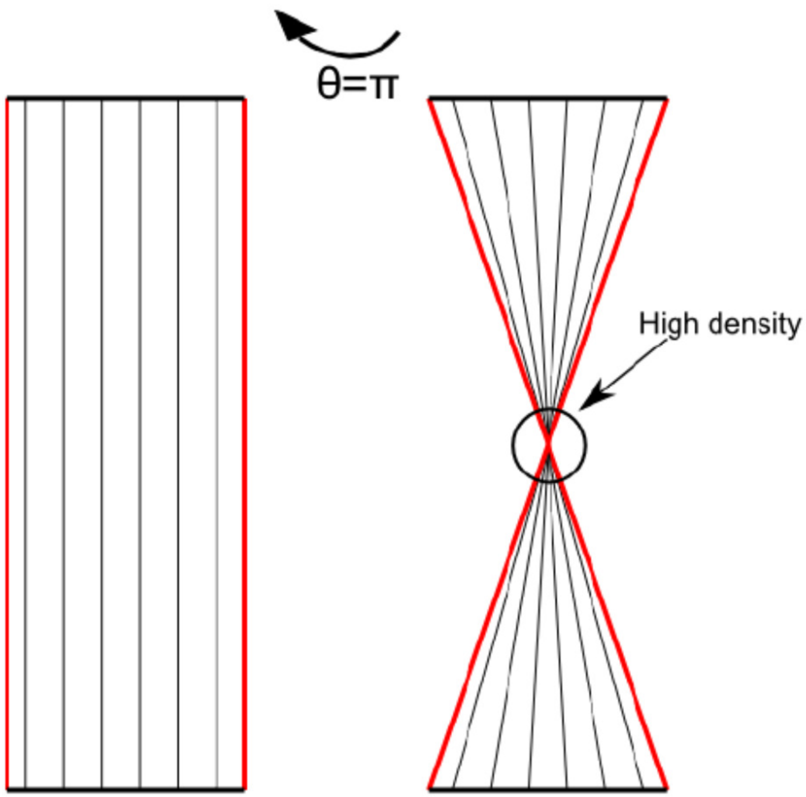
Transverse buckling

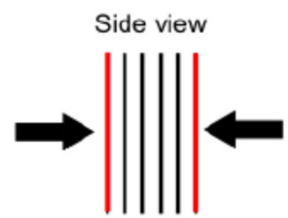
(central transverse cross section)



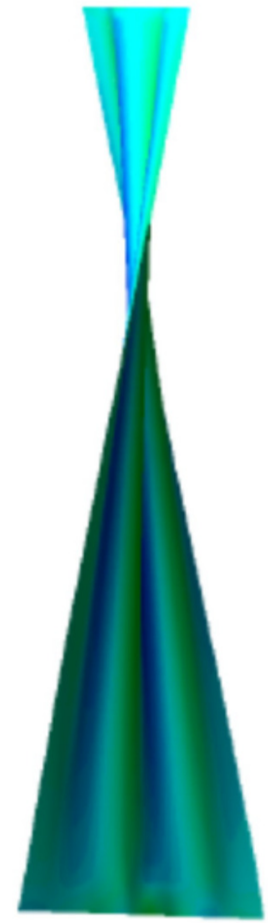


Transverse buckling





Confinement pressure
induced by the edges



Covariant form of F-vK and Ribbon Buckling

J. Chopin et al., J. Elasticity, **119**, 137 (2015)
B. Davidovitch *et al*., PNAS **108**,18227 (2011)

$$\sigma^{sr} = 0,$$
$$\partial_s \sigma^{ss} = 0,$$
$$\partial_r \sigma^{rr} \boxed{- \eta^2 r \sigma^{ss}} = 0.$$

$$\sigma^{ss}(r) = T + \frac{\eta^2}{2}\left(r^2 - \frac{1}{12}\right),$$

→ $$\sigma^{rr}(r) = \frac{\eta^2}{2}\left(r^2 - \frac{1}{4}\right)\left[T + \frac{\eta^2}{4}\left(r^2 + \frac{1}{12}\right)\right].$$

Scaling analysis for transverse buckling

J. Chopin et al., J. Elasticity, **119**, 137 (2015)

Stretching(orthogonal) : $\Delta U_S^{\perp} \sim \sigma^{rr} \left(\frac{A}{\lambda}\right)^2,$

Stretching(longitudinal) : $\Delta U_B^{||} \sim \sigma^{ss} \left(\frac{A}{L}\right)^2,$

Bending(orthogonal) : $\Delta U_B^{\perp} \sim B \left(\frac{A}{\lambda^2}\right)^2.$

The scalings for stresses are : $\sigma^{ss} \sim T\,;\ \sigma^{rr} \sim \eta^2 T$

Using $\Delta U_S^{\perp} \sim \Delta U_S^{||} \sim \Delta U_B^{\perp}$,

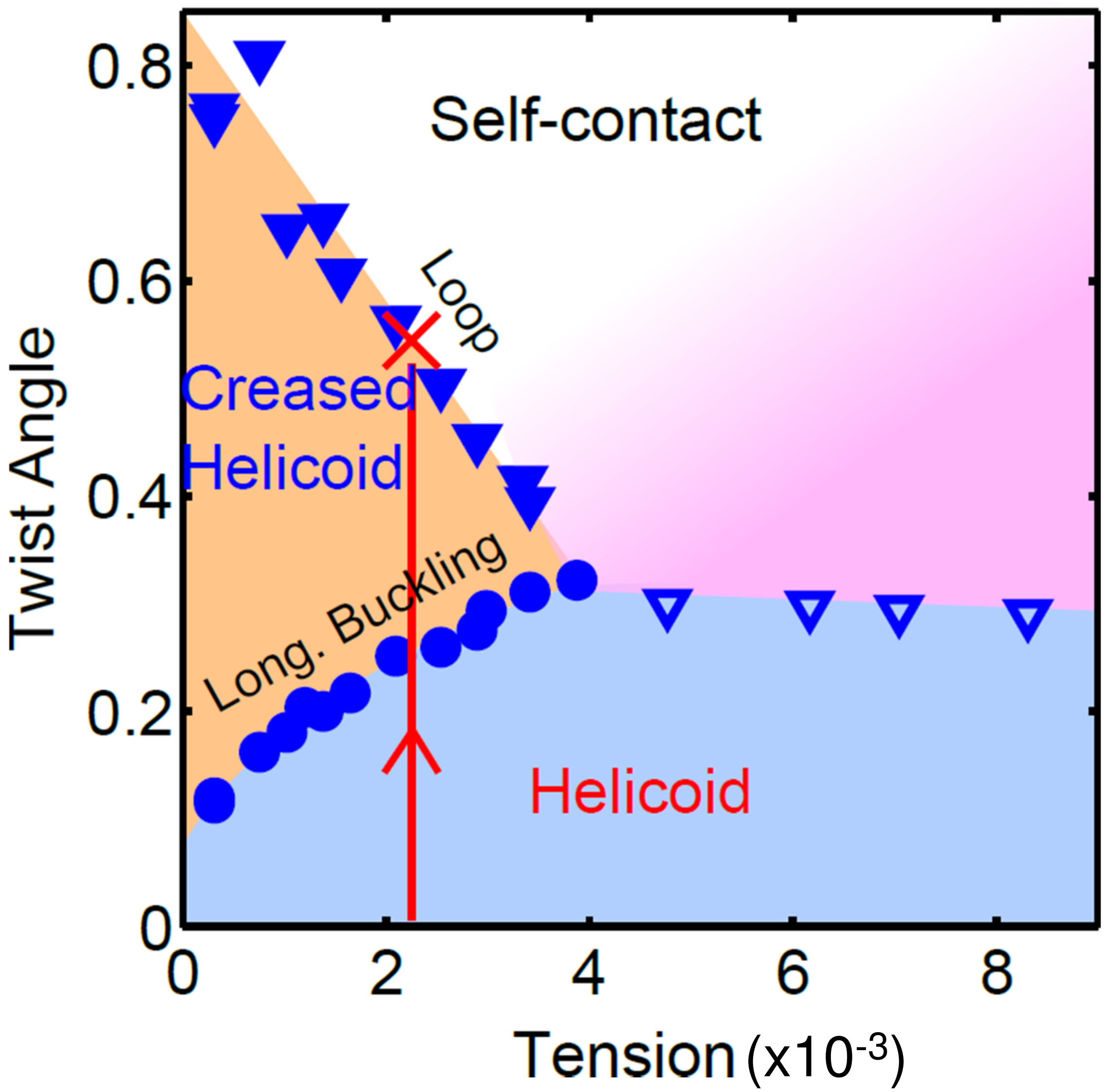
For L finite $\eta_{tr} \sim \sqrt{\frac{t}{L}} T^{-1/4}$

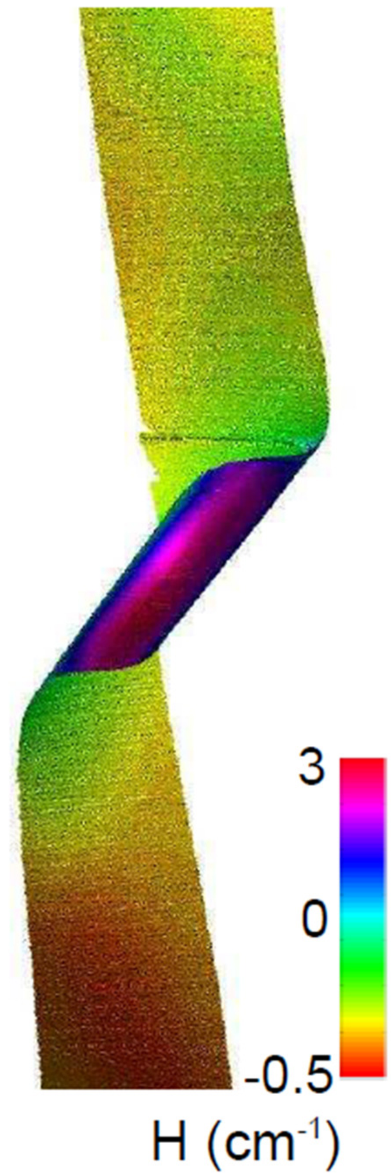
$\lambda_{tr} \sim \sqrt{Lt}\, T^{-1/4}$

For L infinite $\eta_{tr} \sim \frac{t}{\sqrt{T}}$

$\lambda_{tr} \sim 1$

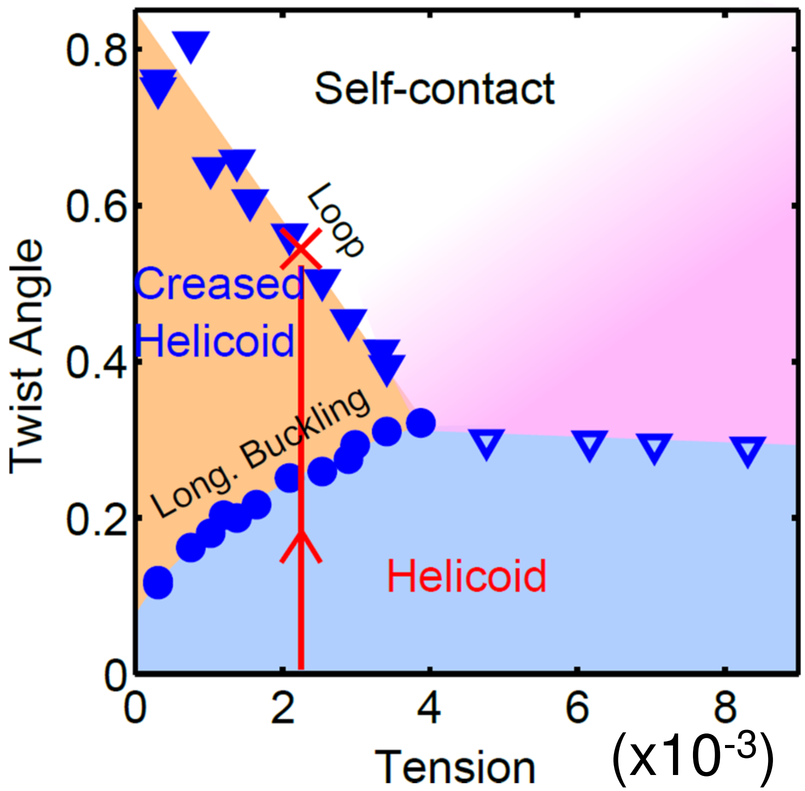
Anomalous loop transition

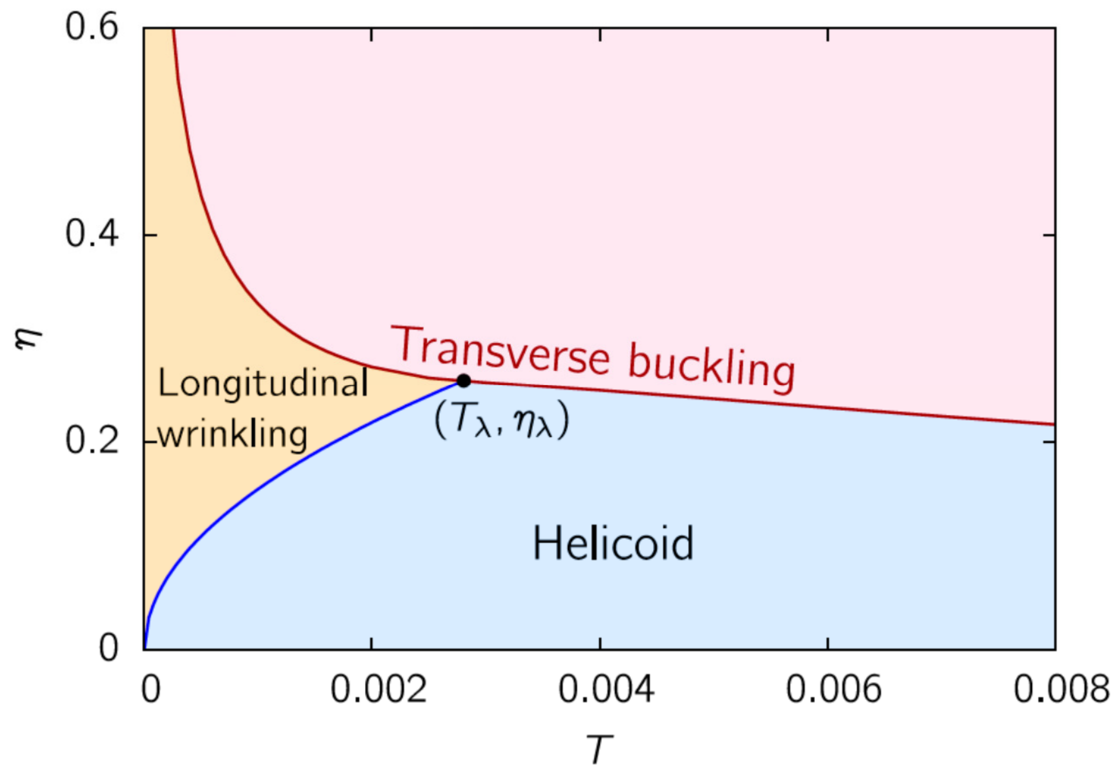




The critical angle decreases with the tension !

Anomalous loop transition





- Loop transition interpreted as a combination of longitudinal and transverse buckling

Roadmap to the morphological instabilities of a stretched twisted ribbon
J. Chopin*, V. Démery* and B. Davidovitch, *J. Elasticity* (2015)

Conclusions

- A stretched twisted ribbon exhibits a rich set of morphologies
- Linear stability analysis explains wrinkling instability near threshold
- Far from threshold approach (compression free stress field) capture some aspect of the morphology an mechanics deep inside the post-buckling regime
- Continuous transition from smooth wrinkled helicoid to a faceted ribbon
- A faceted ribbon corresponds to the shape resulting from interacting e-cones/negative disclinations organized on a triangular lattice.

References:

- Helicoids, Wrinkles, and Loops in Twisted Ribbons, J. Chopin and A. Kudrolli, PRL **111**, 174302 (2013).
- Disclinations, e-cones, and their interactions in extensible sheets, J. Chopin and A. Kudrolli arXiv:1601.00575 (2016).