The mechanics and geometry of real and fictitious elastic charges

Michael Moshe

Syracuse University & Harvard University

KITP - March 2016

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(in 2D)

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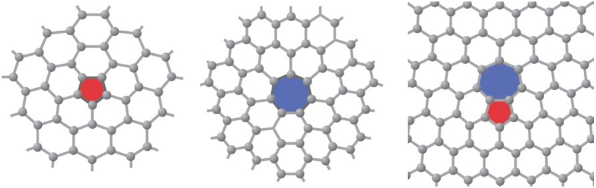
Examples of singular sources of deformations

Defects in crystalline solids



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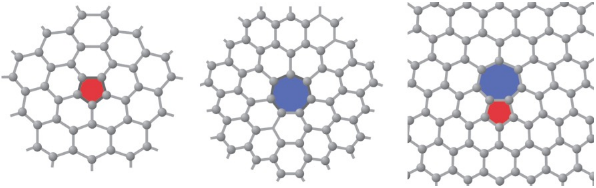
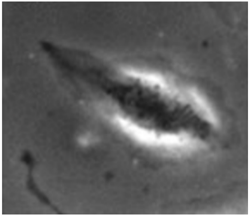


Contractile cells



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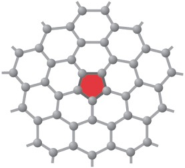

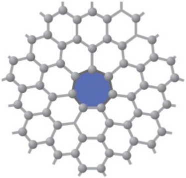

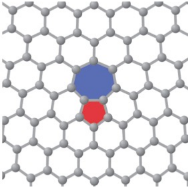

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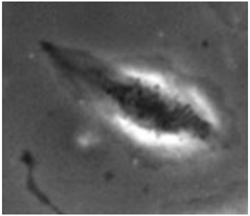

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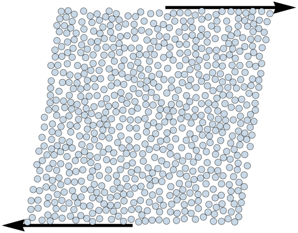

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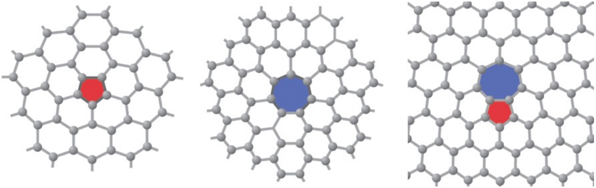

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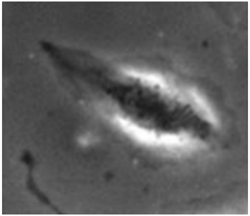

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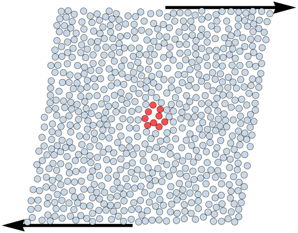

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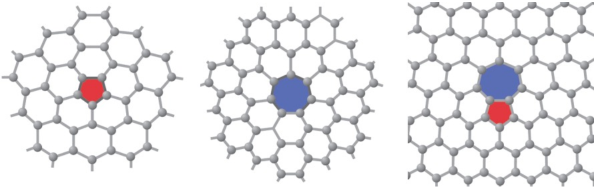

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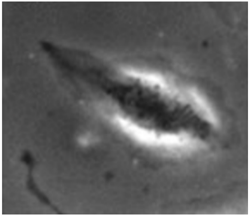

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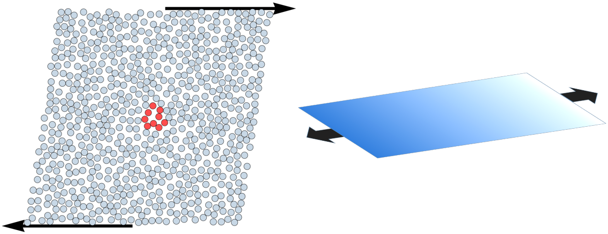

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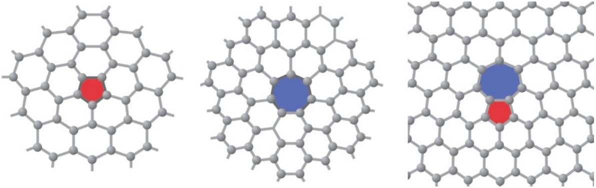

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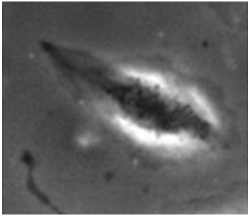

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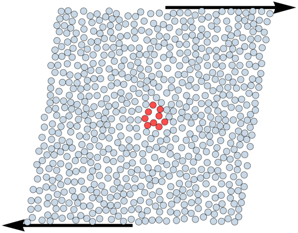
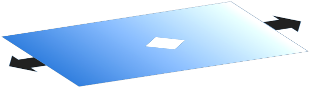
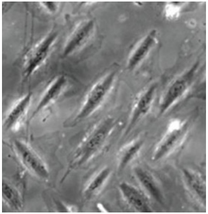


Holes

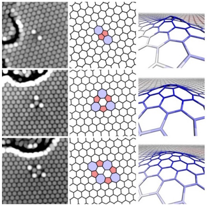


Challenges related with singular sources in elasticity

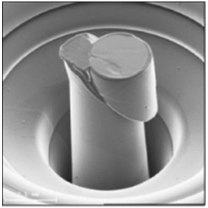
Collective behavior of cells[1]



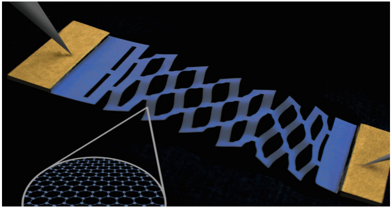
Mechanics of complex defects



Failure of amorphous solids



Kirigami



[1] Livne et al. 2014 Nature Communications

Talk plan

- The formalism
- Geometry of elastic charges
- Mechanics of elastic charges
- Fictitious elastic charges

Geometric formulation of elasticity

Why?

Geometric formulation of elasticity - 2D

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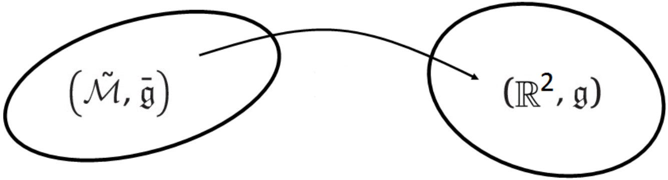

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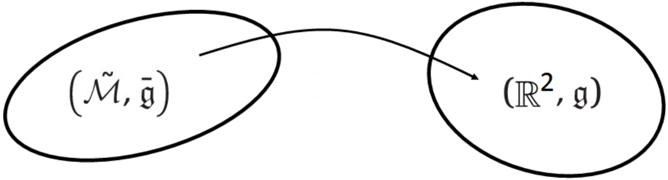

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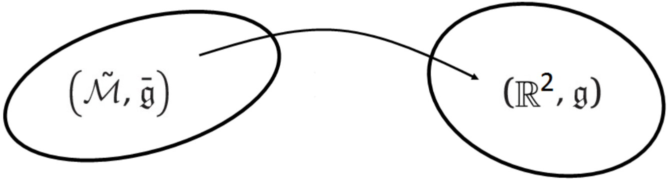
$$u_{ij} = \frac{1}{2}\left(\mathfrak{g}_{ij} - \bar{\mathfrak{g}}_{ij}\right)$$



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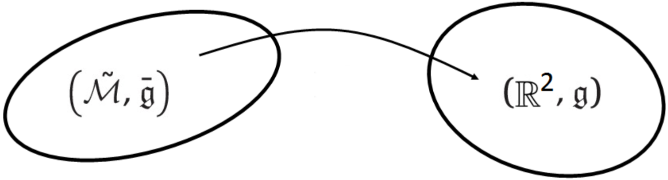
$$E = \int \mathcal{W}\left(\mathfrak{g}, \bar{\mathfrak{g}}\right) \mathrm{d}V$$



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For example - a Hookean constitutive law

$$\mathcal{W}\left(\mathfrak{g}, \bar{\mathfrak{g}}\right) = \frac{1}{2} A^{ijkl} u_{ij} u_{kl}$$

A is the elastic tensor.

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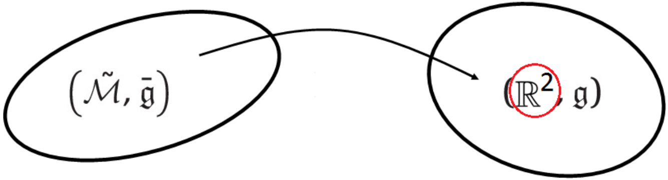
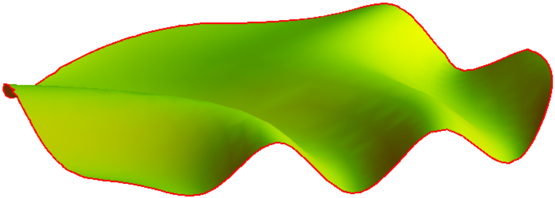


Intuition



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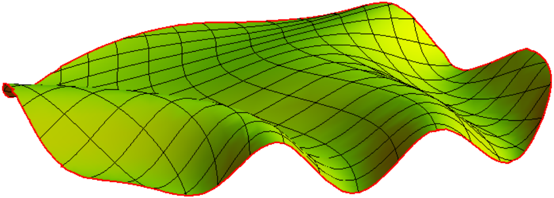

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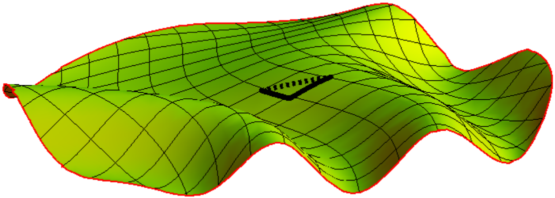

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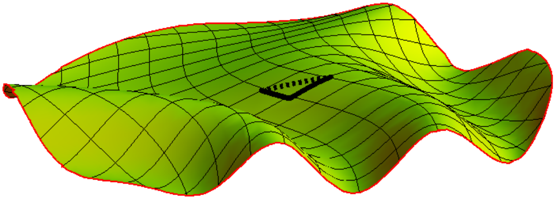

$$\bar{d}l^2 = \bar{\mathfrak{g}}_{11} du^2 + 2\bar{\mathfrak{g}}_{12} du\, dv + \bar{\mathfrak{g}}_{22} dv^2$$

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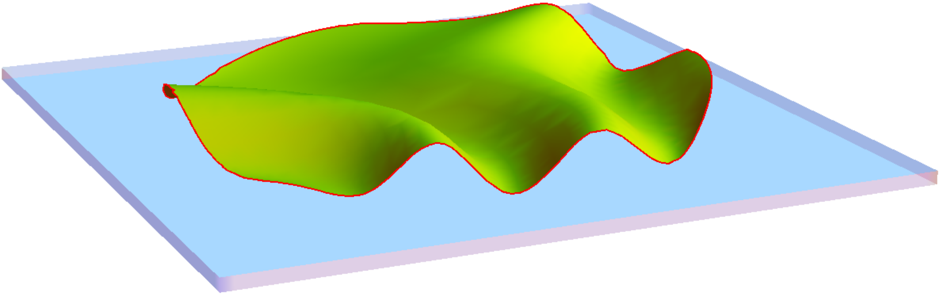

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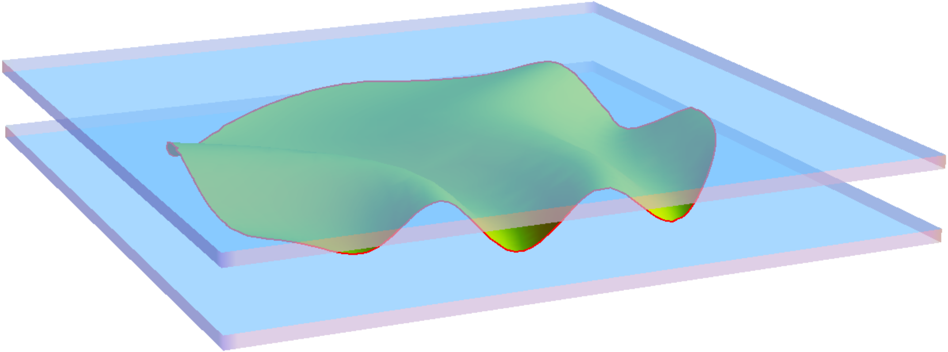

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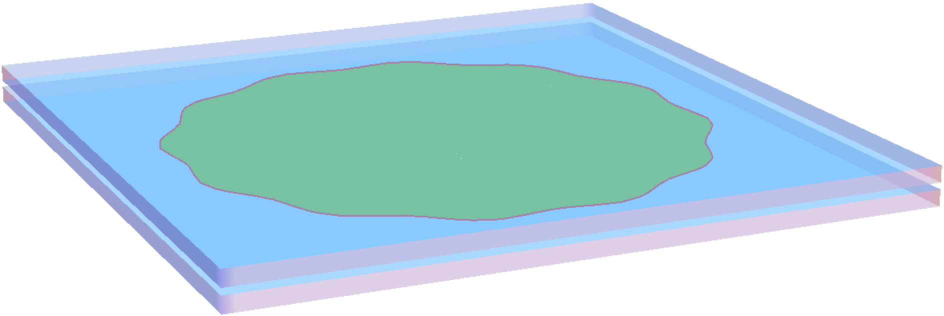

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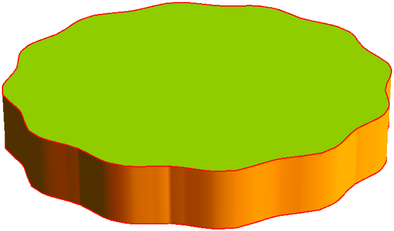

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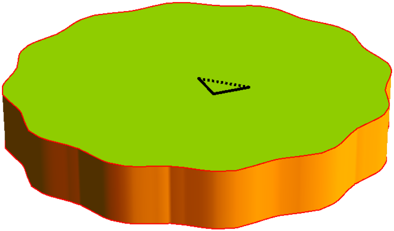

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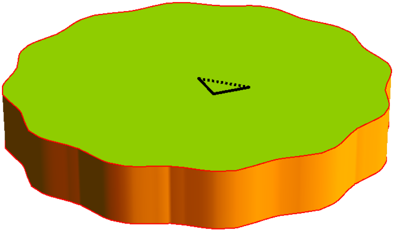

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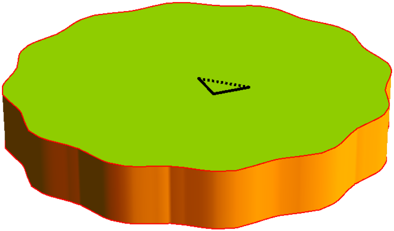

$$\mathfrak{g} = \bar{\mathfrak{g}}?$$

Incompatibility and residual stresses

- Two manifolds are isometric iff their curvatures are identical

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Euclidean $\bar{\mathfrak{g}}$ $\quad \Rightarrow \quad \bar{K}_G = 0 \quad \Rightarrow \quad$ Compatibility

Non-Euclidean $\bar{\mathfrak{g}}$ $\quad \Rightarrow \quad \bar{K}_G \neq 0 \quad \Rightarrow \quad$ Incompatibility

Incompatibility and residual stresses

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Euclidean $\bar{\mathfrak{g}}$ $\Rightarrow$ $\bar{K}_G = 0$ $\Rightarrow$ Compatibility

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$\bar{K}_G$ *is the source for residual stresses*

The elastic problem

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What is the reference metric of an elastic charge? (Geometry)

How this functional can be minimized in practice? (Mechanics)

Geometric description of elastic charges[2]

[2] M.M. et al. 2015 PNAS

The solution

Multipoles of curvature

$$\bar{K} = \mathrm{D}^n \delta\left(\vec{x}\right)$$

$$\mathrm{D}^0 = q, \quad \mathrm{D}^1 = \vec{b} \cdot \vec{\nabla}, \quad \mathrm{D}^2 = \vec{\nabla} \cdot \mathbf{Q} \cdot \vec{\nabla}, \quad ...$$

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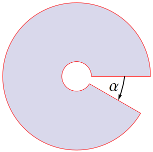
A conformal representation of the reference metric

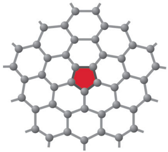
$$\bar{\mathfrak{g}} = e^{2\phi(r,\theta)} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\phi\left(r,\theta\right) = \beta + \alpha \ln(r) + \textstyle\sum_{n=1}^{\infty} r^{-n} \left(\alpha_n \sin\left(n\theta\right) + \beta_n \cos\left(n\theta\right)\right)$$

The monopole

$\phi = \frac{\alpha}{2\pi} \ln r$





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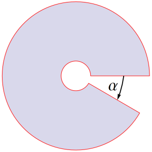

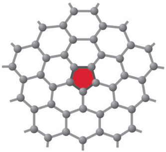

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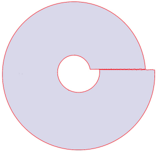

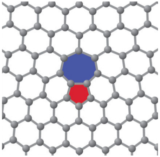

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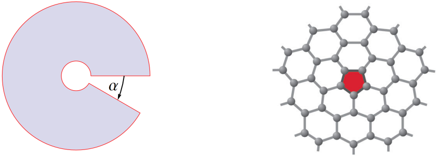

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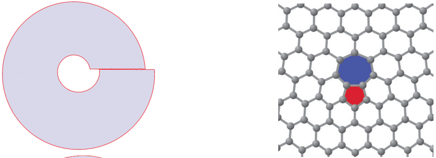
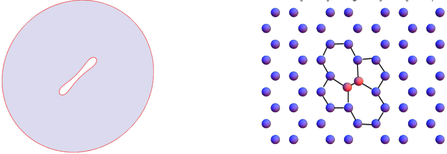


The quadrupole

$\phi = \frac{Q \cos 2\theta}{2\pi r^2}$



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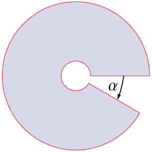

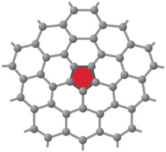

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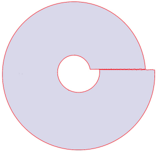

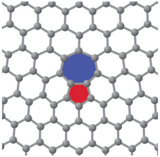

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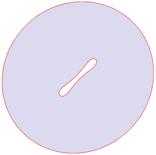
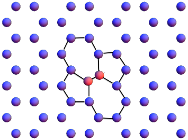
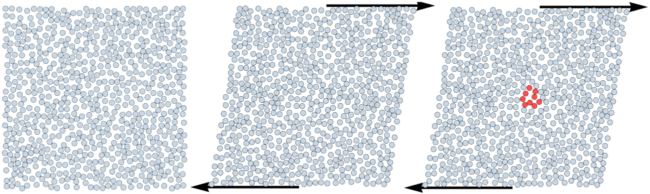




Why is it good?

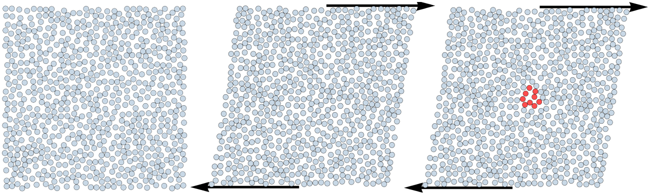
Localized plastic deformation

Defects in amorphous solids



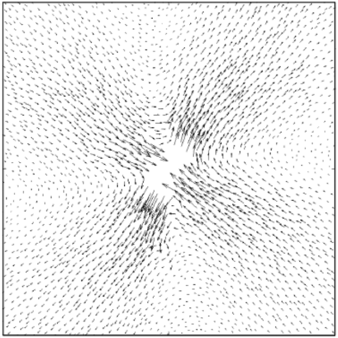
Localized plastic deformation

Defects in amorphous solids



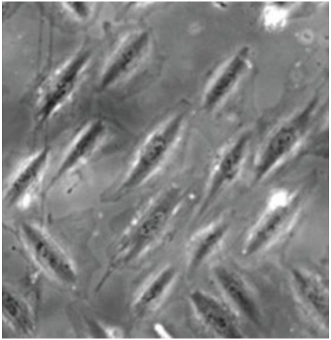
Elastic charge at least of quadrupolar order

Displacement field of a localized plastic deformation[3]



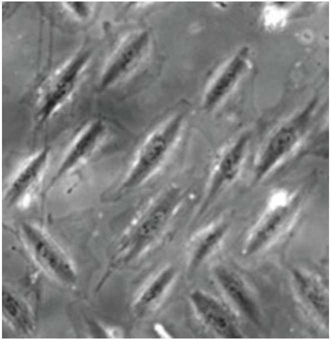
[3] Dasgupta et al. 2012 PRL

The description of several elastic charges[4]



[4] Livne et al. 2014 Nature Communications

The description of several elastic charges[4]



Additivity of the conformal factor

$$\bar{\mathfrak{g}} = e^{2\varphi} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad \varphi = \sum \varphi_i$$

[4] Livne et al. 2014 Nature Communications

Mechanics of elastic charges[5]

[5] M.M., et al. 2015 PRE

The equilibrium equation

Define the stress tensor: $\sigma^{\mu\nu} = \frac{\partial \mathcal{W}(\mathfrak{g},\bar{\mathfrak{g}})}{\partial u_{\mu\nu}}$

The equilibrium equation

$$\left(\nabla_\mu + \bar{\Gamma}^\lambda_{\lambda\mu} - \Gamma^\lambda_{\lambda\mu}\right)\sigma^{\mu\nu} = 0 \qquad \sigma^{\mu\nu} n_\nu = 0$$

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A nonlinear equation for $\mathfrak{g}$!

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A nonlinear equation for $\mathfrak{g}$!

Representation if the solution

$$\sigma^{\mu\nu} = \sqrt{\frac{1}{|\mathfrak{g}|}}\sqrt{\frac{1}{|\bar{\mathfrak{g}}|}}\varepsilon^{\mu\alpha}\varepsilon^{\nu\beta}\nabla_\alpha\bar{\nabla}_\beta\psi$$

Solving the equilibrium equation

$$\mathfrak{g} = \bar{\mathfrak{g}} + \eta \mathfrak{g}^{(1)} + \eta^2 \mathfrak{g}^{(2)} + O(\eta^3)$$

$$\psi = \eta \psi^{(1)} + \eta^2 \psi^{(2)} + O(\eta^3)$$

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$$\frac{1}{Y} \bar{\Delta} \bar{\Delta} \psi^{(1)} + \frac{2\bar{K}_G}{Y} \bar{\Delta} \psi^{(1)} + \frac{1}{Y} \left(1 + \nu_p\right) \bar{\mathfrak{g}}^{\mu\nu} \left(\partial_\mu \bar{K}_G\right) \left(\partial_\nu \psi^{(1)}\right) = \bar{K}_G$$

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$$\bar{K}_G \sim \eta \rightarrow \tfrac{1}{Y} \bar{\Delta} \bar{\Delta} \psi^{(1)} = \bar{K}_G$$

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$$\bar{K}_G \sim \eta \rightarrow \tfrac{1}{Y} \bar{\Delta} \bar{\Delta} \psi^{(1)} = \bar{K}_G$$

$$\boxed{E = \tfrac{1}{2} \int \psi^{(1)} \bar{K}_G \, dS}$$

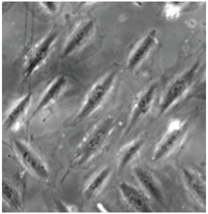
Interacting elastic multipoles

$$\boxed{U_{12} = \int \psi_1 \bar{K}_2 \, dS}$$

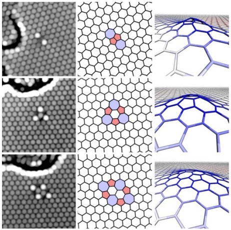
Defect type	ψ	$\bar{K}_G$
Monopole	$\frac{Y}{8\pi} q \lvert\mathbf{x}\rvert^2 \left(\ln \lvert\mathbf{x}\rvert - 1\right)$	$q\,\delta(\mathbf{x})$
Dipole	$\frac{Y}{4\pi} (\mathbf{p} \cdot \mathbf{x}) \ln \lvert\mathbf{x}\rvert$	$(\mathbf{p} \cdot \nabla)\delta(\mathbf{x})$
Quadrupole	$\frac{Y}{16\pi} (\hat{\mathbf{x}}^T \cdot \mathbf{Q} \cdot \hat{\mathbf{x}})$	$\frac{1}{4}(\nabla^T \cdot \mathbf{Q} \cdot \nabla)\delta(\mathbf{x})$
Point	$\frac{Y}{16\pi} (\hat{\mathbf{x}}^T \cdot \mathbf{C} \cdot \hat{\mathbf{x}})$	$\frac{1}{4}(\nabla^T \cdot \mathbf{C} \cdot \nabla)\delta(\mathbf{x})$
External	$\frac{1}{2} \mathbf{x}^T \cdot \mathrm{Cof}(\sigma) \cdot \mathbf{x}$	–

Challenges related with singular sources in elasticity

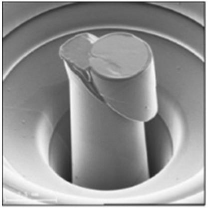
Collective behavior of cells



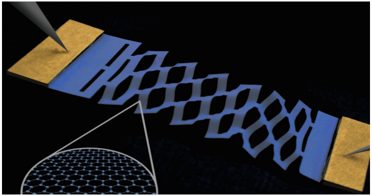
Mechanics of complex defects



Failure of amorphous solids



Interactions between holes



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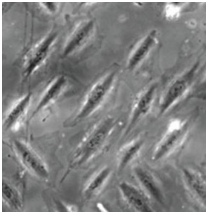

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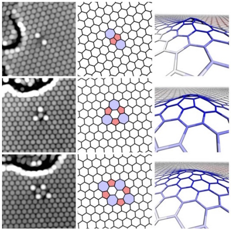

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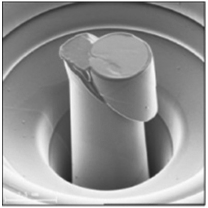
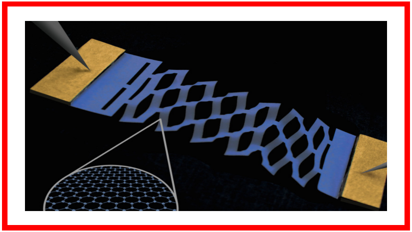


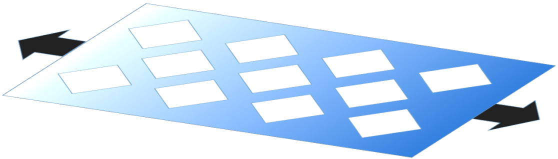
Kirigami



Mechanics of Kirigami

A small taste from a work in progress

Array of frames



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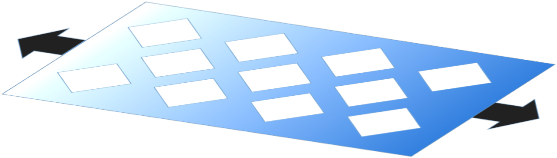
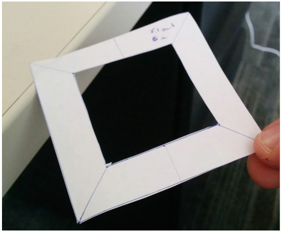
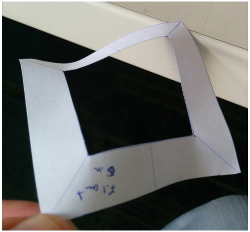
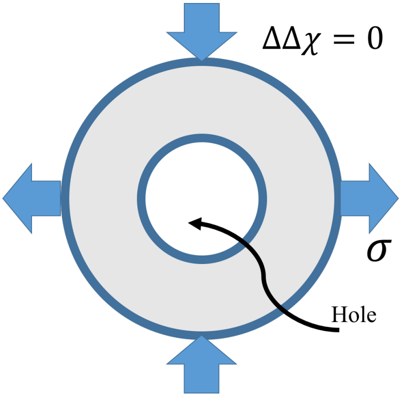






Fictitious elastic charges



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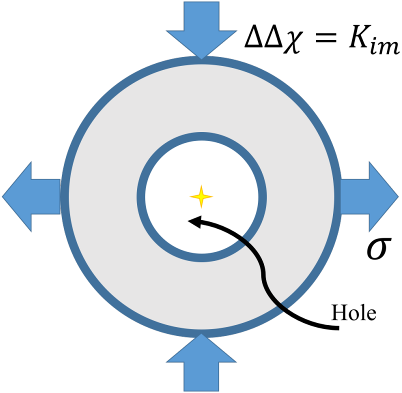
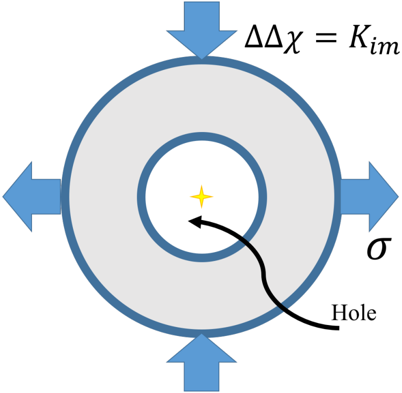


$$\bar{K}_{\mathrm{Im}} = \alpha\tilde{\Delta}\delta\left(\mathbf{x}\right) + \beta\tilde{\Delta}\Delta\delta\left(\mathbf{x}\right)$$

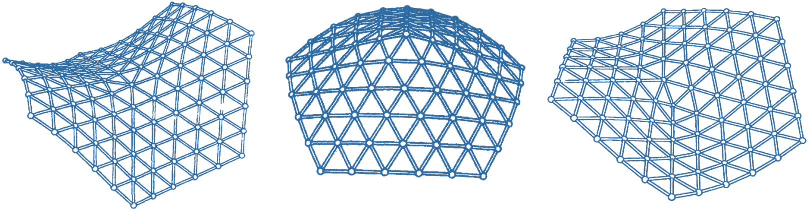
Fictitious elastic charges



$$\bar{K}_{\mathrm{Im}} = \alpha\tilde{\Delta}\delta\left(\mathbf{x}\right) + \beta\tilde{\Delta}\Delta\delta\left(\mathbf{x}\right)$$

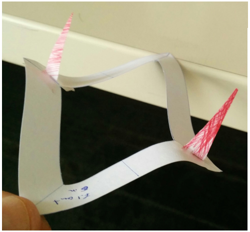
The lowest order of the fictitious charge is always the quadrupole

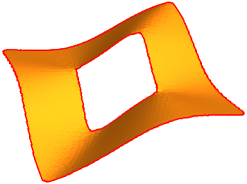
Screening defects (Seung & Nelson 88')



$$\frac{1}{Y}\Delta\Delta\chi = \bar{K} - K$$

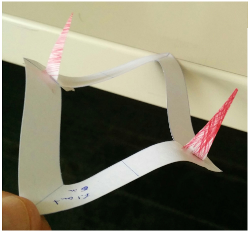
Screening the quadrupole

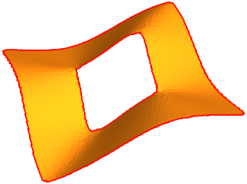




Effective mechanics of pulled frames

Screening the quadrupole





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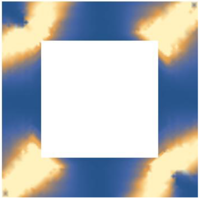
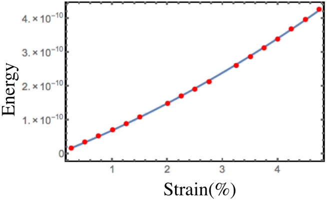
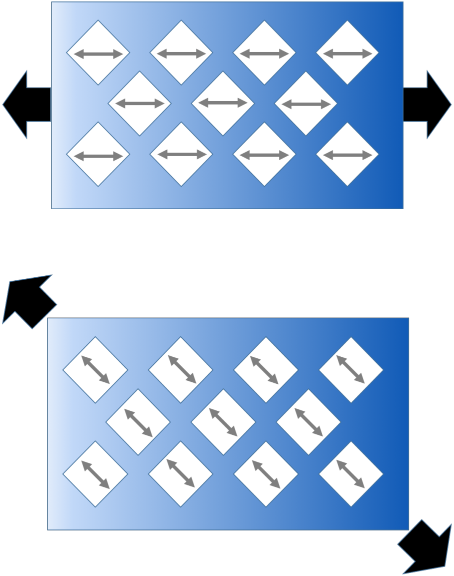




Array of fictitious quadrupoles



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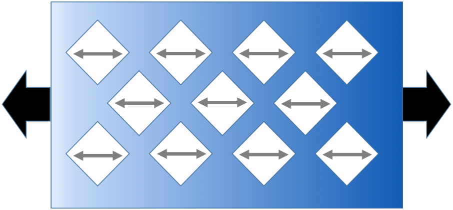
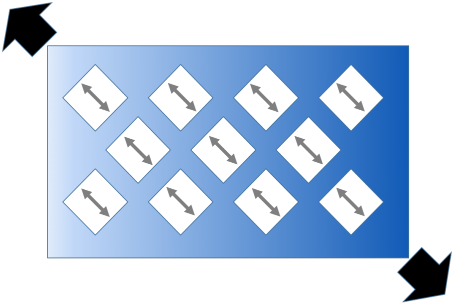


Neighboring frames are compatible



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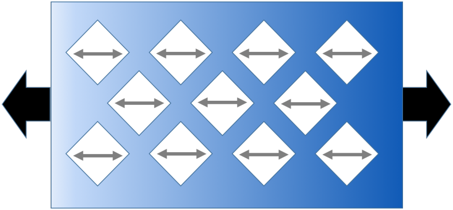

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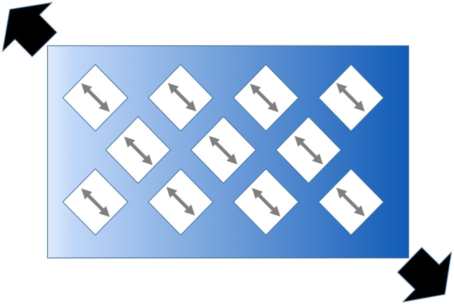
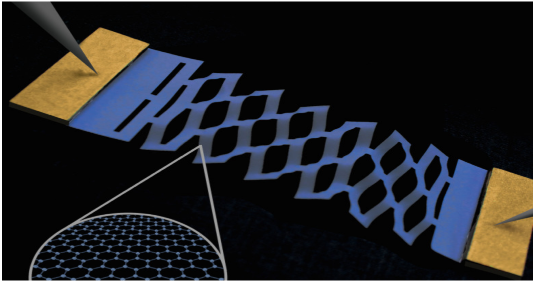
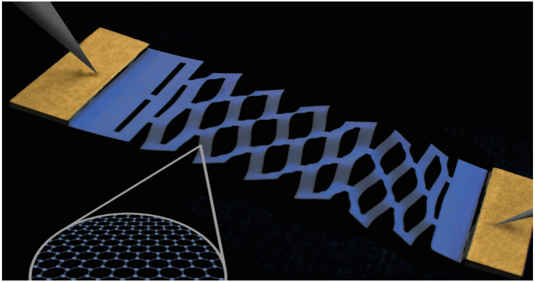


Frames are frustrated

Array of fictitious quadrupoles



Array of fictitious quadrupoles



Collective excitation of screening quadrupoles

Thank you