Twist operators and Self-avoiding loops

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Outline

- Motivation: the measure on self-avoiding loops.
- A brief summary of the loop gas formulation of the O(n) model.
- Twist operators and correlation functions.
- Application to self-avoiding loops.
- Interpretation as a stochastic process.
- Concluding remarks.

Motivation

- Werner (2005) showed that there exists a measure on simple loops on any Riemann surface which has the property of conformal restriction.
- Note that a given loop separates the domain into two disconnected subdomains.
- Given the positions of a pair of points on the Riemann sphere, we may consider the subset of loops which separate them. What is the measure on these loops? Can we generalise this idea to sets of more than two points?
- We use methods of conformal field theory to predict answers to these questions.

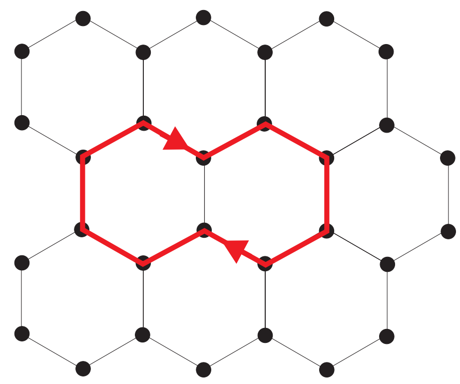
A loop model from the O(n) model:

$$Z = \mathrm{Tr} \prod_{(ij)} \Big(1 + x s_a(r_i) s_a(r_j)\Big)$$

- The $\{s\}$ are n-component spins, on the honeycomb lattice and the product is over pairs of nearest neighbours.
- This partition function is equivalent to the sum over graphs of closed loops on the edges:

$$Z = \sum_G x^l n^v$$

v is the number of loops.



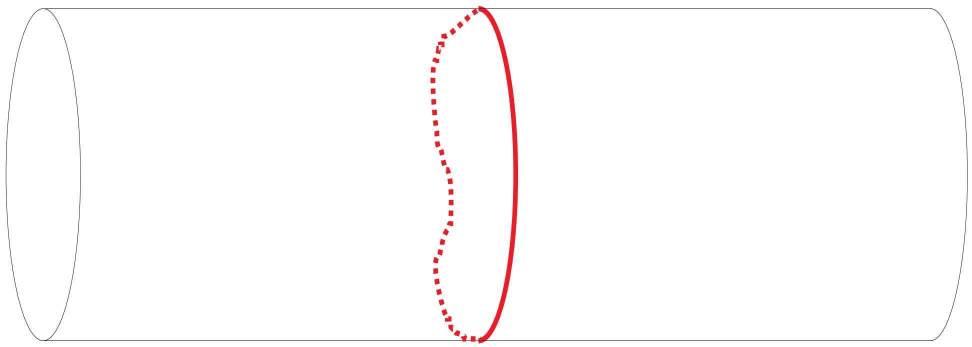
- Single self-avoiding loops weighted only by their length correspond to $O(n^1)$.

To formulate a field theory for this model, the non-local factors of n are made local as follows:

- Assign arbitrary orientations to the loops and sum over these.
- Introduce complex weights for left/right turns, $e^{\pm i\chi}$.
- Each closed clockwise loop is weighted by $e^{6i\chi}$.
- Choose χ such that $e^{6i\chi} + e^{-6i\chi} = n$.
- Map to a height model on the dual lattice; crossing a loop running from left to right leads to a decrease in height of π.
- Such a height model for $x = x_c$ is conjectured to flow under the renormalization group to a Gaussian free field with action:

$$S[\,h(\mathbf{r})] = \frac{g(\chi)}{4\pi} \int (\partial h(\mathbf{r}))^2 d^2\mathbf{r}$$

Topological subtleties: the cylinder

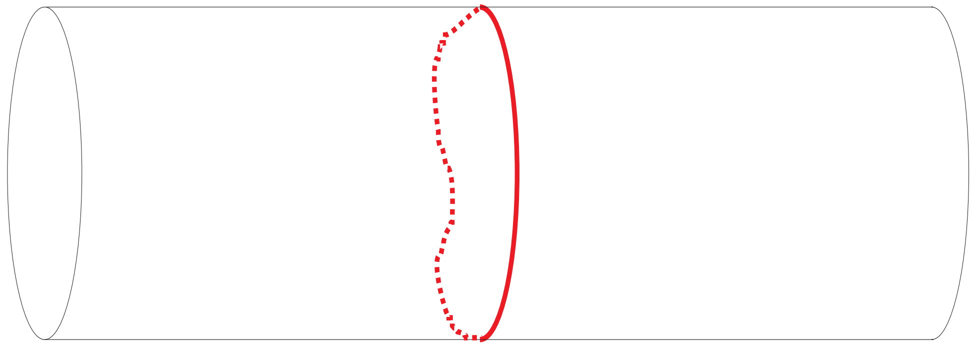


• Loops wrapping around the cylinder are counted incorrectly; they have the same number of left and right turns and hence are counted with weight two.

• Insert vertex operators at both ends to produce desired factors of n .

Topological subtleties: the cylinder

$$e^{-\frac{6i\chi h(-\infty)}{\pi}}$$



$$e^{\frac{6i\chi h(\infty)}{\pi}}$$

- A single loop results in a height difference of $\pm\pi$ between the ends, hence leads to a weight $e^{\pm 6i\chi}$.
- Summing over orientations leads to $e^{6i\chi} + e^{-6i\chi} = n$
- This works for multiple loops around the cylinder.

Modified partition function on the cylinder:

$$Z = Z_{\mathsf{ff}} \lim_{w\to\infty} \langle e^{-i\frac{6\chi h(-w/2)}{\pi}} e^{i\frac{6\chi h(w/2)}{\pi}} \rangle_{\mathsf{ff}}$$

$$= Z_{\mathsf{ff}} \left| \frac{L^2}{4\pi^2} e^{2\pi w/L} \right|^{(6\chi)^2/2\pi^2 g(\chi)}$$

The conformal charge may be ascertained from the free energy per unit length:

$$\frac{F}{w} = -\frac{\ln(Z)}{w} = -\frac{\pi c}{6L}$$

with $c = 1 - \frac{6}{g(\chi)} \left(\frac{6\chi}{\pi}\right)^2$

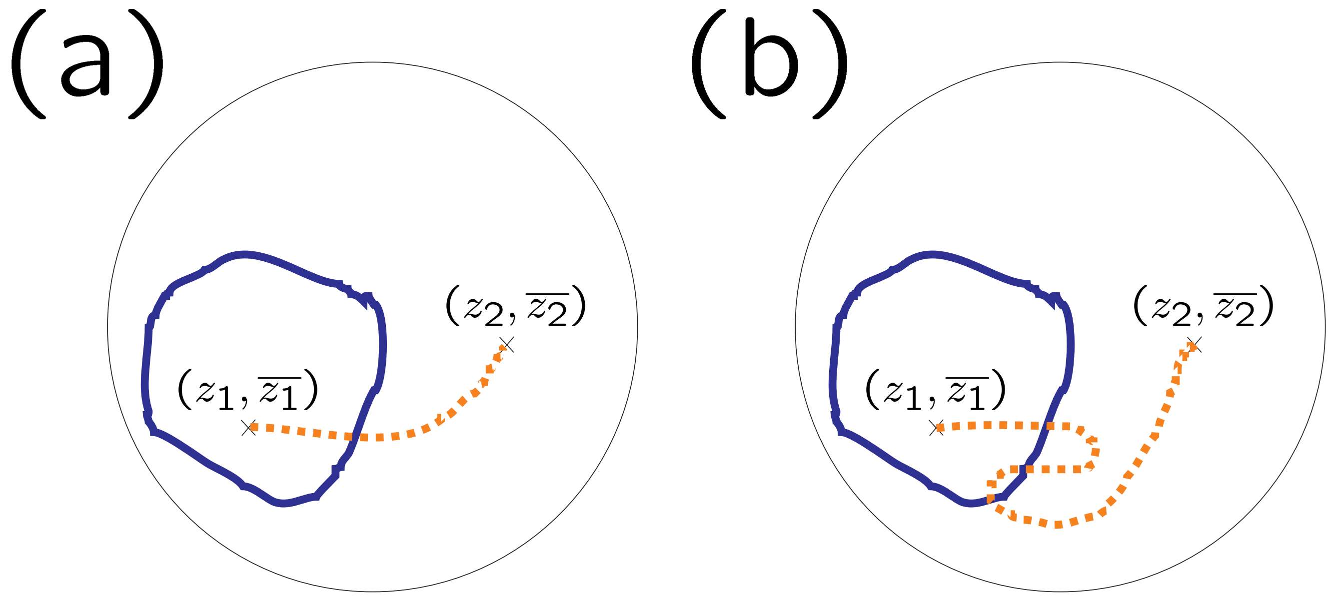
Twist operators:

• Count loops between two points with a weight n' rather than n, corresponding to the insertion of a pair of operators $e^{\pm 6i(\chi'-\chi)h/\pi}$.

• The partition function on the cylinder is again a free field expectation value, so the scaling dimension of these operators may be determined easily.

• The choice $n' = -n$ is of particular interest. Its correlation function returns the expectation value of -1^M, where M is the number of loops separating the two points.

• The picture is that of a defect line running between the two operators. The correlation function counts -1 to the power of the number of crossings of this defect line, which is path independent.



•The scaling dimension of these operators for $n' = -n$ is

$$x = 3/2g(n) - 1$$

Precisely that of a $\phi_{1,2}$ operator, which has a null state at level two and satisfies BPZ partial differential equations.

Correlation functions of twist operators:

Thus we see that $\langle -1^{M} \rangle_{\text{loop gas}} = \left| \frac{z_1 - z_2}{a} \right|^{2-3/g(n)}$.

As we are interested in self avoiding loops, let us expand both sides of this equation in powers of n. The LHS is

$$\Big(1+\sum_{G_{z_1|z_2}} (-1) x_c^l n^1 + \sum_{G_{z_1 z_2|}} x_c^l n^1\Big)\Big(1+\sum_{G_{z_1|z_2}} x_c^l n^1 + \sum_{G_{z_1 z_2|}} x_c^l n^1\Big)^{-1} + O(n^2)$$

The set of graphs with one loop have two subsets: the first contains all graphs separating the two points into different subdomains of the sphere, the other has both in the same subdomain. Thus,

$$1-2\sum_{G_{z_1|z_2}} (-1) x_c^l n^1 + O(n^2) = 1 - \frac{2n}{3\pi} \ln \left| \frac{z_1 - z_2}{a} \right| + O(n^2)$$

We have arrived at the answer to our first question.

$$\sum_{G_{z_1|z_2}} x_c^l = \frac{1}{3\pi} \ln \left| \frac{z_1 - z_2}{a} \right| ,$$

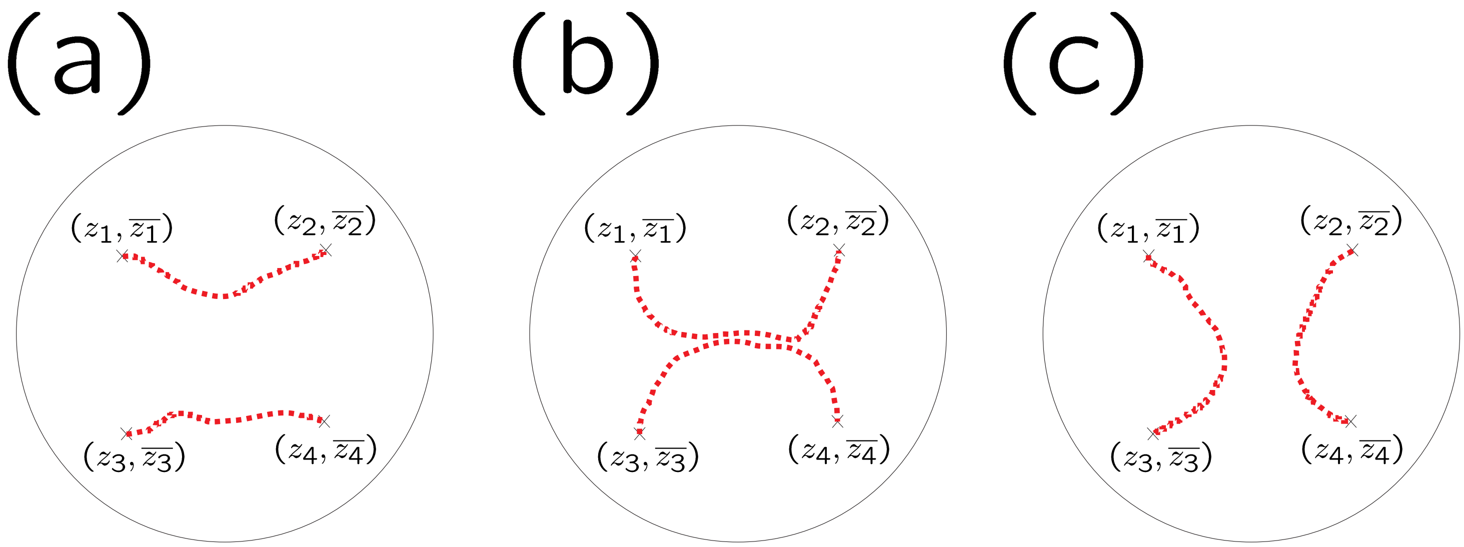
where the sum is over all graphs with a single loop separating points z_1 and z_2 .

• This number diverges for vanishing lattice spacing due to the contribution from small loops around each of the points.

4-point correlation function

• Let us consider next the four point correlation function of twist operators. We should think of two defect lines, running between pairs of operators.

• These defect lines may take any path, and hence may run between any choice of pairs of points as the figure below demonstrates.



Now, we have

$$\langle(-1)^{N_{12}}(-1)^{N_{34}}\rangle_{\text{loop gas}} = \langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle$$

- As before, we expand both sides in powers of n and look at the term of order n^1.
- With four marked points, there are eight topologically distinct subsets of a single loop.
- The left hand side is

$$1-2n\Big[W_{z_1z_2z_3|z_4}+W_{z_1z_2z_4|z_3}+W_{z_1z_3z_4|z_2}+W_{z_1|z_2z_3z_4}\Big]+O(n^2)$$

- The W's are defined as

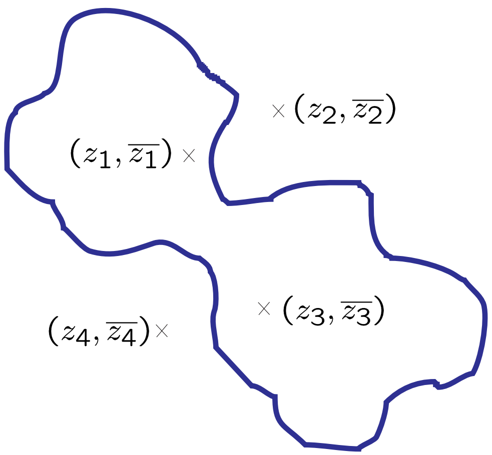
$$W_{z_1z_2z_3|z_4} \equiv \sum_{G_{z_1z_2z_3|z_4}} x_c^l\,,$$

$$\langle(-1)^{N_{12}}(-1)^{N_{34}}\rangle_{\text{loop gas}} = \langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle$$

• The right hand side satisfies a PDE with solutions in terms of hypergeometric functions. Once symmetry under permutation of labels is enforced, the solution is unique up to normalisation.

• By subtracting the appropriate pairs of products of two point functions, particular subsets of loops may be isolated. For example,

$$W_{z_1 z_3|z_2 z_4} = \frac{1}{8n}\Big[\langle\phi\phi\phi\phi\rangle - C_{12}C_{34} + C_{13}C_{24} - C_{14}C_{23}\Big] + O(n)$$

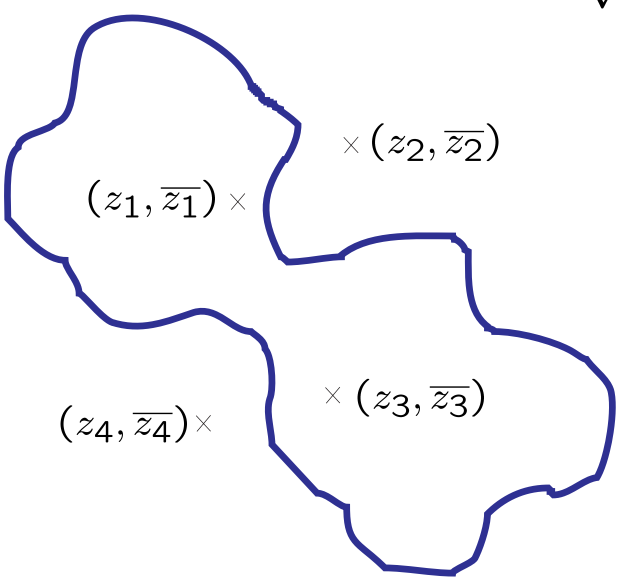


From the known form of the right hand side, the number of weighted loops winding around points 1 & 3 can be determined.

• The average weighted number of loops around two of the four points is a function of the cross-ratios only. It is finite in the limit of vanishing lattice spacing and takes the form:

$$\sum_{G_{z_1 z_3 | z_2 z_4}} x_c^l = -\frac{1}{24\pi}\Big(\eta\, {}_3F_2(1,1,\tfrac{4}{3};2,\tfrac{5}{3};\eta) + \overline{\eta}\, {}_3F_2(1,1,\tfrac{4}{3};2,\tfrac{5}{3};\overline{\eta})\Big)$$

$$+ \frac{2^{1/3}\pi}{3\sqrt{\pi}\Gamma(\frac{1}{6})^2\Gamma(\frac{4}{3})^2}|\eta(1-\eta)|^{2/3}|{}_2F_1(\tfrac{2}{3},1;\tfrac{4}{3};\eta)|^2$$



where $\eta = \dfrac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$

Stochastic formulation

- The four point function satisfies a pair of PDEs for each point.
- The expected number of weighted loops do, also:

$$\left[\frac{3}{2}\partial_{z_1}^2 + \sum_{i\neq 1}\frac{\partial_{z_i}}{z_i - z_1}\right] W_{z_1 z_4|z_2 z_3} = \frac{1}{24\pi}\left[\frac{1}{(z_4 - z_1)^2} + \frac{1}{z_3 - z_1}\left(\frac{1}{z_3 - z_4} + \frac{1}{z_2 - z_3}\right) + \frac{1}{z_2 - z_1}\left(\frac{1}{z_2 - z_4} + \frac{1}{z_3 - z_2}\right) + \frac{1}{z_4 - z_1}\left(\frac{1}{z_4 - z_3} + \frac{1}{z_4 - z_2}\right)\right]$$

- This may be made homogeneous by the substitution

$$\widetilde{W} \equiv W - (24\pi)^{-1} 2\Re e\left[-2\ln(z_4 - z_1) + \ln(z_3 - z_4) + \ln(z_2 - z_4) - \ln(z_2 - z_3)\right]$$

• So we arrive at the following equation (for a linear combination)

$$\left[3\partial_{x_1}^2 + 2\Re e \sum_{i\neq 1} \frac{2\partial_{z_i}}{z_i - z_1}\right]\widetilde{W} = \frac{3}{2}\Delta_{z_1}\widetilde{W}$$

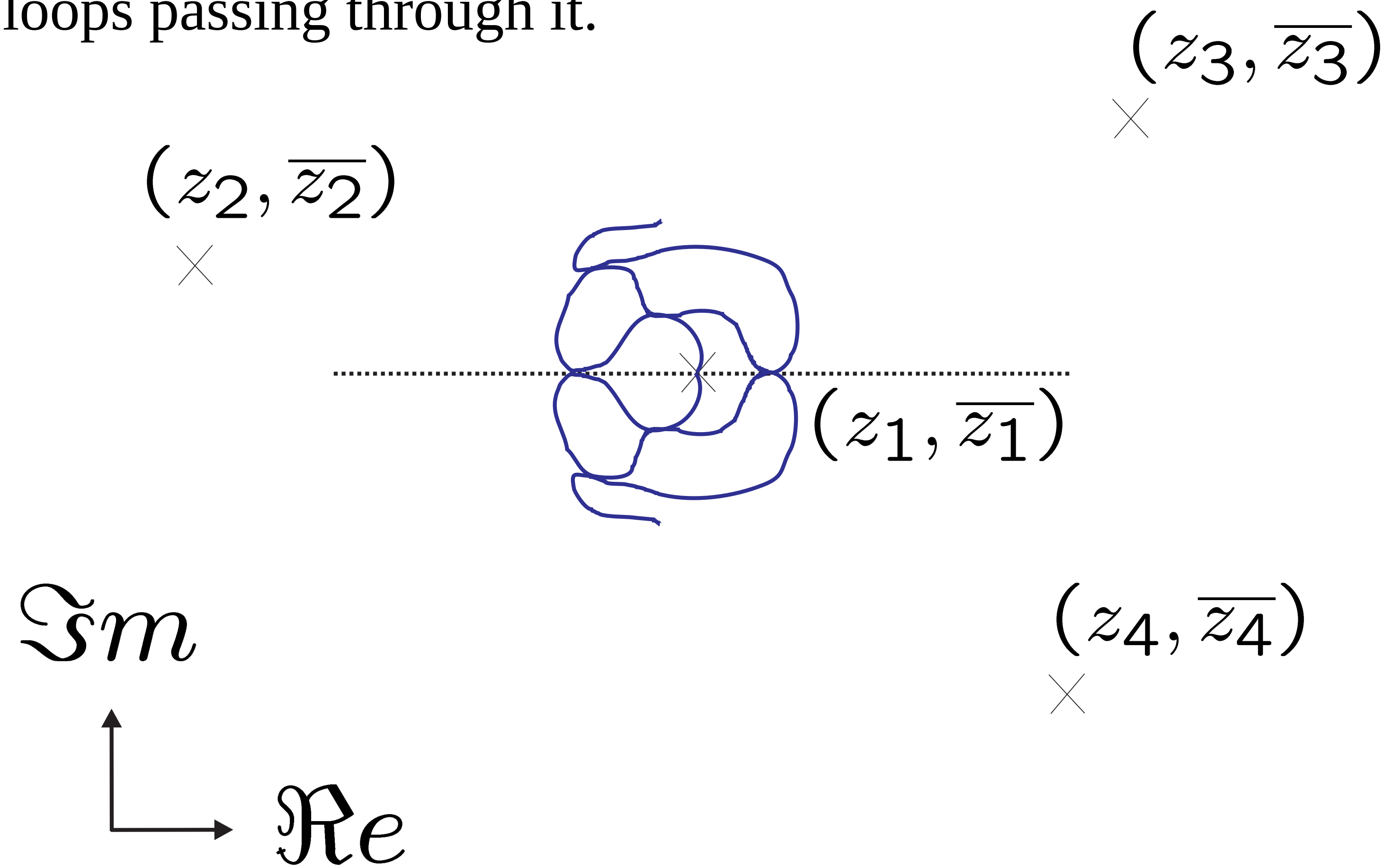
• Consider a sequence of conformal maps which satisfy the stochastic chordal Loewner equation:

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - z_{1t}} \quad \text{with} \quad z_{1t} = z_1 + \sqrt{\kappa}B_t$$

• This describes an SLE growing in the half plane $\mathrm{Im}(z) > \mathrm{Im}(z_1)$ starting from z_1 , together with its reflection in $\mathrm{Im}(z) = \mathrm{Im}(z_1)$

• The term involving the Laplacian describes the extent to which $\widetilde{W}$ fails to be a martingale, for an SLE$_{\kappa = 6}$ process.

- The interior of the hull at time t is mapped to the point $\sqrt{\kappa}B_t$
- Hence the number of weighted loops is not a martingale.
- The rate at which loops disappear is proportional to the Laplacian. If W were constant in some region, there would be no loops passing through it.



2-point correlation function with a boundary

•Any simply connected domain may be mapped to the upper half plane via a conformal transformation

•The two point function in the upper half plane is simply related to the four point function in the bulk theory

•We solve this two point function and find that the number of single self-avoiding loops which wrap both twist operators is

$$-\frac{1}{12\pi}\ln(\eta(1-\eta)) - \frac{1}{12\pi}\eta\,{}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\eta) + \frac{\Gamma(2/3)^2}{6\pi\Gamma(4/3)}(-\eta(1-\eta))^{\frac{1}{3}}\,{}_2F_1(\frac{2}{3},1;\frac{4}{3};\eta)$$

where $\eta = \frac{(z_1 - z_2)(z_1^* - z_2^*)}{(z_1 - z_1^*)(z_2 - z_2^*)}$

In conclusion:

- The O(n) model may be expressed alternatively as a loop model.
- This loop model, as $n \to 0$, describes single self-avoiding loops.
- Twist operators count loops separating them with weight $-n$.
- Their correlation functions yield information about the weighted number of single self-avoiding loops of various classifications.
- There is a stochastic process related to this number, which may be more naturally expressed as a radial or full plane SLE process.