TalkKPP

Titre de la note 22/10/2006

Right BB Functional RG (a set of) 1 method deal many relev. op.

Next week(s) → elastic systems (next week(s)) pinning, glass

Today → 2D, freezing, localisation log systems, marginal glass

classical → XY model + disorder

C. Mudry → fermions localization

developped independently each subfield 95 → connexions

cannot quote everyb XY: Natterman, Korshunov, Tang, Scheidl, Christopher ... fermion side

own works prospect. + David Carpentier

from classical spin model freezing transitions → REM, DPCT mean field hierarchical exact / Derrida – Spohn

free energy T indep glass phase → 2d log-disorder + transl. inv. RG

random gauge XY model → replicated (vector) CG

φ_{diag} ← KPP eq (FRG) ← $n \to 0$ ∞ rel op

CG advantage see the physics

(→ more powerful methods CFT)

other applications → 2d vortex lattice + dislocation

experiments, numerics

→ random Dirac fermions Josephs junct arrays (Lobb)

Liouville / Sinh gordon

LeftBB

XY model 2D $\qquad Z = \int \mathcal{D}\varphi \, e^{-\beta H}$

$$H_{XY} = -\frac{J}{\pi} \sum_{\langle x x' \rangle} \cos\left(\varphi(x) - \varphi(x')\right) \left(-A_{xy}\right) \qquad S(x) = e^{i\varphi(x)}$$

$$\begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \qquad \varphi \in [0, 2\pi] \to \varphi \in \mathbb{R}$$

Spin waves + vortices

$$T < T_{KT} \qquad H_{XY} \simeq \frac{J}{2\pi} \int d^2r \, (\nabla\varphi)^2 \left(-A\right) \qquad \frac{d^2r}{a^2}\, a^2(\nabla\varphi)^2$$

$$\langle e^{i(\varphi_x - \varphi_{x'})} \rangle \sim \frac{1}{|x-x'|^{\eta}} \qquad \eta = \frac{T}{2J} \to T/2J_R$$

One vortex

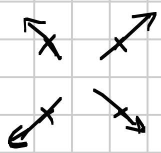
$$z = re^{i\theta} \qquad \varphi = \theta \qquad \vec{\nabla}\varphi = \frac{1}{r}\vec{e}_\theta \qquad J_R \simeq J$$

Screening by dipoles

$$\oint d\vec{l} \cdot \vec{\nabla}\varphi = 2\pi \qquad \nabla\varphi = \nabla^T\psi \qquad \mathcal{E} = \frac{J}{2\pi}\int_a^L 2\pi r\, dr \, \frac{1}{r^2}$$

$$\nabla \wedge \nabla^T \psi = \nabla^2\psi = 2\pi\delta(r) \qquad \simeq J \ln(L/a)$$

$$\psi(r) = \ln\frac{r}{a}$$



$(+E_c$ core energy $)$

$\to$ vortex pair $\qquad (+) \quad (-)$ dipole

$2J \ln R/a \qquad \overleftrightarrow{\;R\;}$

$$T_{KT} = J/2 \quad \text{pairs unbind} \qquad \langle e^{i\varphi_x} e^{-i\varphi_y} \rangle \sim e^{-\frac{|x-y|}{\xi}}$$

entropy $2T\ln(R/a)$

(or $4T \ln R/a$)

$$P(R) \sim \left(R/a\right)^{-(2J)/T} \qquad \text{dipoles start decreasing } J$$

$$\int d^2R \; R^2 \; P(R) = \infty \qquad \frac{2J}{T} < 4$$

Right BB

XY model $\Longleftrightarrow$ CG coulomb gas charges

$r \in$ dual lattice n_r integer

$$Z_{XY} = Z_{SW} Z_{CG}$$

(exact Villain) $e^{-\beta V(\theta)} = \sum_p e^{-\frac{\beta J}{2\pi}(\theta - 2\pi p)^2}$

$$Z_{CG} = \sum_{\{n_r\}} e^{-\beta H}$$

$$G_{latt} = \left(\ln \frac{r-r'}{a} + \gamma\right)\left(1 - \delta^{a}_{rr'}\right)$$

$$H = -\frac{1}{2} \sum_{r \neq r'} 2J\, n_r \underbrace{\ln \frac{|r-r'|}{a}}\, n_{r'}$$

$\ln \frac{L}{a}\left(\sum_r n_r\right)^2$ $\Leftarrow$ neutral $\int 1/q^2$

$2J \ln R$ / dipole $+$ $-$ r_1 r_2 (r_1, r_2) (r_2, r_1) in $\sum_{r \neq r'}$

Right BB

$$Z_{Cont} = \sum_p \sum_{n_1 \dots n_p}{}' \int_{|r_i - r_j| \geq a} \frac{d^2 r_1}{a^2} \cdots \frac{d^2 r_p}{a^2} \underbrace{e^{-\beta H}\, y^{\sum_i n_i^2}}_{e^{-\beta H'}}$$

hard core "disk" charges

$\prod_n \frac{1}{N(n)!}$ each $\neq$ conf counted once

$$\beta H' = -\beta J \sum_{i \neq j} n_i \ln \frac{|r_i - r_j|}{a}\, n_j - \sum_i n_i^2 \ln y$$

$$y = e^{-\beta E_c} \quad \text{fugacity}$$

$$E_c = \gamma J$$

RG integrate $a < r < a e^{dl}$

$$\delta \ln y = -\beta J\, dl$$

$$-J \sum_{i \neq j} n_i \ln \frac{a(1+dl)}{a} n_j = -J\,dl\Big(\underbrace{\big(\textstyle\sum_i n_i\big)^2}_{0} - \sum_i n_i^2\Big) = J\,dl \sum_i n_i^2$$

rescaling $\frac{1}{a^2} y \to \frac{1}{(a e^{dl})^2} \tilde{y}$ $\quad \tilde{y} = y e^{2dl}$

$\Longleftrightarrow$ one loop SG

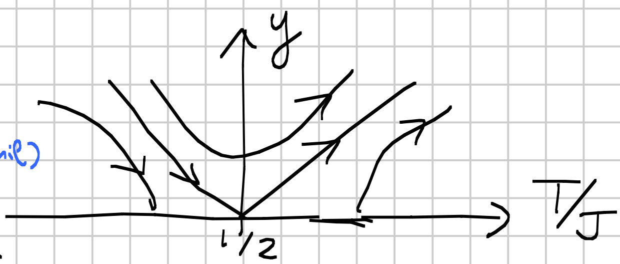
$$\partial_l y = \left(2 - \frac{J}{T}\right) y + O(y^3)$$

$$\partial_l \left(\frac{T}{J}\right) = c\, y^2$$

screening by dipole (annihil.)

details of transition

$J \to J_R$



Disorder   random gauge XY model 2D $Z_A = \int D\varphi\, e^{-\beta H_A}$

$$H_{XY} = -\frac{J}{\pi} \sum_{\langle xx' \rangle} \cos(\varphi_x - \varphi_{x'} - A_{xx'}) \qquad \overline{A^2_{xx'}} = \pi\sigma$$

i.i.d each bond

Spin waves $\quad H \simeq \frac{J}{2\pi} \int d^2x\, (\nabla\varphi - A)^2 \qquad \overline{A_x A_{x'}} = \pi\sigma\, \delta^{(2)}(x-x')$

$$\overline{\langle e^{i\varphi_0} e^{-i\varphi_x} \rangle} \sim \frac{1}{x^\eta} \qquad \eta = \frac{1}{2}\left(\frac{T}{J_R} + \sigma_R\right)$$

$$\overline{\langle e^{i\varphi_0} \rangle \langle e^{-i\varphi_x} \rangle} \sim \frac{1}{x^{\bar\eta}} \qquad \bar\eta = \frac{\sigma_R}{2}$$

vortex ? $\quad A \cdot \nabla\varphi \qquad A(r) = \frac{e_\theta}{r}\, n \quad$ vortex charge n at $r=0$

$$V_r = -2J \sum_{r'} G_{r-r'}\, q_{r'}$$

random dipole $\quad q = \frac{1}{2\pi} \nabla \wedge A$

$$\mathcal{E}^{\text{dis}}_{\text{vortex}} = -V(r_i)\, n_i$$

each vortex sees gaussian RP with log-correlations

$$\overline{(V(r) - V(r'))^2} \sim 4\sigma J^2 \ln \frac{|r-r'|}{a}$$

$$(1/q^2)^2 q^2$$

$$\beta H' = -\beta J \sum_{i \neq j} n_i \ln\left(\frac{|r_i - r_j|}{a_0}\right) n_j - \sum_i n_i \beta V(r_i)$$
$$- \sum_i \ln Y[n_i, r_i]$$

fight BB

$$\ln Y[n, r] = -E_c n^2 + \beta n v_r$$

$$y(r) = e^{-\beta E_c \pm v_r}$$

$\rightsquigarrow$ Replicated Coulomb gas

$\overline{Z^m}$ $\quad$ m replicas

to average over dis $m \to 0$

$$\overline{Z^m_{latt}} = \sum_{\{n_r^a\}} e^{-\beta H^{(m)}_{latt}}$$

$$\beta H^m_{latt} = -\sum_{r \neq r'} K_{ab} n_r^a G_{r-r'} n_{r'}^b$$

$$\beta H^{(m)}_{cont} = -\sum_{i \neq j} K_{ab}\, n_a^i \ln\left(\frac{|r_i - r_j|}{a_0}\right) n_j^b - \sum_i \ln Y[\vec{n}_i]$$

$$\vec{n} = (n^1, \dots n^m)$$

$$K_{ab} = \beta J \delta_{ab} - \sigma \beta^2 J^2$$

vector charges

$$Y[\vec{n}]_{bare} = e^{-n_a \gamma K^{ab} n_b} \qquad \vec{n} \neq \vec{0}$$

RG analysis of vector CG (Nienhuis) $\quad a_0 \to a_0 e^{dl}$

$$\partial_l Y[\vec{n}] = \left(2 - n^a K_{ab} n^b\right) Y[\vec{n}] + c_2 \sum_{\substack{\vec{n}' + \vec{n}'' = \vec{n} \\ \vec{n}', \vec{n}'' \neq 0}} Y[\vec{n}']\, Y[\vec{n}'']$$

rescaling

fusion

$c_2 = \pi$

$c_1 = 2\pi^2$

hard cutoff

$$\partial_l (K_l^{-1})_{ab} = c_1 \sum_{n \neq 0} n^a n^b\, Y[n]\, Y[-n]$$

annihilation / screening

dilute limit

expansion in vect charge fugacities

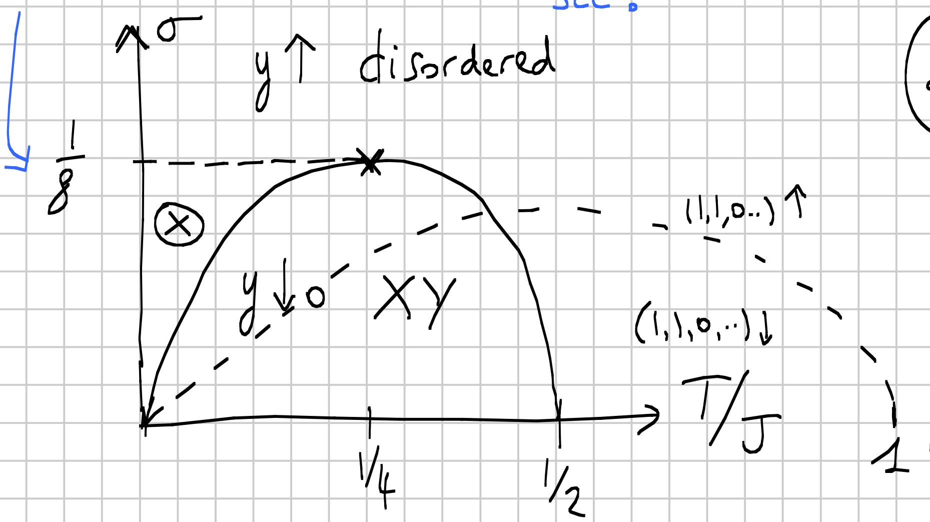
limit $m \to 0$?

look at $\vec{n} = (1, 0, \dots 0)$ $\quad y = Y[\vec{n}]$

$$\partial_\ell y = (2 - K_{11}) y = (2 - \beta J + \sigma \beta^2 J^2) y$$

$y = e^{-\beta E_c}$

aim: understand low T XY phase (freezing) and T=0 transition (vortex prolif) SLE?



$$\left(2 - \frac{J}{T} + \sigma \frac{J^2}{T^2}\right) = 0$$

$$\sigma_c^0 = \tilde{T}^2\left(\frac{1}{\tilde{T}} - 2\right) = \tilde{T}(1 - 2\tilde{T})$$

⊗ Pb non perturb

Rubinstein – Shraiman
Nelson

uniform fugacity

1) Nishimori impossible reentrance

$$2 - K \sum_a n_a^2 + \sigma K^2 \left(\sum_a n_a\right)^2$$

$\langle y^2 \rangle \equiv (1, 1, 0, ..)$

$(1, 1, 0, ..)$ $\quad 2 - 2K + 4\sigma K^2$ $\quad \sigma_c = \frac{1}{4} \tilde{T}(2 - 2\tilde{T})$

XY destroyed?

∞ relevant operators → hopeless? → no because high moments we don't care

→ typical may still be perturbative → FRG / KPP

$m \to 0$, + D Carpentier

Generating function

$n^a = 0, \pm 1$

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \begin{matrix} n_+ \\ \\ n_- \\ \\ m - n_+ - n_- \end{matrix}$$

$$Y[\vec{n}] = \left(z_+^{n_+} z_-^{n_-} \right)_{\phi(z_+, z_-)}$$

$$P = \phi / \int \phi$$

interpretation proba. density

$$\partial_\ell P(z_+, z_-) = O P - 2P$$
$$+ 2 \left\langle \delta\left(z_+ - \frac{z_+' + z_+''}{1 + z_-' z_+'' + z_+' z_-''}\right) \delta\left(z_- - (+ \to -)\right) \right\rangle_{P' P''}$$

fusion of 2 environ.
all charge 1,0 0,1

$$P(z_+, z_-)\, dz_+ dz_- = \tilde{P}(u, v)\, du\, dv$$

$$z_+ = e^{\beta(u+v)} \qquad z_- = e^{\beta(u-v)}$$

$u = -E_c$
local part fct

$$Y[n] = e^{\beta((n_+ + n_-) u + (n_+ - n_-) v)}$$

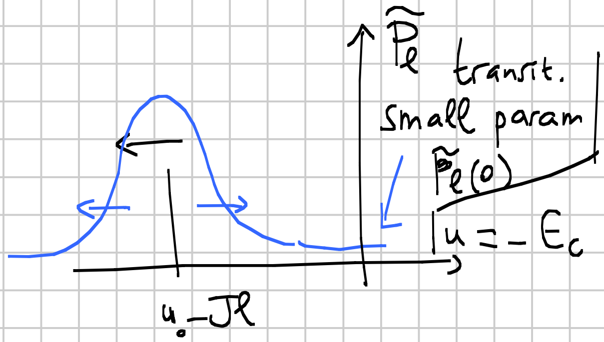
$$2 - n_a K_{ab} n_b = 2 - \beta J (n_+ + n_-) + \sigma \beta^2 J^2 (n_+ - n_-)^2$$

$u \to -u$?

int by part

$$\partial_\ell \tilde{P} = \left(J \partial_u + \sigma J^2 \partial_v^2 \right) \tilde{P}(u, v) + \text{fusion}$$

$T = 0$

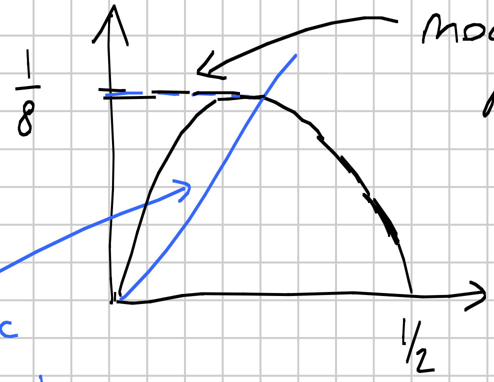


$\sigma = 0 \Rightarrow \delta(u + J\ell)$

if broadening faster $(\sigma > \frac{3}{8})$ than drift

$\Rightarrow$ some charge will have $E_c < 0$
$u > 0$ $z \sim 1$

2 effects



modified transition vortex proliferation

$\longrightarrow$ screening

freezing of single vortex

KPP equation

rare regions $z \simeq 1$

either z_+ or z_-

both

$P(z\simeq 1)^2$

$$(z_+, z_-) \longrightarrow z \qquad P_\ell(z)$$

$$u = -E_c \qquad z = e^{\beta u}$$

$$u = \frac{1}{\beta} \ln\left(e^{\beta u'} + e^{\beta u''}\right) \xrightarrow[T\to 0]{} \max(u', u'')$$

$$\partial_\ell P(u) = \left(J\partial_u + \sigma J^2 \partial_u^2\right) P - 2P + 4P(u) \int_{-\infty}^{u} P(u')\,du'$$

$$G_\ell(x) = \int_{x-J\ell}^{\infty} P_\ell(u)\,du \qquad P_\ell(u) = -\partial_x G(u+J\ell)$$

$$\frac{1}{2}\partial_\ell G(x) = \underbrace{\frac{\sigma J^2}{2}}_{D} \partial_x^2 G(x) + \underbrace{G(1-G)}_{f(G)}$$

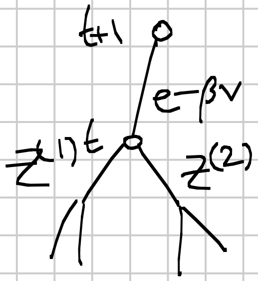
valid any T — depend on T initial condition

$$1 - G_\ell(x) = \int du\; P_\ell(u) \exp\left(-e^{\beta(u - x + J\ell)}\right)$$

$$\theta(u - x + J\ell < 0)$$

DPCT

$$Z(t) = \sum_{\text{paths}} e^{-\beta E_{\text{path}}} \quad \text{on C.T}$$



$$Z(t+1) = e^{-\beta V}\left(Z^{(1)}(t) + Z^{(2)}(t)\right)$$

$$G_t(x) = \left\langle \exp\left(-e^{-\beta x} Z(t)\right)\right\rangle$$

$$\begin{cases} G_{t+1}(x) = \int P(V)\,dV\,\left(G_t(x+V)\right)^2 \\ G_0(x) = \exp\left(-e^{-\beta x}\right) \end{cases}$$

Continuum limit : branching diffusion $\longrightarrow$ KPP

$$\begin{aligned} Z(t+dt) &= e^{-\beta dV} Z(t) && \text{proba } 1-\lambda\,dt \\ &= e^{-\beta dV}\left(Z^{(1)}(t) + Z^{(2)}(t)\right) && \lambda\,dt \end{aligned}$$

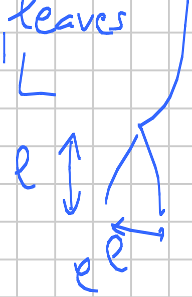
$$\partial_t G = D\,\partial_x^2 G + \lambda\left(G^2 - G\right)$$

intuitive why DPCT $\quad N \sim e^{\ell}$ $\quad$ here transl. inv.

distance on tree $=$ # generation $\simeq \ln N_{\text{leaves}}$
ℓ $\qquad\qquad\qquad\qquad\qquad\qquad\quad \| \; L$

$$\overline{(V_i - V_j)^2} \sim \ell \sim \ln L$$

$\hookrightarrow$ # common branches



KPP eq

Kolmogorov – Fisher

Petrosky Piscounov (37)

Bramson (83)

(proof of convergence to traveling waves)

popul. dynamics / invasion

$$f(G) = G(1-G)$$

$$\frac{1}{2}\partial_\ell G = D\,\partial_x^2 G + f(G)$$

$$f(0) = f(1) = 0 \qquad 0 \leq f \leq 1$$

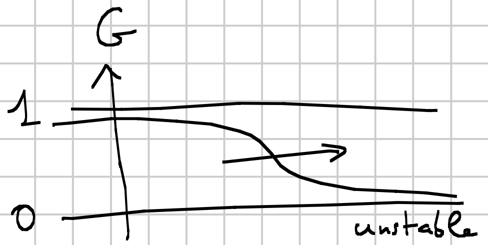
$$f'(0) = 1 \qquad f'(G) \leq 1$$

$$\partial_\ell G = -\frac{\partial F}{\partial G}$$

invasion of unstable by stable

F local maximum in $G=0$ (unstable state)

min $G=1$ (stable)



$$G_\ell(x) \to h(x - m_\ell) \text{ uniformly}$$

front velocity selection $\quad c = \partial_\ell m_\ell$

$$\text{if } G_{\ell=0}(x) \underset{x\to+\infty}{\sim} e^{-\mu x} \sim \langle z \rangle_0 e^{-\beta x}$$

$$D = \sigma J^2/2$$

1) $\beta < \beta_c = \frac{1}{\sqrt{D}} \qquad T > T_g = J\sqrt{\sigma/2}$ flow decay

$$c = c(\beta) = 2(D\beta + \beta^{-1}) = T\left(2 + \frac{\sigma J^2}{T^2}\right)$$

$$P_\ell(u) \to \tilde{p}(u - X_\ell) \qquad \partial_\ell X_\ell = c - J$$

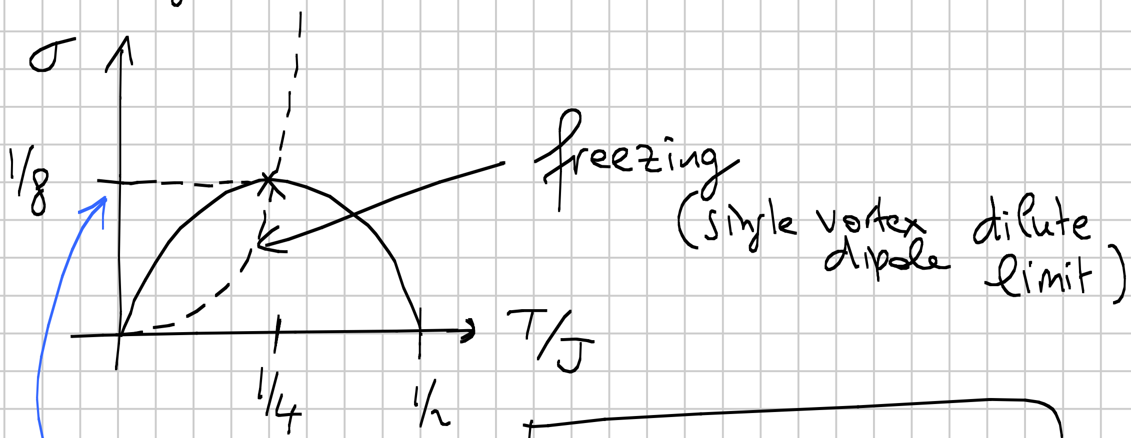
$$-\partial_\ell E^c_{typ} = c - J = T\left(2 - \frac{J}{T} + \frac{\sigma J^2}{T^2}\right)$$

2) $\beta > \beta_c$ $\qquad c = c(\beta^*) = 4\sqrt{D} = J\sqrt{8\sigma}$

$$-\partial_\ell E^c_{typ} = J\left(\sqrt{8\sigma}-1\right)$$

$T>T_g$ $\qquad z^{typ} \sim e^{\left(2-\frac{J}{T}+\frac{\sigma J^2}{T^2}\right)\ell}$ $\qquad T_g = J\sqrt{\sigma/2}$

$T<T_g$ $\qquad z^{typ} \sim e^{-\beta J\left(1-\sqrt{8\sigma}\right)\ell}$



$\boxed{\text{universality of transition}}$

transition

$\xi \sim e^{1/(\sigma_c-\sigma)}$

1) universality of corrections to c in KPP

$$m_\ell = \sqrt{D}\left(\underbrace{4\ell}_{c\ell} - \gamma \ln \ell\right)$$

$\gamma \ln\ln L$ correct. to free energy

$\neq$ univ. classes

$\gamma = 1/2 \quad T_g$

$= 3/2 \quad T<T_g$

$= 0 \quad T>T_g$

2) universal of forward front region
bulk non universal

$P_\ell(z-1)$ at $\sigma = \sigma_c$ $\uparrow$ screening

3) indep in $f(G) \longrightarrow$ indep of cutoff procedure $\quad c_1/c_2^2$ univ ∂k^{-1}

Simple arguments

1 particle in log-correlated landscape

$$Z = \sum_r e^{-\beta V(r)} \qquad \overline{(V(r)-V(0))^2} = 4\sigma \ln \frac{r}{a}$$

(σJ^2)

1) REM approx $\qquad \overline{V(r)V(r')} = 2\sigma \ln \frac{L}{(|r-r'|+a)}$

$$T < T_g = \sqrt{d/\sigma} \qquad \longrightarrow 2\sigma \ln L/a \, \delta_{r,r'}$$

$$F = -T \ln Z \sim V_{min} \simeq 2\sqrt{\sigma d} \ln L \left(+ \gamma \ln \ln L\right)$$

$d=2 \qquad \sigma \to \sigma J^2$

$$\frac{1}{L^2} \sim \int_{-\infty}^{E_{min}} \frac{dV}{\sqrt{4\pi\sigma J^2 \ln L}} \exp\left(-\frac{V^2}{4\sigma J^2 \ln L}\right)$$

$$E_{min} \sim -\sqrt{8\sigma} J \ln L$$

2) DPCT better approx

$$\gamma = 3/2$$

conjecture transl. inv. model described by KPP

CGRG

3) activated classical dynamics $\qquad x \sim t^z$

$$z(T) = 2 + 2\frac{\sigma}{dT^2} \qquad T > T_c = \sqrt{\sigma/d}$$

$$= 4 T_c/T \qquad T < T_c$$

$\longrightarrow$ analog quantum model | Dirac rand. potential

$z = 1+\sigma \qquad \sigma < 2$

$z = \sqrt{8\sigma} - 1$

1) Sine gordon

$K = \beta J$

replicated SG

$$\int d^2r \, \frac{1}{8\pi} \sum_{ab} \left(\frac{1}{K} \delta_{ab} + \sigma \right) \nabla\theta_a \nabla\theta_b$$

$$- \sum_{\vec{n} \neq 0} Y[\vec{n}] \, e^{i \vec{n} \cdot \vec{\theta}}$$

$\xrightarrow{\text{expand in } Y(n)}$

$\sum_a n_a \theta_a$

generates CG

$$\int d^2r \left(\frac{1}{8\pi K} (\nabla\theta)^2 - i \frac{A}{2\pi} \nabla\theta + y \cos\theta \right)$$

$H_D = \vec{\sigma} \cdot (-i\vec{\partial} - A)$

$-E + i\varepsilon$

bosonized form for $K=1$

$y = \varepsilon - iE$

random vect. pot Dirac $\Longleftrightarrow$ random gauge XY + diluted disloc

same freezing transition

random gauge XY

$$2) \int d^2x \left[\underbrace{(\nabla\varphi - A)^2 + \text{vortex}}_{\text{random gauge XY}} + \sqrt{g_M} \cos(\varphi - \zeta(x)) \right]$$

CO model + vortex

lattice + disloc + pinning

$\hookrightarrow g_A$

$\hookrightarrow$ pinning potential (FRG...)

$\downarrow$ ε energy in fermion model (Gade Wegner)

$$\rho(\varepsilon) \simeq \frac{1}{\varepsilon} e^{-(\ln \varepsilon)^{2/3}}$$

Gade Wegner

$$\xi_D \sim a e^{|\ln y_0|^{2/3}}$$

$\hookrightarrow y_0 \sim e^{-\beta E_c}$

KPP TG+PLD

disloc (vortex) always relevant