Majorana Kramers Pairs and $(Z_2)^2$ Fractional Josephson Effects

Fan Zhang (Set Sail)
University of Pennsylvania
My Captains

Charles Kane
(Since Oct. 2011)

Eugene Mele

Allan MacDonald
(Since Jan. 2008)
Conversations with Ken Palley (I)

(Ken Palley is my landlord at KITP.)

Ken: Do you really agree with each other?
Fan: Only when there is one “expert”.

Ken: What about if there are more than one?
Fan: Only when there is “one” electron.

(to understand single-particle physics,
no interaction, no disorder;
but with “spin”, as $\theta^2 = -1$.)
Conversations with Ken Palley (II)

Ken: Is there any unsolved physics problem?
Fan: Yes, e.g., do you want to improve your computer?
Ken: No. I am completely satisfied with my current one. Do you?
Fan: Maybe only when I need to write a proposal.
Ken: Poor boy, let’s go surfing.

(conceptual novelty & experimental realization hopefully, instead of making any device or quantum computing)
Jackiw-Rebbi Model (1976)

1D Massive Dirac Fermion:

\[ H = v k_z \sigma_y + m_0 \text{sgn}(z) \sigma_z \]

a special solution: \( E = 0, \sigma_x = 1, \ z = 0 \)

(gapless, fractionalized, localized)
Topological Insulators and Superconductors

1D Massive Dirac Fermion:

\[ H = v k_z \sigma_y + m_0 \text{sgn}(z) \sigma_z \]

a special solution: \( E = 0, \sigma_x = 1, z = 0 \)

(gapless, fractionalized, localized)

- Band inversions (broadly defined)
- (Topological) boundary condition
- Not topological yet !!!

To make a TI or TSC:

- Add dimensions + impose symmetries
- Are gapless modes protected?
  (topological or trivial)
- If protected, are they still protected if there are two copies? (\( \mathbb{Z} \) or \( \mathbb{Z}_2 \))

Jackiw-Rebbi, PRD 13, 3398 (1976); FZ-Kane-Mele, 2012PRB(R), 2013PRL.
## Periodic Table (free fermions)

<table>
<thead>
<tr>
<th>s</th>
<th>AZ</th>
<th>$\Theta^2$</th>
<th>$\Xi^2$</th>
<th>$\Pi^2$</th>
<th>Dimension ($k$)</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>A</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<tr>
<td>2</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>3</td>
<td>D</td>
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<td>1</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
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<td>0</td>
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</tr>
<tr>
<td>6</td>
<td>C</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>7</td>
<td>CI</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

(Kitaev; Schnyder-Ryu-Furusaki-Ludwig, Teo-Kane; …)

## Outline of My Talk

- Time-reversal-invariant topological SC [in class DIII]
- Topological mirror SC [two copies of class AIII insulators]
- New $\mathbb{Z}_2 \times \mathbb{Z}_2$ Fractional Josephson effects [beyond the table]

**Ref:** FZ-Kane-Mele, PRL 111, 056402 & 056403 (2013); FZ-Kane arXiv:1310.5281.
Review: Class D Topological SC

<table>
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<tr>
<th>Dimensions</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class D</td>
<td>$Z_2$</td>
<td>$Z_2$</td>
<td>$Z$</td>
</tr>
</tbody>
</table>

$H_{BDG} = H_N + H_\Delta$

- In 2D, the goal is to generate the Berry phase $\pi$.
- Replacing the winding number of chiral p wave pairing by the wind number of a single helical band. [Fu-Kane 2008]
- Time reversal symmetry must be broken.
- Dimension reduction: from 2D to 1D (1D TSC)
- 0D Invariant (implies a fractional Josephson effect)

**Q:** Is it possible to engineer a topological SC that respects time reversal symmetry using proximity effect?
Class DIII (TRI) Superconductors

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class DIII</td>
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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>

- Without using any interaction or Josephson effect, pure s-wave pairing is impossible to induce TRI TSC. [FZ-Kane-Mele, PRL (2013)]

- My favorite criterion: [There are other equivalent ones.]
  The DIII $\mathbb{Z}_2$ invariant is determined by whether the pairing has a negative sign on odd number of Fermi surfaces, each of which encloses a TRI momentum. [Qi-Hughes-Zhang, PRB (2010).]
Extended S-wave + Rashba semiconductor

- Time-reversal symmetry: no magnetic perturbation
- No interactions: using proximity effect
- nodeless Iron pnictide SC + extended s-wave

[I. I. Mazin, Nature (2010)]

FZ-Kane-Mele, PRL 111, 056402 (2013)
For simplicity we consider:  \( \Delta_0 = 0 \)

Topological criterion:  \( |\mu| < 2\lambda_R \)

\[ \mathcal{H} = (-2t \cos k_x + 2\lambda_R \sin k_x \sigma_z - \mu) \tau_z + (\Delta_0 + 2\Delta_1 \cos k_x) \tau_x , \]

FZ-Kane-Mele, PRL 111, 056402&056403 (2013)
Evolution of a Majorana Kramers Pair in Zeeman Fields

Breaking mirror symmetry:

Respecting mirror symmetry:

- a MKP forms two ABS
- opposite fermion parity
- Zeeman splitting

signatures in tunneling spectroscopy!

FZ-Kane-Mele, PRL 111, 056402 (2013)
Majorana Kramers Pair

MKP forms a fermion level

\[ |0 \uparrow\rangle \quad |1 \downarrow\rangle \]

TRS = Super Symmetry

TRS

\[ \gamma \quad \eta \]

MF’s: \[ \gamma^\dagger = \gamma, \quad \eta^\dagger = \eta \]

TRS: \[ \gamma \rightarrow \eta \rightarrow -\gamma \]

Define: \[ c_\uparrow = \gamma + i\eta \]

TRS: \[ c_\downarrow = \eta + i\gamma \]

\[ = i(\gamma - i\eta) \]

\[ = ic_\uparrow^\dagger \]

\[ c_\sigma = ic_\bar{\sigma}^\dagger \]

FZ-Kane-Mele, PRL 111, 056402 & 056403 (2013)
Puzzle: 4π Josephson Effects? Non-Abelian Statistics?

- Tunneling Cooper pairs or electrons? [Yes, 4π]
- One minus sign for each Majorana? [No, 2π]

When folded into each other, the Josephson effects can thus be interpreted as the boundary consequences of the bulk invariant of $H(k, \phi)$;

- $H(k, \phi)$ inherits PH, TR, and chiral symmetry constraints.
- How many topological inequivalent $H(k, \phi)$?
Anomalous Pumps $H(k, r, \phi)$

- Class D can be understood by the original table;
- Class AIII, DIII, … cannot !!!

<table>
<thead>
<tr>
<th>$k$</th>
<th>$r$</th>
<th>$\phi$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$</td>
<td>$+$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
</tbody>
</table>

Class AIII:

<table>
<thead>
<tr>
<th>$d_k - d_r$</th>
<th>even</th>
<th>odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_\phi - d_\theta$</td>
<td>even</td>
<td>odd</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$d_k - d_r$</th>
<th>0, 4, 5, 6</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_\phi - d_\theta$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}$</td>
</tr>
</tbody>
</table>

The 1st $\mathbb{Z}_2$ Invariant

- $4\pi$ periodic Josephson effect (non-Abelian statistics of MFs)
- Robust to TRS breaking

(L. Molenkamp, A. Yacoby, R. Du, ...)

Diagram:

- (a) $(\nu, \mu) = (1, 0)$ pump
- $E(\phi)$ vs $\phi$ for different fluxes
- QSH edge states
- Flux $\Phi$
- $s$ wave SC
The 2nd $Z_2$ Invariant

- 4$\pi$ periodic Josephson effect (non-Abelian statistics of Majorana Kramers pairs)

- In the fermion parity (FP: 0 or 1) basis of each “spin”, the adiabatic pumping of FP and “spin” follows: ($\phi$ advances by $\pi$ in each step)

$$|0_{\uparrow0_{\downarrow}}\rangle \rightarrow |1_{\uparrow0_{\downarrow}}\rangle \rightarrow |1_{\uparrow1_{\downarrow}}\rangle \rightarrow |1_{\uparrow0_{\downarrow}}\rangle \rightarrow |0_{\uparrow0_{\downarrow}}\rangle$$

The 3rd $Z_2$ Invariant

(a) $(\nu, \mu) = (1, 0)$ pump

(b) $(\nu, \mu) = (0, 1)$ pump

(c) $(\nu, \mu) = (1, 1)$ pump

Models for Andreev Bound States

\[ H_a = \delta \phi \sigma_y \]
\[ H_b = \delta \phi \sigma_y \tau_z \]
\[ H_c = -\delta \phi \sigma_y \]

**PHS:** \( \mathcal{E} H(k, \phi) \mathcal{E}^{-1} = -H(-k, \phi); \)

**TRS:** \( \theta H(k, \phi) \theta^{-1} = H(-k, -\phi) \)

Choose a **gauge** in which: \( \mathcal{E} = K \) \( \text{ (with } \mathcal{E}^2 = +1) \)
\( \theta = \sigma_y K \) \( \text{ (with } \theta^2 = -1) \)

P. Joyez Group
Nature Physics (2010)

- CNT
- STM
Conclusions

Theory: How many kinds of topological states there are?

Experiment: When will we observe one Majorana fermion?
Majoranas?

Merry Christmas

Thank you and have a nice trip!