Bäcklund transformations and fast moving strings

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Motivation

The theory of integrable systems is usually based on the monodromy matrix

$$M(\lambda) = P \exp \int d\sigma L(\sigma, \lambda)$$

This monodromy matrix is generally speaking a nonlocal functional of the worldsheet fields. In AdS/CFT program we are especially interested in the local dynamics of the worldsheet:

- If we want to relate to the CFT side, it would be nice to compare the string equations to the Feynman diagramms, which are local in the large N limit
- If we want to use the integrability in higher genus amplitudes

The string σ-model has local conserved charges. The Hamiltonian flows of these charges are nonlocal transformations of the worldsheet, but there are special linear combinations of charges which generate Bäcklund transformations which are described by the local equations.

Motivation

Bäcklund transformations correspond to special “shifts of times”. In the language of τ-functions they are related to the creation of fermions. A typical example is the Kontsevich matrix model:

$$\mathcal{F} = \int dX e^{-\frac{1}{3}\mathrm{tr}\, X^3 + \mathrm{tr}\, M^2 X}$$

The “times” which in KdV correspond to the local integrals of motion are

$$T_n = \frac{1}{n}\mathrm{tr}\, M^{-n} = \frac{1}{n}\sum_{k=1}^{N} \mu_k^{-n}$$

The creation of fermion corresponds to adding an eigenvalue μ_{N+1}:

$$T_n \to T_n + \frac{\mu_{N+1}^{-n}}{n}$$

Q-operator

The quantum analogue of the Bäcklund transformation is the Q-operator. It commutes with the monodromy matrix and satisfies the Baxter equation:

$$t(\lambda)\mathcal{Q}(\lambda) = (\cdots)\mathcal{Q}(\lambda+1) + (\cdots)\mathcal{Q}(\lambda-1) \tag{1}$$

I will discuss properties of Bäcklund transformations for classical strings in $AdS_5 \times S^5$, in the sector of fast moving strings.

Plan of the talk

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1. General facts about Bäcklund transformations (BT)
2. Fast moving strings
3. Bäcklund transformations of classical strings
4. Sine-Gordon sector
5. Plane wave limit
6. Remarks on finite-gap solutions

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A general idea of Bäcklund transformation

Suppose that ϕ_{old} is some solution of some partial differential equation. We can try to define a new function ϕ_{new} by the equation of the form:

$$d\phi_{new} = \Phi_\gamma(\phi_{new}, \phi_{old}, d\phi_{old})$$

We can ask what are the compatibility conditions for this equation (sometimes they are equivalent to the PDE on ϕ_{old}) and if ϕ_{new} satisfies the same PDE as ϕ_{old}.

The principal chiral model

(After J. Harnad, Y. Saint-Aubin and S. Shnider)

Phase space is the space of functions $g(\tau^+, \tau^-)$ satisfying the equations of motion:

$$\partial_+(\partial_- g g^{-1}) + \partial_-(\partial_+ g g^{-1}) = 0 \tag{2}$$

The Bäcklund transformation is of the form:

$$g \mapsto \tilde{g} = Ug \tag{3}$$

where $U = U(\tau^+, \tau^-, \lambda)$ is a unitary matrix depending on the parameter λ. (In the case we are interested in, $\lambda \in \mathbf{C}$ and $|\lambda| = 1$.)

The principal chiral model

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This unitary matrix has the following form:

$$U = \mathbf{1} - P + \bar{\lambda}/\lambda P \tag{4}$$

where P is the projector on the k-dimensional subspace $L \subset \mathbf{C}^n$ satsisfying the equations

$$\begin{aligned} \partial_+ L &= \tfrac{1}{1+\bar{\lambda}} \partial_+ g g^{-1} . L \\ \partial_- L &= \tfrac{1}{1-\bar{\lambda}} \partial_- g g^{-1} . L \end{aligned} \tag{5}$$

This equation is called the *linearization* of the Bäcklund transformation. It is the specification of the zero curvature equation on $Gr(k, n, \mathbf{C})$.

Bäcklund transformations as canonical transformations

(after V.B. Kuznetsov and E.K. Sklyanin)

Suppose that the algebra of functions on the phase space is generated by the entries of the 2×2-matrix $T(\lambda)$ with the Poisson brackets

$$\{T(\lambda_1) \overset{\otimes}{,} T(\lambda_2)\} = [r(\lambda_1 - \lambda_2), T(\lambda_1) \otimes T(\lambda_2)]$$

Introduce an auxiliary phase space generated by the elements of another 2×2 matrix $M(\lambda)$ and with the Poisson brackets

$$\{T(\lambda_1) \overset{\otimes}{,} M(\lambda_2)\} = 0$$

$$\{M(\lambda_1) \overset{\otimes}{,} M(\lambda_2)\} = [r(\lambda_1 - \lambda_2), M(\lambda_1) \otimes M(\lambda_2)]$$

Bäcklund transformations as canonical transformations

(after V.B. Kuznetsov and E.K. Sklyanin)

Notice that the product TM satisfies the same Poisson brackets as T and M. Consider the canonical transformation $(T, M) \mapsto (\widetilde{T}, \widetilde{M})$ defined by this formula:

$$TM = \widetilde{M}\widetilde{T}$$

This defines a canonical transformation $\mathcal{T} \times \mathcal{M} \to \mathcal{T} \times \mathcal{M}$. Now impose the constraint $\widetilde{M} = M$. Suppose that this constraint allows us to express M through T. Then, on this constraint, $T \mapsto \widetilde{T}$ determines the canonical transformation $\mathcal{T} \to \mathcal{T}$ which is the “abstract” Bäcklund transformation.

Bäcklund transformations as canonical transformations

(after V.B. Kuznetsov and E.K. Sklyanin)

One class of models for which this scheme works are the classical mechanical lattice models. In these models the phase space is the product

$$\mathcal{T} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$$

over the sites of the 1-dimensional lattice (chain), and the L-matrix is

$$T(\lambda) = \ell_n(\lambda) \cdots \ell_1(\lambda)$$

For example, for the Toda chain $\ell_k(\lambda) = \begin{pmatrix} \lambda + p_j & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix}$ and we choose $M(\lambda) = \begin{pmatrix} \lambda + \tilde{p} & -e^{\tilde{x}} \\ e^{-\tilde{x}} & 0 \end{pmatrix}$

Bäcklund transformations as canonical transformations

(after V.B. Kuznetsov and E.K. Sklyanin)

The Bäcklund transformation $(x_i, p_i) \mapsto (x_i', p_i')$ is given by the equation:

$$p_j = e^{x_j - x_j'} + e^{x_{j+1}' - x_j} - \lambda \tag{6}$$

$$p_j' = e^{x_j - x_j'} + e^{x_j' - x_{j-1}} - \lambda \tag{7}$$

Although the transformation $(x, p) \mapsto (x', p')$ is by itself nonlocal (a difference equation for (x', p')), but the equations (6) and (7) themselves are local.

Rigid strings

The study of fast moving classical strings was started in the works of S. Frolov and A. Tseytlin. They studied a special class of string worldsheets now known as “rigid strings”. They are called rigid because the for these solutions the shape of the string does not depend on the time. Let T denotes the global time of the AdS_5. The profile of the rigid string at $T = T_0$ is related to the profile at $T = 0$ by a global rotation of AdS.

Rigid strings

The simplest examples of the rigid solutions are those worldsheets which project to the timelike geodesic in AdS_5. These strings "move only in S^5". Let us parametrize S^5 by three complex numbers Z_1, Z_2, Z_3 satisfying $|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$. The profile of the rigid string is given by the equation:

$$Z_I(\tau, \sigma) = e^{iw_I \tau} Z_I^{(0)}(\sigma)$$

where w_I, $I = 1, 2, 3$ are some real constants and $Z_I^{(0)}(\sigma)$ solves the eqs. of motion of the Neumann integrable system, with the constraint $\sum_{J=1}^{3} w_J \overline{Z}_J \partial_\sigma Z_J = 0$. Also $Z_I(\sigma)$ should be periodic modulo the "overall phase" ϕ: $Z_J(\sigma = 2\pi) = e^{iw_J \phi} Z_J(\sigma = 0)$.

The observation of Frolov and Tseytlin

For each set $(w_1, w_2, w_3) \in \mathbf{R}^3$ there will be a discreet set of the periodic (modulo the "overall phase") trajectories, therefore a discreet set of string worldsheets.
For each worldsheet we can compute the momenta of $U(1)^3 \subset U(3) \subset SO(6)$, and parametrize the solution by the momentum (J_1, J_2, J_3). It turns out that the energy is given by

$$E = J\left[1 + \frac{\lambda}{J^2}c_1 + \left(\frac{\lambda}{J^2}\right)^2 c_2 + \dots\right]$$

Frolov and Tseytlin conjectured that this expansion in powers of $\frac{\lambda}{J^2}$ correpsonds to the Yang-Mills perturbative expansion, and $c_1, c_2, \dots$ are the coefficients of the anomalous dimension. They depend on the ratios $\frac{J_1}{J_2}, \frac{J_2}{J_3}$.

“Speeding strings”

Rigid strings correspond to operators extremizing the anomalous dimension in the sector with the given charges. For all these cases where the field theory anomalous dimension was reproduced, it is true that every point on the worldsheet is moving almost with the speed of light. In fact, for the comparison with the field theory, it is more important that the string moves very fast, rather than the rigidity. Therefore we can consider a more general class of fast moving strings. In the limit when every point of the string moves with the speed of light, the string worldsheet becomes a null-surface. Null-surface is a degenerate worldsheet, and the worldsheet of the fast moving string is nearly degenerate.

"Speeding strings"

The null-surface perturbation theory

It turns out that there is a perturbation theory in powers of $\sqrt{1-v^2}$ for the fast moving string as a perturbation (or "resolution") of the null-surface.

One way to construct this perturbation theory is to fix a special set of worldsheet coordinates (τ,σ) such that the worldsheet metric is $\simeq \varepsilon^2 d\tau^2 - d\sigma^2$, ε is a small parameter and and $\partial_\tau x$, $\partial_\sigma x$ are finite in a reasonable coordinate system in the target space. The worldsheet action becomes $\int d\sigma d\tau \left[(\partial_\tau x)^2 - \varepsilon^2(\partial_\sigma x)^2\right]$ and term with ε^2 can be considered a perturbation. The unperturbed system $\int d\tau d\sigma(\partial_\tau x)^2$ is a free theory ("string bits").

It was conjectured that the Yang-Mills perturbation theory corresponds to considering the worldsheet of the fast moving string as a perturbation of the null-surface. (Null-surfaces themselves correspond to operators in the free theory.)

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An important property of the string theory on $AdS_5 \times S^5$ is integrability. Integrability leads to the existence of infinitely many local conserved charges (Pohlmeyer charges).
We will now discuss it from the point of view of the null-surface perturbation theory.

Pohlmeyer charges

Let us turn off the fermionic degrees of freedom and consider just a bosonic string on $AdS_5 \times S^5$. Let us parametrize S^5 by $Y \in \mathbf{R}^6$, $(Y, Y) = 1$. The first Pohlmeyer charge exists simply because the target space is a direct product of two manifolds:

$$Q^{[1]} = \oint_C d\tau^+ \sqrt{(\partial_+ Y)^2} \tag{8}$$

This is the "left" charge. There is also a "right" charge:

$$\widetilde{Q}^{[1]} = \oint_C d\tau^- \sqrt{(\partial_- Y)^2} \tag{9}$$

The special properties of S^5 allow us to construct the second conserved charge:

$$Q^{[2]} = \int \left[2\frac{d\tau^-}{|\partial_+ Y|}(\partial_- Y, \partial_+ Y) + \frac{d\tau^+}{|\partial_+ Y|} \left(D_+ \frac{\partial_+ Y}{|\partial_+ Y|}, D_+ \frac{\partial_+ Y}{|\partial_+ Y|} \right) \right] \tag{10}$$

In fact there is an infinite tower of charges $Q^{[n]}$, $\widetilde{Q}^{[n]}$, and a similar tower for AdS_5.

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Pohlmeyer charges

Considering a higher conserved charge $Q^{[n]}$ as a Hamiltonian, we can compute the correpsonding Hamiltonian vector field which we will denote ξ_{2n}.

The explicit formulas for ξ_{2n} can be very complicated.

But it turns out that there are special linear combinations of these Pohlmeyer charges:

$$\xi_\gamma = \sum_n t_{2n}(\gamma)\xi_{2n}$$

The Hamiltonian flows of these special linear combinations are Bäcklund transformations. They can be described by explicit first order equations.

Any canonical transformation generated by the local conserved charges can be approximated by a composition $B_{\gamma_1} \cdots B_{\gamma_N}$ with large enough N. In this sense, the Bäcklund transformations fully characterize the Hamiltonian flows of the local charges.

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Definition of BT for $O(N)$

1.Spectrality

We introduce the Bäcklund transformations for $O(N)$ model "axiomatically", by requiring three basic properties:

1. Spectrality. BT are generated by the Hamiltonian flows of the Pohlmeyer charges:

$$B_\gamma Y = \exp\left(\sum_n t_{2n}(\gamma)\xi_{2n}\right).Y = \exp(\xi_\gamma).Y \qquad (11)$$

$$\tilde{B}_{\tilde{\gamma}} Y = \exp\left(\sum_n \tilde{t}_{2n}(\tilde{\gamma})\tilde{\xi}_{2n}\right).Y = \exp(\tilde{\xi}_{\tilde{\gamma}}).Y \qquad (12)$$

with some coefficients t_{2n} and $\tilde{t}_{2n}$ depending on γ and $\tilde{\gamma}$.

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BT of $O(N)$ model

2. Bäcklund equations

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$$\begin{aligned} \partial_-(B_\gamma Y - Y) &= -\tfrac{1}{2}(1+\gamma^2)(B_\gamma Y, \partial_- Y)(B_\gamma Y + Y) \\ \partial_+(B_\gamma Y + Y) &= \tfrac{1}{2}(1+\gamma^{-2})(B_\gamma Y, \partial_+ Y)(B_\gamma Y - Y) \end{aligned} \quad (13)$$

$$\begin{aligned} \partial_+(\tilde{B}_{\tilde{\gamma}} Y - Y) &= -\tfrac{1}{2}(1+\tilde{\gamma}^{-2})(\tilde{B}_{\tilde{\gamma}} Y, \partial_+ Y)(\tilde{B}_{\tilde{\gamma}} Y + Y) \\ \partial_-(\tilde{B}_{\tilde{\gamma}} Y + Y) &= \tfrac{1}{2}(1+\tilde{\gamma}^{2})(\tilde{B}_{\tilde{\gamma}} Y, \partial_- Y)(\tilde{B}_{\tilde{\gamma}} Y - Y) \end{aligned} \quad (14)$$

BT of $O(N)$ model.

3. When $\gamma \to 0$ and $\tilde{\gamma} \to \infty$

3. *For small* γ *and large* $\tilde{\gamma}$ we should have:

$$B_\gamma Y = Y - \gamma \frac{\partial_+ Y}{|\partial_+ Y|} + o(\gamma) \tag{15}$$

$$\tilde{B}_{\tilde{\gamma}} Y = Y + \frac{1}{\tilde{\gamma}} \frac{\partial_- Y}{|\partial_- Y|} + o\left(\frac{1}{\tilde{\gamma}}\right) \tag{16}$$

“Spectral” BT vs. general BT

The traditional definition of BT in the literature uses only Bäcklund equations and does not require spectrality. In fact, the traditional definition is more general than our definition. Most of the Bäcklund transformations considered usually in the literature do not satisfy the spectrality property.

One can see that the Bäcklund equations do not determine $B_\gamma Y$ (or $\tilde{B}_{\tilde{\gamma}} Y$) unambiguously for a given Y. Indeed, to solve these equations we have to choose the integration constant. In this sense the traditional Bäcklund transformations are not single-valued. And for most of the choices of the integration constant the spectrality is just not true.

But one particular choice of the integration constant will give $B_\gamma Y$ satisfying spectrality. We can call it a “spectral” Bäcklund transformation although this is not a standard terminology. How can we characterize this special choice of a solution of the Bäcklund equations?

"Spectral" solution = perturbative solution

For small γ we can solve the Bäcklund equations perturbatively as a series in γ starting with Eq. (15) as the first approximation. If the series converges, then so defined $B_\gamma Y$ is the "spectral" solution. The series $\sum_n t_{2n}(\gamma)\xi_{2n}$ of Eq. (11) can be thought of as the series in powers of γ.

When we solve Bäcklund eqs. perturbatively in γ we get B_γ a well-defined transformation: $B_\gamma = \mathrm{Id} + \gamma\Delta_1 + \gamma^2\Delta_2 + \ldots$. We can formally take a logarithm of it: $\log B_\gamma = \gamma\delta_1 + \gamma^2\delta_2 + \ldots$. We have not proven that B_γ is a canonical transformation. But if we assume it, then δ_n whould be Hamiltonian vector fields. One can see that δ_n commute with $\mathcal{E}_2$, therefore the Hamiltonian of δ_n should belong to the maximal set of commuting Hamiltonians. And since all δ_n are manifestly local, the corresponding Hamiltonians should also be local. Thus δ_n are generated by the local commuting Hamiltonians.

BT and fast moving strings

In the sector of fast moving strings we could also use an alternative perturbation theory for computing $B_\gamma Y$, the null-surface perturbation theory.

BT and fast moving strings

Consider the null-surface limit. Each point of the string moves on a separate equator of S^5, and different points do not interact with each other. In this limit $|\partial_\tau Y| >> |\partial_\sigma Y|$ and the Bäcklund equations can be solved explicitly:

$$B_\gamma Y = \frac{1-\gamma^2}{1+\gamma^2} Y - \frac{2\gamma}{1+\gamma^2} \frac{\partial_\tau Y}{|\partial_\tau Y|} \tag{17}$$

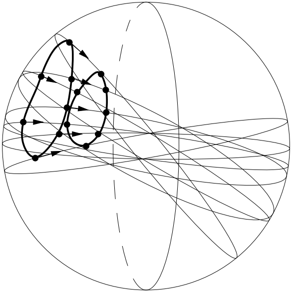
$$\tilde{B}_{\tilde{\gamma}} Y = \frac{1-\tilde{\gamma}^{-2}}{1+\tilde{\gamma}^{-2}} Y + \frac{2\tilde{\gamma}^{-1}}{1+\tilde{\gamma}^{-2}} \frac{\partial_\tau Y}{|\partial_\tau Y|} \tag{18}$$

This means that for the null-surface we have:

$$\xi_\gamma = -2\frac{\arctan\gamma}{|\partial_\tau Y|}\frac{\partial}{\partial\tau} \quad \text{and} \quad \tilde{\xi}_{\tilde{\gamma}} = 2\frac{\arctan(1/\tilde{\gamma})}{|\partial_\tau Y|}\frac{\partial}{\partial\tau}$$

BT and fast moving strings

This picture illustrates the Bäcklund transformation B_γ of a null-surface:



Each point on the string shifts along the corresponding geodesic by the angle $2\arctan\gamma$.

BT and fast moving strings

For the fast moving string the BT can be constructed perturbatively in the null-surface perturbation theory. The small parameter is $1/|\partial_\tau Y|$. The coefficients are rational functions of γ.

We conjecture that the series of the null-surface perturbation theory converge and determine $B_\gamma Y$ when the string worldsheet moves fast enough. This is related to the convergence of series in the spectrality property of the BT: the series $\sum_n t_{2n}(\gamma)\xi_{2n}$ should be convergent.

Conjecture of "improved charges": It is possible to choose the local charges in such a way that $\xi_{2n} \simeq |\partial_\tau Y|^{1-k}$. Therefore $\sum_n t_{2n}(\gamma)\xi_{2n}$ can be thought of as the series of the null-surface perturbation theory.

Properties of BT for $O(N)$

Permutability and tangent rule

Bäcklund transformations commute:

$$B_{\gamma_1} B_{\gamma_2} = B_{\gamma_2} B_{\gamma_1} \tag{19}$$

This is called “permutability theorem”. We should stress that BT do not form a one-parameter group of transformations. It is not true that $B_{\gamma_1} B_{\gamma_2} = B_{\gamma_3}$ (wrong) for some γ_3.

An important consequence of the commutativity is the “tangent rule” for the composition of the Bäcklund transformations:

$$B_{\gamma_1} B_{\gamma_2} Y - Y = \frac{(Y, B_{\gamma_2} Y) - (Y, B_{\gamma_1} Y)}{1 - (B_{\gamma_1} Y, B_{\gamma_2} Y)} (B_{\gamma_1} Y - B_{\gamma_2} Y) \tag{20}$$

The composition of the Bäcklund transformations is given by the algebraic expression (no derivatives; do not have to solve a differential equation). For the sine-Gordon model this “tangent rule” is closely related to the bilinear identity.

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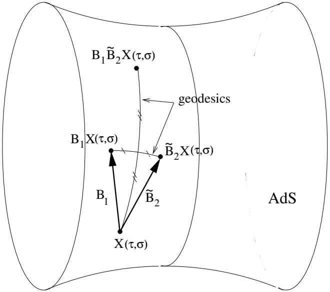
Tangent rule



This picture illustrates the tangent rule for $B_\gamma Y$ and $\tilde{B}_{\tilde{\gamma}} Y$. (The AdS part is shown)

$$B_{\gamma_1}\tilde{B}_{\gamma_2}Y + Y = \frac{(Y,\tilde{B}_{\gamma_2}Y) + (Y,B_{\gamma_1}Y)}{1 + (B_{\gamma_1}Y,\tilde{B}_{\gamma_2}Y)}(B_{\gamma_1}Y + \tilde{B}_{\gamma_2}Y)$$

In other words

$$(B_{\gamma_1}\tilde{B}_{\gamma_2}Y + Y) \wedge (B_{\gamma_1}Y + \tilde{B}_{\gamma_2}Y) = 0$$

Action variable

We have defined B_γ by an expansion near $\gamma = 0$ and $\tilde{B}_{\tilde{\gamma}}$ by an expansion near $\gamma = \infty$. But the null-surface perturbation theory allows us to define B_γ and $\tilde{B}_{\tilde{\gamma}}$ for finite values of γ and $\tilde{\gamma}$.

Then Eqs. (13) and (14) imply for $\gamma = \tilde{\gamma}$:

$$B_\gamma Y = -\tilde{B}_\gamma Y \tag{21}$$

Given that $B_\gamma = e^{\xi_\gamma}$ and $\tilde{B}_\gamma = e^{\tilde{\xi}_\gamma}$, what does it imply for ξ_γ and $\tilde{\xi}_\gamma$? The naive guess $\xi_\gamma - \tilde{\xi}_\gamma = 0$ is wrong.
In fact, $\xi_\gamma - \tilde{\xi}_\gamma$ is nonzero but generates periodic trajectories on the phase space:

$$\exp(\xi_\gamma - \tilde{\xi}_\gamma) = 1 \tag{22}$$

Therefore, if H_γ and $\tilde{H}_\gamma$ are the corresponding Hamiltonians, then $H_\gamma - \tilde{H}_\gamma$ does not depend on γ and is an action variable.

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Action variable

We have defined B_γ by an expansion near $\gamma = 0$ and $\tilde{B}_{\tilde{\gamma}}$ by an expansion near $\gamma = \infty$. But the null-surface perturbation theory allows us to define B_γ and $\tilde{B}_{\tilde{\gamma}}$ for finite values of γ and $\tilde{\gamma}$.

Then Eqs. (13) and (14) imply for $\gamma = \tilde{\gamma}$:

$$B_\gamma Y = -\tilde{B}_\gamma Y \tag{21}$$

Given that $B_\gamma = e^{\xi_\gamma}$ and $\tilde{B}_\gamma = e^{\tilde{\xi}_\gamma}$, what does it imply for ξ_γ and $\tilde{\xi}_\gamma$? The naive guess $\xi_\gamma - \tilde{\xi}_\gamma = 0$ is wrong.
In fact, $\xi_\gamma - \tilde{\xi}_\gamma$ is nonzero but generates periodic trajectories on the phase space:

$$\exp(\xi_\gamma - \tilde{\xi}_\gamma) = \mathbf{1} \tag{22}$$

Therefore, if H_γ and $\tilde{H}_\gamma$ are the corresponding Hamiltonians, then $H_\gamma - \tilde{H}_\gamma$ does not depend on γ and is an action variable.

AdS part

Deck transformation

We can also consider the AdS part of the σ-model. Let us denote B^A_γ and $\tilde{B}^A_{\tilde{\gamma}}$ the Bäcklund transformations of the AdS part of the string and ξ^A_γ, $\tilde{\xi}^A_{\tilde{\gamma}}$ the corresponding Hamiltonian vector fields. Then

$$(B^A_\gamma(\tilde{B}^A_\gamma)^{-1})^2 = \text{deck transformation}$$

This suggests that if we construct the quantum versions $\mathcal{Q}^A(\gamma)$ and $\tilde{\mathcal{Q}}^A(\tilde{\gamma})$ of the Bäcklund transformations (such that $\mathcal{Q}^A(0) = \tilde{\mathcal{Q}}^A(\infty) = 1$) then the anomalous dimension will be given by:

$$e^{\pi i \Delta E} = \lim_{\gamma \to \tilde{\gamma}} \mathcal{Q}^A(\gamma)\tilde{\mathcal{Q}}^A(\tilde{\gamma})^{-1}$$

String on $\mathbf{R} \times S^2$

Consider a particular case when the motion of the string is restricted to $\mathbf{R} \times S^2 \subset AdS_5 \times S^5$.
The worldsheet equations of motion are:

$$(\partial_\tau^2 - \partial_\sigma^2)\vec{n} = -[(\partial_\tau \vec{n})^2 - (\partial_\sigma \vec{n})^2]\vec{n} \tag{23}$$

These equations of motion follow from the constraints:

$$\left(\frac{\partial \vec{n}}{\partial \tau}\right)^2 + \left(\frac{\partial \vec{n}}{\partial \sigma}\right)^2 = 1 \tag{24}$$

$$\left(\frac{\partial \vec{n}}{\partial \tau}, \frac{\partial \vec{n}}{\partial \sigma}\right) = 0 \tag{25}$$

String on $\mathbf{R} \times S^2$

Map to sine-Gordon

The map to the sine-Gordon model (K. Pohlmeyer, 1976) is given by:

$$\cos 2\phi = \left(\frac{\partial \vec{n}}{\partial \tau}\right)^2 - \left(\frac{\partial \vec{n}}{\partial \sigma}\right)^2 \tag{26}$$

In other words

$$\left|\partial_\tau \vec{n}\right| = |\cos\phi|, \qquad \left|\partial_\sigma \vec{n}\right| = |\sin\phi| \tag{27}$$

The Virasoro constraints (24), (25) imply the sine-Gordon equation:

$$\left[\partial_\tau^2 - \partial_\sigma^2\right]\phi = -\frac{1}{2}\sin 2\phi \tag{28}$$

String on $\mathbf{R} \times S^2$

Map to sine-Gordon

It is not true that the canonical Poisson structure of the sine-Gordon corresponds to the canonical Poisson structure of the string on $\mathbf{R} \times S^2$.

In the free field limit (when ϕ is very small) the canonical Poisson structure of sine-Gordon is $\{\phi(\sigma_1), \dot\phi(\sigma_2)\} = \delta(\sigma_1 - \sigma_2)$. This corresponds to the plane wave limit

$$\vec{n} = \cos(x_+)\vec{e}_1 + \sin(x_+)\vec{e}_2 + y\vec{e}_3$$

But $\phi \simeq |\partial_\sigma \vec{n}| \simeq \partial_\sigma y$ and therefore the corresponding Poisson structure of the $O(3)$ model would be $\{\partial_\sigma y(\sigma_1), \partial_\sigma \dot{y}(\sigma_2)\} = \delta(\sigma_1 - \sigma_2)$. We suspect that this is the second Poisson structure of the $O(3)$ model.

String on $\mathbf{R} \times S^2$

BT in sine-Gordon and $O(3)$

BT for sine-Gordon are defined by the following equations:

$$\partial_+(B_\gamma\phi + \phi) = -\frac{1}{2}\gamma^{-1}\sin(B_\gamma\phi - \phi) \tag{29}$$

$$\partial_-(B_\gamma\phi - \phi) = \frac{1}{2}\gamma\sin(B_\gamma\phi + \phi) \tag{30}$$

If $\vec{n}$ is related to ϕ by Eq. (26) then $B_\gamma\vec{n}$ defined by this formula:

$$B_\gamma\vec{n} = \frac{1-\gamma^2}{1+\gamma^2}\vec{n} - \frac{4\gamma}{1+\gamma^2}\left(\frac{\sin(\phi - B_\gamma\phi)}{\sin(2\phi)}\partial_-\vec{n} + \frac{\sin(\phi + B_\gamma\phi)}{\sin(2\phi)}\partial_+\vec{n}\right)$$

satisfies the Bäcklund equations (13) for the $O(N)$ model (with $N = 3$, $Y = \vec{n}$).

String on $\mathbf{R} \times S^2$

Higher times of SG

Let us solve Bäcklund equations of SG perturbatively in γ. Suppose that the series converges. This defines a spectral BT. It is generated by some infinite linear combination of the local conserved charges of the SG model. The Hamiltonian vector fields of the higher conserved charges are usually thought of as "translations of higher times" $\frac{\partial}{\partial t_j}$. For SG model, there are "left" and "right" local conserved charges, and therefore two series of times which are usually denoted $t_1, t_3, \ldots, t_{2p+1}, \ldots$ and $\tilde{t}_1, \tilde{t}_3, \ldots, \tilde{t}_{2p+1}, \ldots$. Let us define these charges in such a way that the BT is

$$\phi(\{t_{2p+1}\}, \{\tilde{t}_{2q+1}\}) \stackrel{B_\gamma}{\mapsto} \phi\left(\left\{t_{2p+1} - \frac{\gamma^{2p+1}}{2p+1}\right\}, \{\tilde{t}_{2q+1}\}\right) \tag{31}$$

We also put $t_1 = \frac{1}{4}(\tau + \sigma)$ and $\tilde{t}_1 = \frac{1}{4}(\tau - \sigma)$.

String on $\mathbf{R} \times S^2$

Tangent rule and definition of τ

The tangent rule for BT of the SG model is:

$$\tan \frac{B_{\gamma_1} B_{\gamma_2} \phi - \phi}{2} = \frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2} \tan \frac{B_{\gamma_1} \phi - B_{\gamma_2} \phi}{2} \tag{32}$$

Similar tangent rules can be written for $\tilde{B}_{\gamma_1} B_{\gamma_2}$ and $\tilde{B}_{\gamma_1} \tilde{B}_{\gamma_2}$.

To better understand the structure of the tangent rule, we introdice the τ-functions τ_+ and τ_- so that:

$$\phi = i \log \frac{\tau_+}{\tau_-} \tag{33}$$

We define $B_\gamma \tau$ as the shift of times, as in Eq. (31). Let us require that τ_+ and τ_- satisfy the bilinear identities:

String on $\mathbf{R} \times S^2$

Bilinear identity

$$B_\mu B_\nu \tau_+ \tau_- + \tau_+ B_\mu B_\nu \tau_- = B_\mu \tau_+ B_\nu \tau_- + B_\nu \tau_+ B_\mu \tau_-$$

$$\frac{\nu-\mu}{\nu+\mu}(B_\mu B_\nu \tau_+ \tau_- - \tau_+ B_\mu B_\nu \tau_-) = \\ = B_\mu \tau_+ B_\nu \tau_- - B_\nu \tau_+ B_\mu \tau_-$$

$$\tilde{B}_{\tilde{\mu}} \tilde{B}_{\tilde{\nu}} \tau_+ \tau_- + \tau_+ \tilde{B}_{\tilde{\mu}} \tilde{B}_{\tilde{\nu}} \tau_- = \tilde{B}_{\tilde{\mu}} \tau_+ \tilde{B}_{\tilde{\nu}} \tau_- + \tilde{B}_{\tilde{\nu}} \tau_+ \tilde{B}_{\tilde{\mu}} \tau_-$$

$$\frac{\tilde{\nu}-\tilde{\mu}}{\tilde{\nu}+\tilde{\mu}}(\tilde{B}_{\tilde{\mu}} \tilde{B}_{\tilde{\nu}} \tau_+ \tau_- - \tau_+ \tilde{B}_{\tilde{\mu}} \tilde{B}_{\tilde{\nu}} \tau_-) = \\ = -\tilde{B}_{\tilde{\mu}} \tau_+ \tilde{B}_{\tilde{\nu}} \tau_- + \tilde{B}_{\tilde{\nu}} \tau_+ \tilde{B}_{\tilde{\mu}} \tau_-$$

$$B_\mu \tilde{B}_{\tilde{\nu}} \tau_+ \tau_+ + B_\mu \tilde{B}_{\tilde{\nu}} \tau_- \tau_- = B_\mu \tau_+ \tilde{B}_{\tilde{\nu}} \tau_+ + B_\mu \tau_- \tilde{B}_{\tilde{\nu}} \tau_-$$

$$\frac{\tilde{\nu}-\mu}{\tilde{\nu}+\mu}(B_\mu \tilde{B}_{\tilde{\nu}} \tau_- \tau_- - \tau_+ B_\mu \tilde{B}_{\tilde{\nu}} \tau_+) = \\ = B_\mu \tau_+ \tilde{B}_{\tilde{\nu}} \tau_+ - \tilde{B}_{\tilde{\nu}} \tau_- B_\mu \tau_-$$

String on $\mathbf{R} \times S^2$

Bilinear identity

Then the tangent rule tangent rule (32) follows from these bilinear identities. In fact these bilinear identities are equivalent to the SG equations of motion. The bilinear equations can be solved by interpreting them as the Plücker identities for determinants:

$$\tau_\pm = \det(1 \pm \mathcal{V})$$

$$\mathcal{V}_{jk} = 2ib_j b_k \frac{\sqrt{\lambda_j \lambda_k}}{\lambda_j + \lambda_k} \times$$
$$\times \exp\left[\sum_p t_{2p+1}(\lambda_j^{2p+1} + \lambda_k^{2p+1}) - \sum_p \tilde{t}_{2p+1}(\lambda_j^{-2p-1} + \lambda_k^{-2p-1})\right]$$

Here b_j and λ_j, $j = 1, \ldots, N$ are parameters characterizing the solution.

String on $\mathbf{R} \times S^2$

Free fermions

Introduce free fermions $\psi(\mu) = \sum_{m\in\mathbf{Z}} \psi_m \mu^{m-1/2}$ and $\tilde\psi(\mu) = \sum_{m\in\mathbf{Z}} \tilde\psi_m \mu^{-m+1/2}$, $\{\psi_m, \tilde\psi_n\} = \delta_{mn}$.
The "vacuum vectors" are labeled by $k \in \mathbf{Z}$:
$\langle k|\psi(\lambda_1)\tilde\psi(\lambda_2)|k\rangle = \left(\frac{\lambda_1}{\lambda_2}\right)^k \frac{\sqrt{\lambda_1\lambda_2}}{\lambda_1-\lambda_2}$ for $|\lambda_1| > |\lambda_2|$.

The τ-function is:
$$\tau_\pm = \frac{\langle k_\pm|e^{H(\{t\})} g e^{\tilde H(\{\tilde t\})}|k_\pm\rangle}{\langle k_\pm|e^{H(\{t\})} e^{\tilde H(\{\tilde t\})}|k_\pm\rangle} \qquad (34)$$

where $k_+ = 0$, $k_- = 1$,
$$H(\{t\}) = \sum t_{2p+1}\psi_n\tilde\psi_{n+2p+1}$$
$$\tilde H(\{\tilde t\}) = \sum \tilde t_{2p+1}\psi_n\tilde\psi_{n-2p-1}$$
$$g = \prod_{j=1}^{N}\left[1 + 2b_j^2\psi(\lambda_j)\tilde\psi(-\lambda_j)\right]$$

String on $\mathbf{R} \times S^2$

Bosonization

We can say that the vacuum k has “fermionic charge” k. It turns out that any state with the fermionic charge k can be obtained from $|k>$ by acting on $|k>$ with some combination of $H(\{t\})$ and $\tilde{H}(\{\tilde{t}\})$. This can be seen from the bosonization formulas:

$$\begin{aligned} &\exp \sum \frac{\lambda^k}{k} \psi_n \psi_{n-k} \, |k> = \lambda^{-k+1/2} \psi(\lambda) |k-1> \\ &< k| \, \exp \sum \frac{\lambda^{-k}}{k} \psi_n \tilde{\psi}_{n+k} = < k-1| \, \tilde{\psi}(\lambda) \lambda^{k-1/2} \end{aligned} \tag{35}$$

This formulas tell us that B_γ corresponds to the insertion of $\psi(\gamma)$ and $\tilde{B}_{\tilde{\gamma}}$ inserts $\tilde{\psi}(\tilde{\gamma}^{-1})$.

String on $\mathbf{R} \times S^2$

Bilinear identities

The bilinear identity reads:

$$(\gamma_1^{-1} - \gamma_2^{-1})B_{\gamma_1} B_{\gamma_2}\tau B_{\gamma_3}\tau + \mathrm{cycl}(1,2,3) = 0$$

It is a consequence of the Wick theorem:

$$\begin{aligned}
&(\gamma_1^{-1} - \gamma_2^{-1})B_{\gamma_1} B_{\gamma_2} < 0|e^H g|0 > < 0|e^H g|0 >= \\
&=< 0|\psi_{-1}\psi_0\tilde{\psi}(\gamma_1)\tilde{\psi}(\gamma_2)e^H g|0 > < 0|e^H g|0 >= \\
&=< 0|\psi_0\tilde{\psi}(\gamma_1)e^H g|0 >< 0|\psi_{-1}\tilde{\psi}(\gamma_2)e^H g|0 > - \\
&- < 0|\psi_0\tilde{\psi}(\gamma_1)e^H g|0 >< 0|\psi_{-1}\tilde{\psi}(\gamma_2)e^H g|0 >
\end{aligned}$$

The bilinear identities for sine-Gordon $\tau_\pm$ follow if we take into account that

$$\tau_- = \lim_{\gamma\to\infty} B_\gamma \tau_+ \qquad (36)$$

String on $\mathbf{R} \times S^2$

Summary

To summarize, the Bäcklund transformation in the S^2 sector (or sine-Gordon) correspond to the insertion of a fermion:

$$\exp \sum \frac{\lambda^k}{k} \psi_n \psi_{n-k} \, |k>= \lambda^{-k+1/2} \psi(\lambda)|k-1> \simeq \tag{37}$$

$$\simeq \exp \left[\int_0^\lambda d\lambda \tilde{\psi}(\lambda)\psi(\lambda) \right] |k> \tag{38}$$

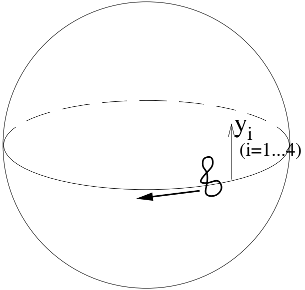
Tangent rule corresponds to the bilinear identity which follows from the Wick theorem.

BT in the plane wave limit

In the plane wave limit the string is fast moving *and* is located near the equator of S^5:

Therefore there are two small parameters: the parameter $1/M$ is the small parameter of the null-surface perturbation theory and the parameter ϵ which measures the deviation from the equator.

$$Y_i = \epsilon y_i \quad (i = 1 \cdots 4)$$



The relation between them is $M = \epsilon^2 J + o(\epsilon^2)$ where J is the BMN angular momentum.

BT in the plane wave limit

In the plane wave limit we can decompose x^I in oscillators (Fourier modes) α_n^I ($I = 1, \cdots, 4$). It is straightforward to compute the action of the Bäcklund transformations on oscillators explicitly:

$$B_\gamma \alpha_n = \frac{M - i\gamma(\omega_n + n)}{M + i\gamma(\omega_n + n)} \alpha_n \tag{39}$$

$$B_\gamma x_+ = x_+ - \arctan\gamma \tag{40}$$

$$B_\gamma x_- = x_- + \frac{2}{\epsilon^2} \arctan\gamma \tag{41}$$

Here

$$\omega_n = \sqrt{M^2 + n^2}$$

BT in the plane wave limit

Vector field ξ_γ

This means that the vector field $\xi_\gamma = \sum_n t_{2n}(\gamma)\xi_{2n}$ defined as the logarithm (see Eq. (11)) of the Bäcklund transformation B_γ acts in the plane wave limit as follows:

$$\xi_\gamma \alpha_n = \log\left[\frac{M - i\gamma(\omega_n + n)}{M + i\gamma(\omega_n + n)}\right]\alpha_n \tag{42}$$

$$\xi_\gamma x_+ = -\arctan\gamma \tag{43}$$

$$\xi_\gamma x_- = \frac{2}{\epsilon^2}\arctan\gamma \tag{44}$$

This vector field ξ_γ is generated by the Hamiltonian H_γ:

$$H_\gamma = \epsilon^2\left[-2\arctan(\gamma)J - i\sum_n \log\left[\frac{M - i\gamma(\omega_n + n)}{M + i\gamma(\omega_n + n)}\right]\alpha_n\overline{\alpha_n}\right] \tag{45}$$

BT in the plane wave limit

Generating function in the plane wave limit

We have argued that the Bäcklund transformations are generated by the local conserved charges. In fact the local conserved charges can be explicitly constructed from the Bäcklund transformations:

$$\mathcal{E}(\gamma) = \frac{1}{2\pi} \int d\sigma \left[\gamma(B_\gamma Y, \partial_+ Y) + \gamma^3 (B_\gamma Y, \partial_- Y) \right] \qquad (46)$$

This is the generating function; $\mathcal{E}(\gamma)$ can be expanded in the even powers of γ and the coefficients are the local conserved charges.

An explicit computation in the plane wave limit gives:

$$\mathcal{E}(\gamma) = -\epsilon^2 \gamma^2 \left[J + (1+\gamma^2) \sum_{n=-\infty}^{\infty} \sum_{l=1}^{4} \frac{M(\omega_n + n)}{M^2 + \gamma^2 (\omega_n + n)^2} \alpha_n^l \overline{\alpha_n^l} \right] \qquad (47)$$

BT in the plane wave limit

Relation between H_γ and $\mathcal{E}(\gamma)$

A comparison of $\mathcal{E}(\gamma)$ of the previous slide with Eq. (45) for H_γ gives

$$\frac{\partial H_\gamma}{\partial \gamma} = \frac{2}{\gamma^2(1+\gamma^2)}\mathcal{E}(\gamma) \qquad (48)$$

When $\gamma = 0$ we should have $H_\gamma = 0$. Therefore

$$H_\gamma = 2\int_0^\gamma \frac{d\gamma}{\gamma^2(1+\gamma^2)}\mathcal{E}(\gamma) \qquad (49)$$

This is the expansion of the Hamiltonian of ξ_γ in terms of the Pohlmeyer charges.

We will now argue that Eq. (49) should be valid for a general fast moving string and not only in the plane wave limit.

BT in the plane wave limit

Relation between H_γ and $\mathcal{E}(\gamma)$

Indeed, $\mathcal{E}(\gamma) = \sum \gamma^{2n}\mathcal{E}_{2n}$ and Eq. (49) gives

$$H_\gamma = 2\sum_n \int_0^\gamma \frac{d\gamma\gamma^{2n}}{1+\gamma^2}\mathcal{E}_{2n} = \sum_n \frac{2}{2n+1}(\gamma^{2n+1} + \ldots)\mathcal{E}_{2n}$$

which is just an expansion of the form (11). The coefficients of this expansion are just fixed functions of γ. There are many ways to find these functions. Here we have derived them from the plane wave limit.

As we have already mentioned, it is possible to view the series in γ actually as series of the null-surface perturbation theory. This is because of the existence of the "improved" charges, which we will now discuss.

BT in the plane wave limit

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BT in the plane wave limit

Improved charges

Let us introduce the angle α: $\tan\alpha = \gamma$. The generating function Eq. (47) can be decomposed in powers of $\sin^2\alpha = \frac{\gamma^2}{1+\gamma^2}$:

$$\mathcal{E}(\gamma) = \tan^2\alpha \sum_{k=1}^{\infty} (-1)^{k-1} (\sin\alpha)^{2k}\, \mathcal{G}_{2k} \qquad (50)$$

where

$$\mathcal{G}_{2k} = \delta_{k,1}\epsilon^2 E + \epsilon^2 \sum_{n=-\infty}^{\infty} \sum_{l=1}^{4} \frac{\omega_n + n}{M} \left[\frac{(\omega_n + n)^2}{M^2} - 1 \right]^{k-1} \alpha_n^l \overline{\alpha_n^l}$$

That is, $\mathcal{G}_{2k}$ is a local conserved charge which is of the order $1/M^{k-1}$. We will call such charges “improved”.

BT in the plane wave limit

Improved charges

We conjecture that the charges $\mathcal{G}_{2k}$ defined from the Pohlmeyer generating function by (50) are "improved" not only in the plane wave limit, but actually exactly improved. It would be interesting to prove it rigorously. (We have to prove the existence of the improved charges.)
It would be interesting to derive (50) using the finite-gap technique.

If we substitute Eq. (50) in Eq. (49) we get:

$$H_\gamma = \sum_{k=0}^{\infty}(-)^k \int_0^\gamma \sum_{k=0}^{\infty}(-)^k \frac{d\gamma\, \gamma^{2k}}{(1+\gamma^2)^{2k+1}} \mathcal{G}_{2k}$$

Therefore the integral representation is in fact equivalent to H_γ being an infinite sum of the improved charges.

BT in the plane wave limit

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Therefore the integral representation is in fact equivalent to H_γ being an infinite sum of the improved charges.

Comparison with the results of the finite-gap approach

Monodromy matrix

Definition of the monodromy matrix:

$$T_R = P\exp\int\left[\frac{\rho_R(\partial_+ Y\wedge Y)}{1-x}d\tau^+ + \frac{\rho_R(\partial_- Y\wedge Y)}{1+x}d\tau^-\right]$$

where ρ_R is some representation of $so(6)$ and $dY \wedge Y$ is considered an element of $so(6)$. For the definition of the quasimomenta we take ρ a spinor representation and define $p_1(x), \cdots, p_4(x)$ as the logarithms of the eigenvalues of T:

$$T(x) = U\,\mathrm{diag}(e^{ip_1}, e^{ip_2}, e^{ip_3}, e^{ip_4})\; U^{-1}$$

Comparison with the results of the finite-gap approach

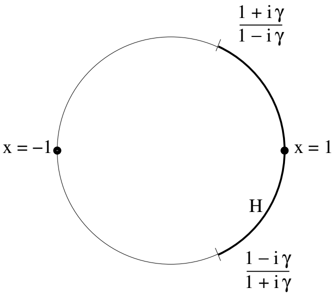
Results of Arutyunov and Zamaklar

$$\begin{aligned}\mathcal{E}(\gamma) = -\frac{i}{2\pi}\frac{\gamma^3}{1+\gamma^2} \quad & \left[p_1\left(\frac{1-i\gamma}{1+i\gamma}\right) + p_2\left(\frac{1-i\gamma}{1+i\gamma}\right) + \right. \\ & \left. + p_3\left(\frac{1+i\gamma}{1-i\gamma}\right) + p_4\left(\frac{1+i\gamma}{1-i\gamma}\right)\right]\end{aligned} \tag{51}$$

Introduce the notation ε_i: $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, 1, -1, -1)$. Substitution of (51) into Eq. (49) gives:

Comparison with the results of the finite-gap approach

Hamiltonian of ξ_γ and quasimomenta



$$H_\gamma = -\frac{i}{16\pi} \int_{x=\frac{1+i\gamma}{1-i\gamma}}^{x=\frac{1-i\gamma}{1+i\gamma}} d\left(x + \frac{1}{x}\right) \sum_{j=1}^{4} \varepsilon_j p_j(x) \tag{52}$$

Comparison with the results of the finite-gap approach

Hamiltonian of ξ_γ and quasimomenta

This suggests that in some sense

$$\sum_{j=1}^{4} \varepsilon_j p_j(x) \simeq \psi(\lambda)\tilde{\psi}(\lambda)$$

Summary

We discussed some aspects of the integrability of the string sigma model, which are local on the worldsheet.

- We have to prove the canonicity of Bäcklund transformations
- It would be interesting to understand the role of the τ-function in the worldsheet sigma-model; is there a description in terms of free fermions?
- We should think about Q-operator. Can we guess the Q-operator for the worldsheet σ-model? Can we guess it on the field theory side? Perhaps, there is an analogue of Bäcklund equations on the field theory side?